

Transcendentality in string amplitudes

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Bibliography

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Building on earlier work

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Introduction

- **Quantum Field Theory and String amplitudes**
 - exhibit structure beyond the implications from known symmetries.
- **QFT amplitudes are periods** (in the sense of Kontsevich and Zagier)
 - integrals whose integrand and boundary conditions are *algebraic*,
 - include multiple polylogarithms graded by transcendental weight,
 - planar $\mathcal{N} = 4$ SYM exhibits *uniform transcendentality* (i.e. *grading*),
- **Closed Type II superstring amplitudes**
 - **Can one define transcendentality ? If yes, is it a grading ?**
- **This talk: genus-one four-graviton amplitude in low energy expansion**
 - **evaluate analytic and non-analytic contributions**
 - **transcendentality grading exists (plus finer arithmetic properties)**
- **Related goal: KLT beyond tree-level**

Tree-level amplitude

- Four-graviton amplitude in Type II string perturbation theory

$$\mathbf{A}(k_i, \varepsilon_i) = \sum_{h=0}^{\infty} g_s^{2h-2} \mathcal{A}^{(h)}(s_{ij}) \mathcal{R}^4 \quad 1 \leq i, j \leq 4$$

- External momenta k_i with $k_i^2 = 0$ and $\sum_i k_i = 0$
- Dimensionless Lorentz-invariants $s_{ij} = -\alpha'(k_i + k_j)^2/4$
- all dependence on polarization tensors ε_i captured by \mathcal{R}^4

- Tree-level amplitude (cfr. Virasoro-Shapiro)

$$\mathcal{A}^{(0)}(s_{ij}) = \frac{1}{stu} \frac{\Gamma(1-s)\Gamma(1-t)\Gamma(1-u)}{\Gamma(1+s)\Gamma(1+t)\Gamma(1+u)}$$

- $s = s_{12}$, $t = s_{13}$, $u = s_{14}$ with $s + t + u = 0$
- simple poles in s, t, u at non-negative integers;
- Γ -factors holomorphic for $|s|, |t|, |u| < 1$ and admit Taylor series expansion

Low energy expansion of tree-level amplitude

- Low energy expansion of $\mathcal{A}^{(0)}(s_{ij})$ in powers of s, t, u

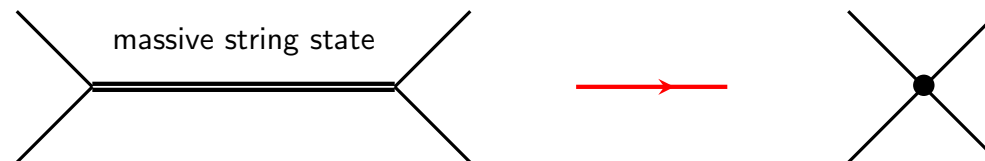
$$\mathcal{A}^{(0)}(s_{ij}) = \frac{1}{stu} \exp \left\{ - \sum_{n=1}^{\infty} \frac{2\zeta(2n+1)}{2n+1} (s^{2n+1} + t^{2n+1} + u^{2n+1}) \right\}$$

- Generates non-local massless exchange plus local effective interactions

$$\mathcal{A}^{(0)}(s_{ij}) = \frac{1}{stu} + 2\zeta(3) + \zeta(5)(s^2 + t^2 + u^2) + 2\zeta(3)^2 stu + \dots$$

massless
 \mathcal{R}^4
 $D^4\mathcal{R}^4$
 $D^6\mathcal{R}^4$

- Exchange of massive string states produces local effective interactions.



Multiple zeta values and polylogarithms

- Appearance of Riemann zeta-values is familiar from QFT amplitudes
 - at loop order, but not at tree-level in contrast with string amplitudes
- At low order QFT amplitudes are expressible in polylogarithms

$$\text{Li}_{a_1, \dots, a_\ell}(z_1, \dots, z_\ell) = \sum_{n_1 > n_2 > \dots > n_\ell > 0} \frac{z_1^{n_1} \dots z_\ell^{n_\ell}}{n_1^{a_1} \dots n_\ell^{a_\ell}}$$

- Multiple zeta values (MZV) $\zeta(a_1, \dots, a_\ell) = \text{Li}_{a_1, \dots, a_\ell}(1, \dots, 1)$
- Polylogs and MZVs satisfy shuffle and stuffle relations; e.g. stuffle

$$\text{Li}_a(y)\text{Li}_b(z) = \text{Li}_{a,b}(y, z) + \text{Li}_{b,a}(z, y) + \text{Li}_{a+b}(yz)$$

$$\zeta(a)\zeta(b) = \zeta(a, b) + \zeta(b, a) + \zeta(a + b) \quad (\text{Euler 1730})$$

- For $a, b \in \mathbb{N}$ and $a + b$ odd $\zeta(a, b)$ is always reducible to zeta-values
- For $a + b$ even $\zeta(a, b)$ is not always reducible to zeta-values

Assigning transcendental weight

- The transcendental *weight* w for polylogs and MZVs is defined by,

$$w = a_1 + \cdots + a_\ell \quad \text{e.g. } w[\zeta(a)] = a, \quad w[\pi] = 1$$

- The weight is conserved by the shuffle and stuffle relations and provides a *grading* on the algebra of polylogs and MZVs

- Can we assign a definite weight to a string amplitude ?

- Yes, for tree-level, recalling the 4-graviton amplitude,

$$\mathcal{A}^{(0)}(s_{ij}) = \frac{1}{stu} \exp \left\{ - \sum_{n=1}^{\infty} \frac{2\zeta(2n+1)}{2n+1} (s^{2n+1} + t^{2n+1} + u^{2n+1}) \right\}$$

- has definite weight iff we assign $w[s] = w[t] = w[u] = -1$.
- Generalizes to tree-level amplitudes with more states using MZVs.

(Schlotterer, Stieberger 2013-14)

- How about genus-one amplitudes ?

- no explicit formula available \Rightarrow much more complicated/interesting.

Genus-one four-graviton amplitude in Type II

- The reduced amplitude is given by an integral (Green, Schwarz 1982)
 - of a $SL(2, \mathbb{Z})$ -invariant function $\mathcal{B}(s_{ij}|\tau)$ of $\tau \in \mathbb{H} = \{\tau \in \mathbb{C}, \text{Im}(\tau) > 0\}$,

$$\mathcal{A}^{(1)}(s_{ij}) = \int_{\mathcal{M}} \frac{d^2\tau}{\tau_2^2} \mathcal{B}(s_{ij}|\tau)$$

- over the moduli space $\mathcal{M} = SL(2, \mathbb{Z}) \backslash \mathbb{H}$ of tori, represented by

$$\mathcal{M} = \left\{ \tau = \tau_1 + i\tau_2, \tau_1, \tau_2 \in \mathbb{R}, \tau_2 > 0, |\tau| \geq 1, |\tau_1| \leq 1/2 \right\}$$

- \mathcal{B} is an integral over 4 copies of the torus $\Sigma_\tau = \mathbb{C}/\Lambda_\tau$ with $\Lambda_\tau = \mathbb{Z}\tau + \mathbb{Z}$

$$\mathcal{B}(s_{ij}|\tau) = \left(\prod_{i=1}^4 \int_{\Sigma_\tau} \frac{d^2 z_i}{\tau_2} \right) \exp \left\{ \sum_{1 \leq i < j \leq 4} s_{ij} G(z_i - z_j|\tau) \right\}$$

- The scalar Green function $G(z|\tau)$ on Σ_τ is defined by

$$\tau_2 \partial_z \partial_{\bar{z}} G(z|\tau) = -\pi \delta(z) + \pi \quad \int_{\Sigma_\tau} d^2 z G(z|\tau) = 0$$

Low energy expansion of $\mathcal{B}(s_{ij}|\tau)$

- For fixed modulus τ the integrals defining $\mathcal{B}(s_{ij}|\tau)$
 - are absolutely convergent for $\text{Re}(s_{ij}) < 1$ and analytic near $s_{ij} = 0$;
 - have convergent Taylor expansion for $|s|, |t|, |u| < 1$;
 - in terms of symmetric functions $\sigma_k = s^k + t^k + u^k$ for $k = 2, 3$,

$$\mathcal{B}(s_{ij}|\tau) = \sum_{p,q=0}^{\infty} \mathcal{B}_{(p,q)}(\tau) \frac{\sigma_2^p \sigma_3^q}{p! q!}$$

- Assigning the same weight to s, t, u as for tree-level, namely -1 ,
 - we can assign a definite weight, namely 0 , to $\mathcal{B}(s_{ij}|\tau)$ iff

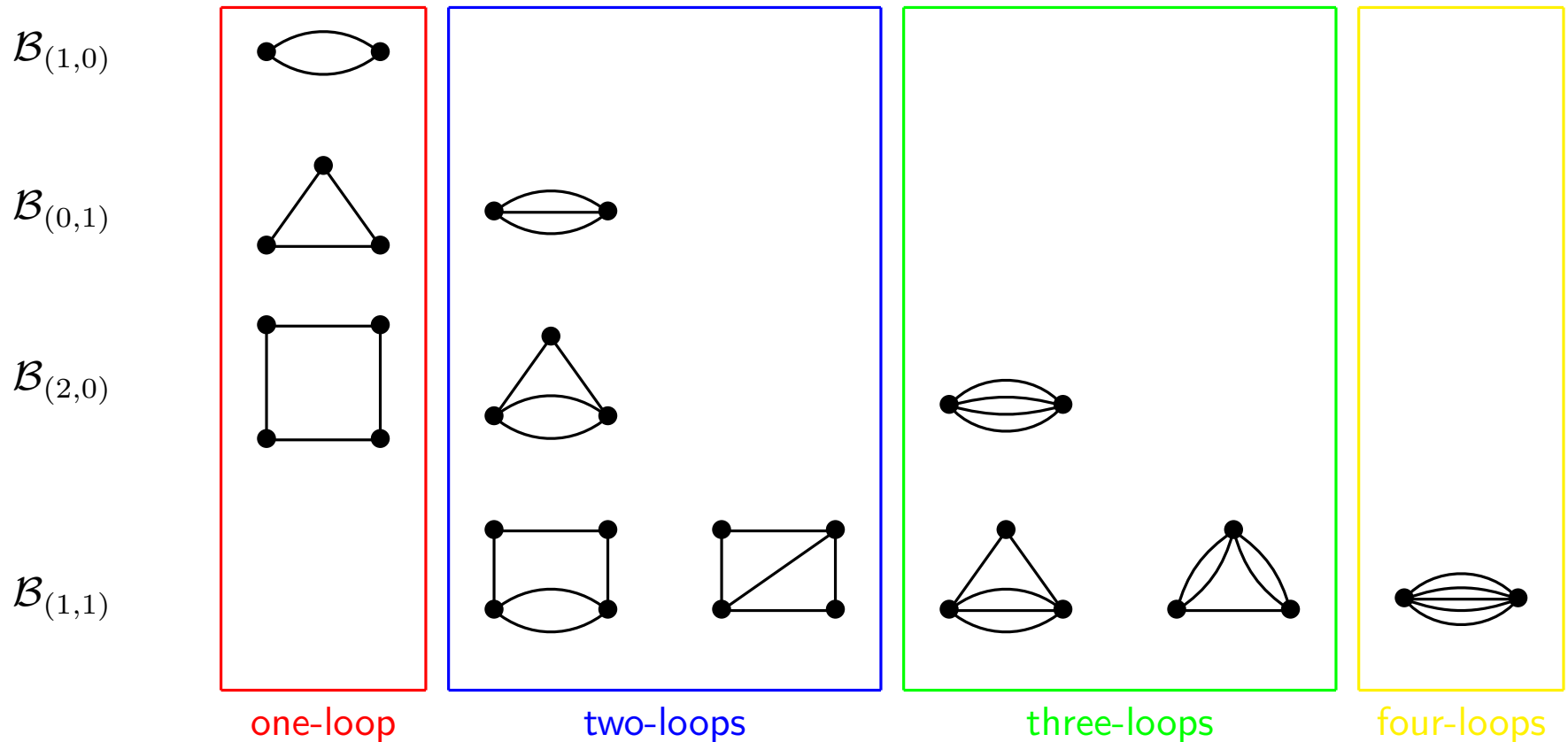
$$w[\mathcal{B}_{(p,q)}(\tau)] = 2p + 3q$$

- The coefficients $\mathcal{B}_{(p,q)}(\tau)$ admit a natural graphical expansion
 - an integration point z_i is represented by a vertex of the graph,
 - a Green functions $G(z_i - z_j|\tau)$ by an edge joining z_i and z_j ;
 - a Green function cannot begin and end on the same vertex.

Modular graph functions

- To every graph corresponds a modular function

Green function : $\begin{array}{c} \bullet \text{---} \bullet \\ z_i \quad z_j \end{array} = G(z_i - z_j | \tau)$



One-loop modular graph functions

- Fourier series for the scalar Green function,

$$G(z|\tau) = \sum_{p \in \Lambda'_\tau} \tau_2 \frac{e^{2\pi i(n\beta - m\alpha)}}{\pi |p|^2} \quad \begin{cases} z = \alpha + \beta\tau & \alpha, \beta \in \mathbb{R} \\ p = m\tau + n & m, n \in \mathbb{Z} \end{cases}$$

- One-loop graphs with $a \geq 2$ vertices $z_1, z_2, \dots, z_a, z_{a+1} = z_1$

$$\prod_{i=1}^a \int_{\Sigma} \frac{d^2 z_i}{\tau_2} G(z_i - z_{i+1}|\tau) = \sum_{p \in \Lambda'} \frac{\tau_2^a}{\pi^a |p|^{2a}} = E_a(\tau)$$

- E_a is the “real analytic Eisenstein series”
- satisfies $\Delta E_a = a(a-1)E_a$ with $\Delta = 4\tau_2^2 \partial_{\bar{\tau}} \partial_{\tau}$
- invariant under $SL(2, \mathbb{Z})$
- asymptotics in $y = \pi\tau_2$ near the cusp $\tau \rightarrow i\infty$,

$$E_a(\tau) = \frac{2\zeta(2a)}{\pi^{2a}} y^a + \frac{\Gamma(2a-1)\zeta(2a-1)}{2^{2a-3} \Gamma(a)^2 y^{a-1}} + \mathcal{O}(e^{-2y})$$

assigning $w[\tau] = 0$ and $w[y] = 1$ gives both terms weight a .

- assign weight $w[E_a(\tau)] = a$ for all $a \in \mathbb{N}$

Two-loop modular graph functions

- Two-loop graphs evaluate to multiple Kronecker-Eisenstein sums

$$C_{a,b,c}(\tau) = \sum_{p_r \in \Lambda'_\tau} \delta\left(\sum_r p_r\right) \frac{\tau_2^{a+b+c}}{\pi^{a+b+c} |p_1|^{2a} |p_2|^{2b} |p_3|^{2c}}$$

- convergent for $a, b, c \geq 1$
- invariant under $SL(2, \mathbb{Z})$
- assign weight $w[C_{a,b,c}] = a + b + c = w$
- satisfy differential identities which conserve the weight

$$\begin{aligned} 2\Delta C_{a,b,c} = & a(a-1) C_{a,b,c} + 2ab C_{a+1,b-1,c} - 4ab C_{a+1,b,c-1} \\ & + ab C_{a+1,b+1,c-2} + 5 \text{ permutations of } (a, b, c) \end{aligned}$$

- asymptotics in $y = \pi\tau_2$ near the cusp conserves the weight

$$C_{a,b,c}(\tau) = c_w y^w + c_{2-w} y^{2-w} + \sum_{k=1}^{w-1} c_{w-2k-1} \zeta(2k+1) y^{w-2k-1} + \mathcal{O}(e^{-2y})$$

- $c_w, c_{w-2k-1} \in \mathbb{Q}$, c_{2-w} is bilinear in odd zeta values of weight $2w - 2$.

(ED & Bill Duke 2017)

Higher-loop modular graph functions

- **Higher-loop graphs all evaluate to multiple Kronecker-Eisenstein sums**
 - RULE of THUMB: summing over momenta, each πp_r has weight zero !

e.g.
$$E_a(\tau) = \sum_{p \in \Lambda'} \frac{(\pi \tau_2)^a}{|\pi p|^{2a}}$$

- All differential (and algebraic) identities conserve the weight.

- **Evaluation of coefficients $\mathcal{B}_{(p,q)}$ in terms of Kronecker-Eisenstein sums**
 - gives a linear combination of modular graph functions,
 - each one of which has weight $w = 2p + 3q$,
 - asymptotics near cusp generally involves MZVs (Zerbini 2017)

⇒ **Consistent assignment of transcendental weight**

$$w[\mathcal{B}_{(p,q)}(\tau)] = 2p + 3q$$

- String integrand $\mathcal{B}(s_{ij}|\tau)$ consistently has weight 0.

Genus-one Type II string amplitude

- Obtained by integrating $\mathcal{B}(s_{ij}|\tau)$ over modulus τ

$$\mathcal{A}^{(1)}(s_{ij}) = \int_{\mathcal{M}} \frac{d^2\tau}{\tau_2^2} \mathcal{B}(s_{ij}|\tau)$$

- Inherits the OPE pole singularities of $\mathcal{B}(s_{ij}|\tau)$ at $s_{ij} \in \mathbb{N}$.
- Further singularities due to the non-uniform behavior of G in τ ;
- To see this, extract from G the constant Fourier mode in α ,

$$G(z|\tau) = 2\pi\tau_2 \left(\beta^2 - |\beta| + \frac{1}{6} \right) + g(z|\tau) \quad z = \alpha + \tau\beta$$

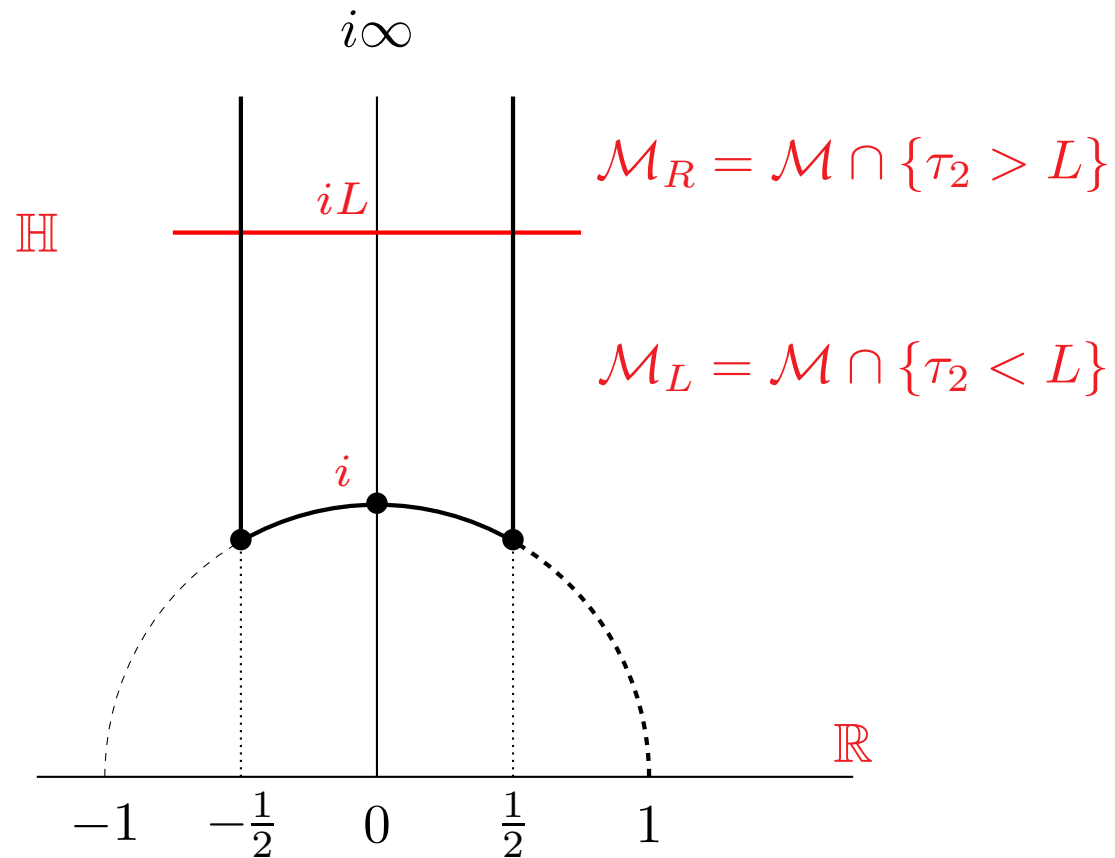
where the non-constant Fourier modes are given by $g(z|\tau)$,

$$g(z|\tau) = \sum_{m \neq 0} \sum_{k \in \mathbb{Z}} \frac{1}{|m|} e^{2\pi i m (\alpha + \tau_1(k+\beta)) - 2\pi\tau_2 |m(k+\beta)|}$$

- Absolute convergence of τ -integral requires s_{ij} purely imaginary
 - analytic continuation in s_{ij} was shown to exist (ED, Phong 1994)
 - produces poles and branch cuts in s_{ij} expected from unitarity.

Partitioning the moduli space of the torus

- Isolate a neighborhood of the cusp, for $L > 1$,



Partitioning the moduli space of the torus (cont'd)

- The contribution from \mathcal{M}_L is analytic in s, t, u for $|s|, |t|, |u| < 1$

$$\mathcal{A}_L(L; s_{ij}) = \sum_{p,q=0}^{\infty} \mathcal{A}_{(p,q)}(L) \frac{\sigma_2^p \sigma_3^q}{p! q!} \quad \mathcal{A}_{(p,q)}(L) = \int_{\mathcal{M}_L} \frac{d^2\tau}{\tau_2^2} \mathcal{B}_{(p,q)}(\tau)$$

– recall $\sigma_k = s^k + t^k + u^k$

- The contribution from \mathcal{M}_R is non-analytic in s, t, u for $|s|, |t|, |u| < 1$

$$\mathcal{A}_R(L; s_{ij}) = \int_L^{\infty} \frac{d\tau_2}{\tau_2^2} \int_0^1 d\tau_1 \mathcal{B}(s_{ij}|\tau)$$

– requires analytic continuation in s_{ij}

- All L -dependence cancels in $\mathcal{A}_L(L; s_{ij}) + \mathcal{A}_R(L; s_{ij})$

Partitioning the moduli space of the torus (cont'd)

- Use the freedom in choosing L to organize calculations

- Choosing $L \gg 1$ allows one to omit exponentials of L .
- E.g. the asymptotics of modular graph functions for $w = 2p + 3q$

$$\mathcal{B}_{(p,q)}(\tau) = \sum_{k=1-w}^w \mathfrak{b}_k \tau_2^k + \mathcal{O}(e^{-2\pi\tau_2})$$

- determines \mathcal{A}_L , up to an L -independent constant $\mathcal{A}_{(p,q)}^{(0)}$

$$\mathcal{A}_{(p,q)}(L) = \mathcal{A}_{(p,q)}^{(0)} + \mathfrak{b}_1 \ln L + \sum_{\substack{k=1-w \\ k \neq 1}}^w \frac{\mathfrak{b}_k L^{k-1}}{k-1} + \mathcal{O}(e^{-2\pi L})$$

- Since L -dependence cancels in total amplitude, only $\mathcal{A}_{(p,q)}^{(0)}$ contributes
- for later use, also keep $\ln L$ terms

Calculating the analytic part

- Exploit algebraic and differential identities to simplify $\mathcal{B}_{(p,q)}$

(ED, Green 2016; ED, Kaidi 2016; Broedel, Sclotterer, Zerbini 2018)

- Expose well-understood Eisenstein series
- Expose Laplacian of modular functions
- Laplacian of modular function f integrates by Stokes's theorem

$$\int_{\mathcal{M}_L} \frac{d^2\tau}{\tau_2^2} \Delta f(\tau) = \int_0^1 d\tau_1 \partial_{\tau_2} f(\tau) \Big|_{\tau_2=L}$$

- Up to weight 6 we have $\mathcal{B}_{(0,0)} = 1$, $\mathcal{B}_{(1,0)} = E_2$, $3\mathcal{B}_{(0,1)} = 5E_3 + \zeta(3)$ and

$$\begin{aligned} \mathcal{B}_{(2,0)} &= \Delta C_{2,1,1} - 10E_4 + 2E_2^2 \\ 180 \mathcal{B}_{(1,1)} &= 70 \Delta C_{3,1,1} - 1612E_5 + 580E_2E_3 + 60E_2\zeta(3) + 29\zeta(5) \\ 6 \mathcal{B}_{(3,0)} &= \Delta \left(-9C_{2,2,1,1} + 18E_3^2 + 9E_2E_4 + 6C_{4,1,1} + 156C_{3,2,1} + 41C_{2,2,2} \right) \\ &\quad + 72E_2C_{2,1,1} - 12E_3^2 - 36E_2E_4 - 2652E_6 \\ 27 \mathcal{B}_{(0,2)} &= \Delta \left(9C_{2,2,1,1} - 18E_3^2 - 9E_2E_4 - 6C_{4,1,1} + 258C_{3,2,1} + 64C_{2,2,2} \right) \\ &\quad - 36E_2C_{2,1,1} + 483E_3^2 + 30\zeta(3)E(3) + 18E_2E_4 + 6E_2^3 - 3186E_6 + 3\zeta(3)^2 \end{aligned}$$

Calculating the analytic part (cont'd)

- Integrals over \mathcal{M}_L of product of Eisenstein series (see also Zagier 1981)
 - may be evaluated using the eigenvalue equations and Stokes's theorem

$$\begin{aligned} \int_{\mathcal{M}_L} \frac{d^2\tau}{\tau_2^2} (E_a \Delta E_b - E_b \Delta E_a) &= (b(b-1) - a(a-1)) \int_{\mathcal{M}_L} \frac{d^2\tau}{\tau_2^2} E_a E_b \\ &= \int_0^1 d\tau_1 (E_a \partial_{\tau_2} E_b - E_b \partial_{\tau_2} E_a) \Big|_{\tau_2=L} \end{aligned}$$

- Using the Laurent series of E_a, E_b the integral evaluates to,

$$\int_{\mathcal{M}_L} \frac{d^2\tau}{\tau_2^2} E_a E_b = \#L^{a+b-1} + \#L^{a-b} + \#L^{b-a} + \#L^{1-a-b} + \mathcal{O}(e^{-4\pi L})$$

- For $a \neq b \geq 2$ there are no terms of order L^0 and $\ln L$, thus set to 0.
- For $a = b \geq 2$ contributions of order L^0 and $\ln L$ are given by

$$\frac{16\Gamma(2a-1)\zeta(2a)\zeta(2a-1)}{(2\pi)^{2a-1}\Gamma(a)^2} \left[\ln(2L) + \frac{\zeta'(2a)}{\zeta(2a)} - \frac{\zeta'(2a-1)}{\zeta(2a-1)} - \Psi(2a-1) + \Psi(a) \right]$$

- where $\Psi(a) = \Gamma'(a)/\Gamma(a)$.
- Similarly for integrals of E_2^3 and $E_2 C_{2,1,1}$.

Calculating the analytic part (cont'd)

- **Summary of contributions from analytic part up to weight 6**
 - Omitting all non-zero power and exponential contributions in L
 - Dividing by the volume of moduli space $\text{Vol}(\mathcal{M}) = \frac{\pi}{3}$

$$\mathcal{A}_{(0,0)} = 1 \quad \mathcal{A}_{(0,1)} = \frac{\zeta(3)}{3} \quad \mathcal{A}_{(1,0)} = 0 \quad \mathcal{A}_{(1,1)} = \frac{29\zeta(5)}{180}$$

which all manifestly conserve transcendentality. Next, we have,

$$\mathcal{A}_{(2,0)} = \frac{4\zeta(3)}{15} \left[\ln(2L) + \frac{\zeta'(4)}{\zeta(4)} - \frac{\zeta'(3)}{\zeta(3)} - \frac{1}{4} \right]$$

$$\mathcal{A}_{(3,0)} = \frac{\zeta(5)}{105} \left[33 \ln(2L) - 2 \frac{\zeta'(6)}{\zeta(6)} - 33 \frac{\zeta'(5)}{\zeta(5)} + 70 \frac{\zeta'(4)}{\zeta(4)} - 35 \frac{\zeta'(2)}{\zeta(2)} + \frac{13}{2} \right]$$

$$\mathcal{A}_{(0,2)} = \frac{\zeta(5)}{135} \left[18 \ln(2L) + 23 \frac{\zeta'(6)}{\zeta(6)} - 18 \frac{\zeta'(5)}{\zeta(5)} - 10 \frac{\zeta'(4)}{\zeta(4)} + 5 \frac{\zeta'(2)}{\zeta(2)} - \frac{31}{12} \right] + \frac{\zeta(3)^2}{9}$$

- Transcendentality assignments are clearly more tricky !
- Postpone until the non-analytic part has been evaluated.

Calculating the non-analytic part

- **Non-analytic part has branch cuts in s, t, u starting at integers ≥ 0**
 - In low energy expansion, non-analyticity in s, t, u starts at zero.
 - Split Green function into constant and non-constant Fourier modes

$$G(z|\tau) = 2\pi\tau_2 \left(\beta^2 - |\beta| + \frac{1}{6} \right) + g(z|\tau) \quad z = \alpha + \beta\tau$$

- Treat constant mode exactly, and expand in powers of $g(z|\tau)$
- **To disentangle absolute values: partition into 6 channels (ED, Phong 1994)**

$$\mathcal{A}_R(L; s_{ij}) = \mathcal{A}_*(L; s, t) + 5 \text{ permutations of } s, t, u$$

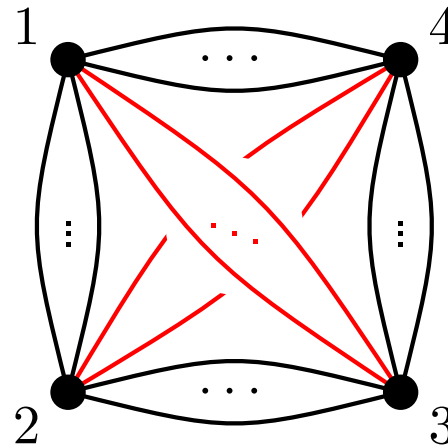
$$\mathcal{A}_*(L; s, t) = \int_L^\infty \frac{d\tau_2}{\tau_2^2} \int [dx] e^{4\pi\tau_2(sx_1x_3 + tx_2x_4)} \mathcal{F}(s_{ij}; x_i, \tau_2)$$

- changed variables from β_i to differences x_i of β_i
- measure $[dx] = dx_1 \cdots dx_4 \delta(1 - x_1 \cdots - x_4)$ with $x_i \in [0, 1]$

$$\mathcal{F}(s_{ij}; x_i, \tau_2) = \int_0^1 d\tau_1 \int_0^1 d\alpha_1 \cdots \int_0^1 d\alpha_4 \exp \left\{ \sum_{i < j} s_{ij} g(z_i - z_j | \tau) \right\}$$

Two Lemmas

- Graphical expansion in powers of $g_{ij} = g(z_i - z_j|\tau)$ in the s -channel

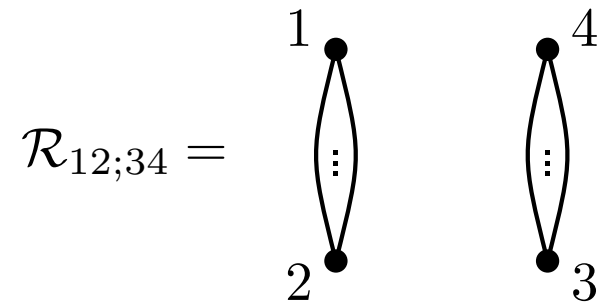
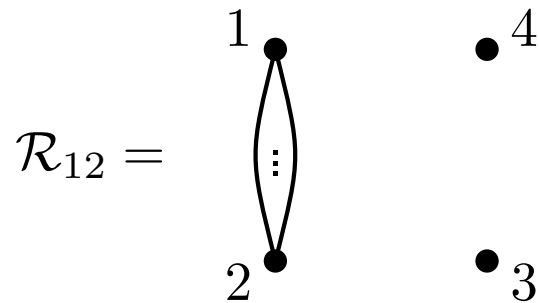


Lemma 1 Any contribution to $\mathcal{A}_*(L; s, t)$ from the Taylor expansion of \mathcal{F} in powers of s, t, u involving at least one factor of g_{13} or g_{24} is of order $\mathcal{O}(L^{-3})$.

Lemma 2 Any contribution to $\mathcal{A}_*(L; s, t)$ from the expansion of \mathcal{F} in powers of s, t, u involving at least one factor of $g_{ij}g_{ik}$ with $k \neq j$ is of order $\mathcal{O}(L^{-3})$.

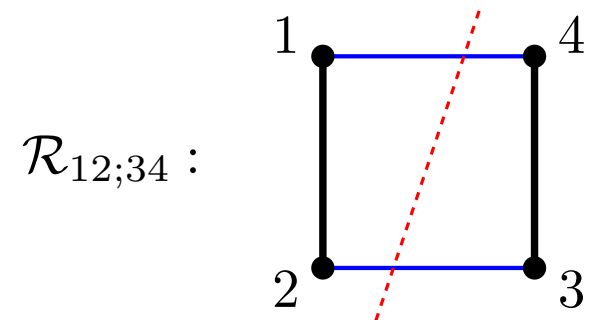
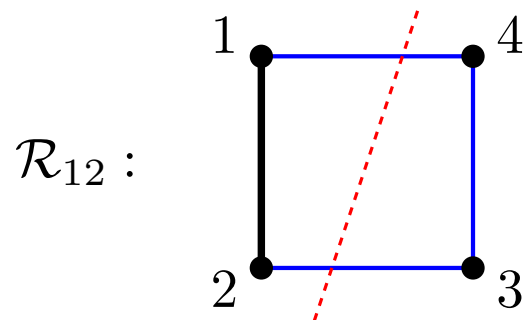
Only two contributions remain in the s -channel

- Representing the non-constant Fourier mode by full black lines



- Compare with massless cut structure in s -channel

- massive exchanges in black lines
- massless exchanges in blue lines
- s -channel cut in red lines



Non-analytic part

Theorem 1. To all orders in s, t , and orders $L^0, \ln L$ we have

$$A_*(L; s, t) = \sum_{N=2}^{\infty} \frac{s^N}{N!} \mathcal{R}_{12}^{(N)}(L; s, t) + \sum_{M, N=2}^{\infty} \frac{s^{M+N}}{M! N!} \mathcal{R}_{12;34}^{(M, N)}(L; s, t) + (s \leftrightarrow t)$$

– The coefficient functions are given by

$$\mathcal{R}_{12}^{(N)}(L; s, t) = \sum_{k=0}^{\infty} \sum_{\ell=0}^k C_{k, \ell} S(N, k+1) s^{k-\ell+2} t^{\ell}$$

$$\times \left[\ln(-4\pi Ls) + \Psi(k - \ell + 2) - 2\Psi(k + 7) \right]$$

$$\mathcal{R}_{12;34}^{(M, N)}(L; s, t) = \sum_{k, m, n=0}^{\infty} D_{k, m, n} S(M, k+m+1) S(N, k+n+1) s^{k+m+n+3} t^k$$

$$\times \left[\ln(-4\pi Ls) + \Psi(k + m + n + 4) - 2\Psi(2k + m + n + 8) \right]$$

where $C_{k, \ell}$ and $D_{k, m, n}$ are rational combinatorial coefficients, and,

$$S(N, k) = \sum_{\substack{m_r \neq 0 \\ r=1, \dots, N}} \frac{\delta(\sum_r m_r)}{|m_1 \cdots m_N| (|m_1| + \cdots + |m_N|)^k}$$

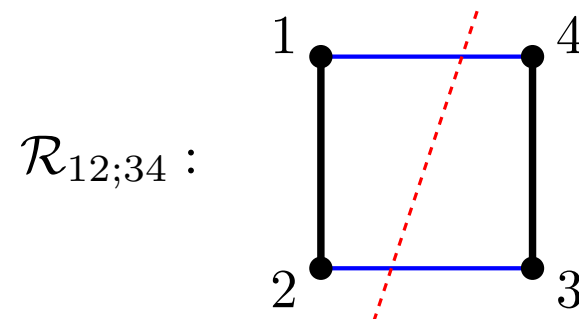
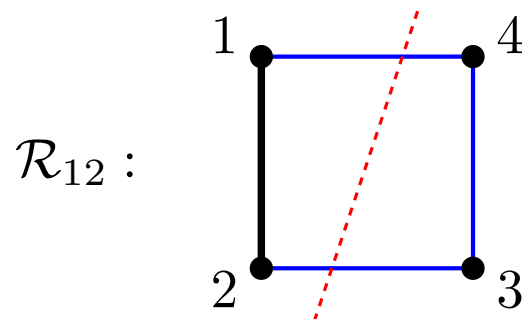
Multi-zeta value decomposition

- The coefficients $S(N, k)$ are linear combinations of MZVs (Zagier 2008)

$$S(N, k) = \sum_{\substack{a_1, \dots, a_r \\ a_1 + \dots + a_r = N-2}} N! 2^{2r+2-N-k} \zeta(k+2, a_1, \dots, a_r)$$

THEOREM 2. The s -channel partial amplitude $\mathcal{A}_*(L; s, t) + \mathcal{A}_*(L; s, u)$

- is free of irreducible multiple zeta-values
i.e. it is in the \mathbb{Q} -ring generated by odd zeta-values
- its discontinuity is the product of tree-level 4-point amplitudes



Transcendentality counting for $D^8\mathcal{R}^4$

- $D^8\mathcal{R}^4$ is the lowest order at which non-analyticities appear

$$\mathcal{A}_L = \frac{4\zeta(3)\sigma_4}{15} \left[\ln(2L) + \frac{\zeta'(4)}{\zeta(4)} - \frac{\zeta'(3)}{\zeta(3)} - \frac{1}{4} \right]$$

$$\mathcal{A}_R = \frac{4\zeta(3)s^4}{15} \left[-\ln(-4\pi Ls) - \gamma + \frac{17}{5} \right] + 2 \text{ perms of } s, t, u$$

– Their sum gives (we have factored out the volume $\text{Vol}(\mathcal{M}) = \frac{\pi}{3}$)

$$\mathcal{A}_L + \mathcal{A}_R = \frac{4\zeta(3)s^4}{15} \left[-\ln(-2\pi s) + \frac{\zeta'(4)}{\zeta(4)} - \frac{\zeta'(3)}{\zeta(3)} - \gamma + \frac{63}{20} \right] + 2 \text{ perms of } s, t, u$$

RULES towards conserving weight

✓ The argument $-2\pi s$ of the log consistently has weight 0;

✓ Assign weight 1 to $\ln(-2\pi s)$

✓ Assign weight 1 to the combination $\frac{\zeta'(4)}{\zeta(4)} - \frac{\zeta'(3)}{\zeta(3)} - \gamma$

?? Assign weight 1 to $\frac{63}{20}$

Transcendentality counting for $D^{10}\mathcal{R}^4$

- $D^{10}\mathcal{R}^4$ is purely analytic so that $\mathcal{A}_R = 0$

$$\mathcal{A}_{(1,1)\sigma_2\sigma_3} = \frac{29 \zeta(5) \sigma_2 \sigma_3}{180}$$

– manifestly conserves transcendentality

Transcendentality counting for $D^{12}\mathcal{R}^4$

- **Lowest order where there are two different kinematic invariants**
 - We introduce the following combinations

$$Z_n \equiv \frac{\zeta'(n)}{\zeta(n)} - \frac{\zeta'(n-1)}{\zeta(n-1)} - \gamma \quad n \geq 3$$

- The full amplitude to this order is given by

$$\begin{aligned} \mathcal{A}_L + \mathcal{A}_R = & -\frac{84s^6 + 2\sigma_2 s^4}{420} \zeta(5) \ln(-2\pi s) + 2 \text{ perms of } s, t, u \\ & + \frac{\zeta(5)\sigma_2^3}{18} \left[-\frac{2}{35} Z_6 - Z_5 + Z_4 + Z_3 + \frac{8339}{2450} \right] \\ & + \frac{\zeta(5)\sigma_3^2}{54} \left[\frac{23}{5} Z_6 + Z_5 - Z_4 - Z_3 + \frac{51011}{420} \right] + \frac{\zeta(3)^2 \sigma_3^2}{18} \end{aligned}$$

RULES towards conserving weight

- ✓ The argument $-2\pi s$ of the log consistently has weight 0;
- ✓ Assign weight 1 to $\ln(-2\pi s)$;
- ✓ Assign weight 1 to the combination Z_n for all $n \geq 3$
- ?? Assign weight 1 to other rationals ?

Interpreting the combination $Z_n = \frac{\zeta'(n)}{\zeta(n)} - \frac{\zeta'(n-1)}{\zeta(n-1)} - \gamma$

- Is Z_n a period ?
- Does it make sense to assign weight 1 to Z_n ?
 - Look at series representation

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \quad \zeta'(z) = - \sum_{n=1}^{\infty} \frac{\ln n}{n^z}$$

- Since $\ln n$ is assigned weight 1 for $n \in \mathbb{Q}$ it “adds” weight 1
- The following combination can be assigned weight 1

$$Z_m - Z_n = \frac{d}{dz} \ln \frac{\zeta(m+z)\zeta(n-1+z)}{\zeta(m-1+z)\zeta(n+z)} \Big|_{z=0}$$

- The argument of the log has weight zero
- The log itself produces weight 1.
- What is the role of γ ?
 - Can one relate this to $\zeta(1+z) = z^{-1} + \gamma + \mathcal{O}(z)$?

The weight of finite harmonic sums

- An isolated rational number must have weight 0
- **The rational numbers all arise from series of finite harmonic sums**
 - **non-analytic part** from $\mathcal{R}_{12}^{(N)}$ (similarly for analytic part up to weight 6)

$$\sum_{k,\ell} \#(k, \ell) s^{k-\ell+2} t^\ell \left(\ln(-4\pi Ls) + \Psi(1) + \Psi(k - \ell + 2) - 2\Psi(k + 7) \right)$$

- differences of Ψ -functions at integers are finite harmonic sums

$$\Psi(m + n) - \Psi(n) = \sum_{k=0}^{m-1} \frac{1}{k + n}$$

- assign weight 1 when they occur in a series, e.g.

$$\zeta(a, 1) = \sum_{m > n \geq 1} \frac{1}{m^a n} = \sum_{m=2}^{\infty} \frac{1}{m^a} \sum_{n=1}^{m-1} \frac{1}{n}$$

(Kotikov, Lipatov 2002; Beccaria, Forini 2009), (conversations with Broadhurst, Zagier)

Conclusions and outlook

- **Explicit evaluation of non-analytic part of one-loop string amplitude**
 - Local effective interactions require its normalization;
 - General structure of analytic part known

- **Consistent assignment of conserved transcendental weight**

- **Simple, but striking, observation at order $D^{12}\mathcal{R}^4$**

- recall two invariants: σ_2^3 and σ_3^2

$$\frac{\zeta(3)^2 \sigma_3^2}{18} + 0 \cdot \zeta(3)^2 \sigma_2^3$$

- First term can arise from “square of tree-level”, but not the second term !
- Why ? Is there a pattern ?

- **Extension to higher-point functions at genus one and higher genus**

- Cuts involve tree-level higher-point functions and MZVs
- Higher genus modular graph functions (ED, Green, Pioline 2018; ED, Schlotterer 2020)

Thank You !