Modular Graph Functions, Forms and Tensors

Eric D'Hoker

Mani L. Bhaumik Institute for Theoretical Physics

Department of Physics and Astronomy, UCLA

New connections in number theory and physics

Isaac Newton Institute, Cambridge

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Happy 75-th birthday to Michael

Introduction

- **Modular graph functions**
  - ⋆ map graphs to $SL(2,\mathbb{Z})$-invariant functions on Poincaré half-plane
  - ⋆ generalize non-holomorphic Eisenstein series
  - ⋆ may be generalized to $SL(2,\mathbb{Z})$-covariant modular graph forms

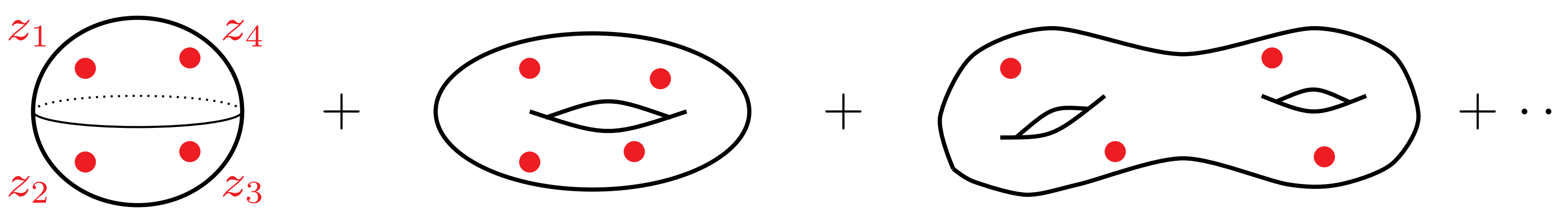
- **Higher genus modular graph functions**
  - ⋆ map graphs to functions on the moduli space of Riemann surfaces
  - ⋆ generalize invariants of Kawazumi and Zhang and Faltings
  - ⋆ may be generalized to modular graph tensors under the modular group

- **Modular graph functions naturally emerge from string theory**
  - ⋆ string perturbation theory is a genus expansion
  - ⋆ low energy expansion in terms of modular graph functions
  - ⋆ compare with predictions from S-duality in string theory

    (see the talks by Steve Miller and Boris Pioline in this conference)

Sums over Riemann surfaces

- **Perturbative string amplitudes are given by a topological expansion**



⋆ each vertex point z_i represents an incoming or outgoing string
⋆ for a given physical process, the number of vertex points N is fixed

⋆ for each genus g integrate
  – the N vertex points over the compact Riemann surface
  – over the moduli space $\mathcal{M}_g$ of compact Riemann surfaces

[for superstrings, integrals originate from moduli space of super Riemann surfaces]

Genus zero

- **The Riemann surface is a sphere** $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$
  - ⋆ Genus-zero graviton amplitudes involve integrals of the type

$$\prod_{i=1}^{N-3} \int_{\mathbb{C}} d^2 z_i \, |z_i|^{-2-2s_{i,N-1}} |1-z_i|^{-2s_{iN}} \prod_{j \neq i}^{N-3} |z_i - z_j|^{-2s_{ij}}$$

  - ⋆ $s_{ij} = s_{ji} \in \mathbb{C}$ are kinematic parameters satisfying $s_{ii} = 0$ and $\sum_j s_{ij} = 0$
  - ⋆ meromorphic in s_{ij} with simple poles at non-negative integers

- **Four-graviton amplitude** (omitting kinematic factors)

$$\mathcal{A}_4^{(0)}(s_{ij}) = \prod_{i=2,3,4} \frac{\Gamma(1-s_{1i})}{s_{1i}\,\Gamma(1+s_{1i})} = \frac{1}{s_{12}s_{13}s_{14}} \exp \sum_{n \in 2\mathbb{N}+1} \frac{2\zeta(n)}{n} (s_{12}^n + s_{13}^n + s_{14}^n)$$

  - ⋆ Low energy expansion in powers of s_{ij} involves only odd zeta-values

- **Amplitudes with more than four gravitons**
  - ⋆ Expansion in powers of s_{ij} involves "single-valued" multiple zeta-values

(conjectured in Schlotterer, Stieberger 2012; Stieberger 2013; Stieberger, Taylor 2014)

(proofs in Schlotterer, Schnetz; Brown, Dupont; Vanhove, Zerbini 2018)

Genus-one

- **The Riemann surface is a torus** $\Sigma = \mathbb{C}/\Lambda$
  - ⋆ $\Lambda = \mathbb{Z}\tau \oplus \mathbb{Z}$ with modulus $\tau \in \mathcal{H}_1 = \{\tau \in \mathbb{C},\ \tau_2 = \mathrm{Im}\,\tau > 0\}$
  - ⋆ volume form $d\mu_z = \frac{i}{2\tau_2} dz \wedge d\bar z$ with unit area $\int_\Sigma d\mu_z = 1$
- **The scalar Green function** g

$$\tau_2 \partial_{\bar z} \partial_z g(z-z'|\tau) = \pi - \pi\delta(z-z'), \qquad \int_\Sigma d\mu_z\, g(z-z'|\tau) = 0$$

  - ⋆ in terms of a double Fourier series ($z = u\tau + v \in \Sigma$ with $u, v \in [0,1]$)

$$g(z|\tau) = \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{\tau_2}{\pi|m\tau+n|^2} e^{2\pi i(mv-nu)}$$

  - ⋆ invariant under the modular group $SL(2,\mathbb{Z})$

$$z \to \frac{z}{\gamma\tau+\delta} \qquad \tau \to \frac{\alpha\tau+\beta}{\gamma\tau+\delta} \qquad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{Z})$$

The genus-one amplitude

- **Define the following integral over N marked points**

$$\mathcal{B}_N^{(1)}(s_{ij}|\tau) = \prod_{k=1}^{N} \int_\Sigma d\mu_{z_k} \; \exp \Big\{ \sum_{1 \leq i < j \leq N} s_{ij} \, g(z_i - z_j|\tau) \Big\}$$

  - ⋆ $s_{ij} \in \mathbb{C}$ are the same kinematic parameters as for genus zero
  - ⋆ poles at $s_{ij} = 1, 2, 3, \cdots$
  - ⋆ holomorphic in s_{ij} for $|s_{ij}| < 1$;
  - ⋆ invariant under $SL(2, \mathbb{Z})$

- **Genus-one four graviton amplitude** (Green, Schwarz, 1982)
  - ⋆ Four graviton amplitude is provided precisely by $\mathcal{B}_4^{(1)}$

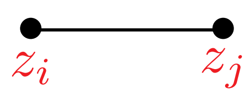
$$\mathcal{A}_4^{(1)}(s_{ij}) = \int_{\mathcal{M}_1} \frac{d^2\tau}{\tau_2^2} \mathcal{B}_4^{(1)}(s_{ij}|\tau) \qquad\qquad \mathcal{M}_1 = PSL(2, \mathbb{Z}) \backslash \mathcal{H}_1$$

  - ⋆ The integral is absolutely convergent only for $\mathrm{Re}\,(s_{ij}) = 0$
  - ⋆ Analytic continuation to $s_{ij} \in \mathbb{C}$ exists and produces branch cuts

Graphical Representation of Taylor series of $\mathcal{B}_N^{(1)}(s_{ij}|\tau)$

- **Absolute convergence of the integral $\mathcal{B}_N^{(1)}(s_{ij}|\tau)$ for $|s_{ij}| < 1$ and fixed τ**
  - ⋆ allows for Taylor expansion in the variables s_{ij}
    - $=$ physically contributes to the "low energy expansion"

- **Represented by Feynman graphs** (Green, Russo, Vanhove 2008)
  - ⋆ Each integration point z_i on Σ is represented by a vertex •
  - ⋆ Each Green function by an edge between vertices z_i and z_j



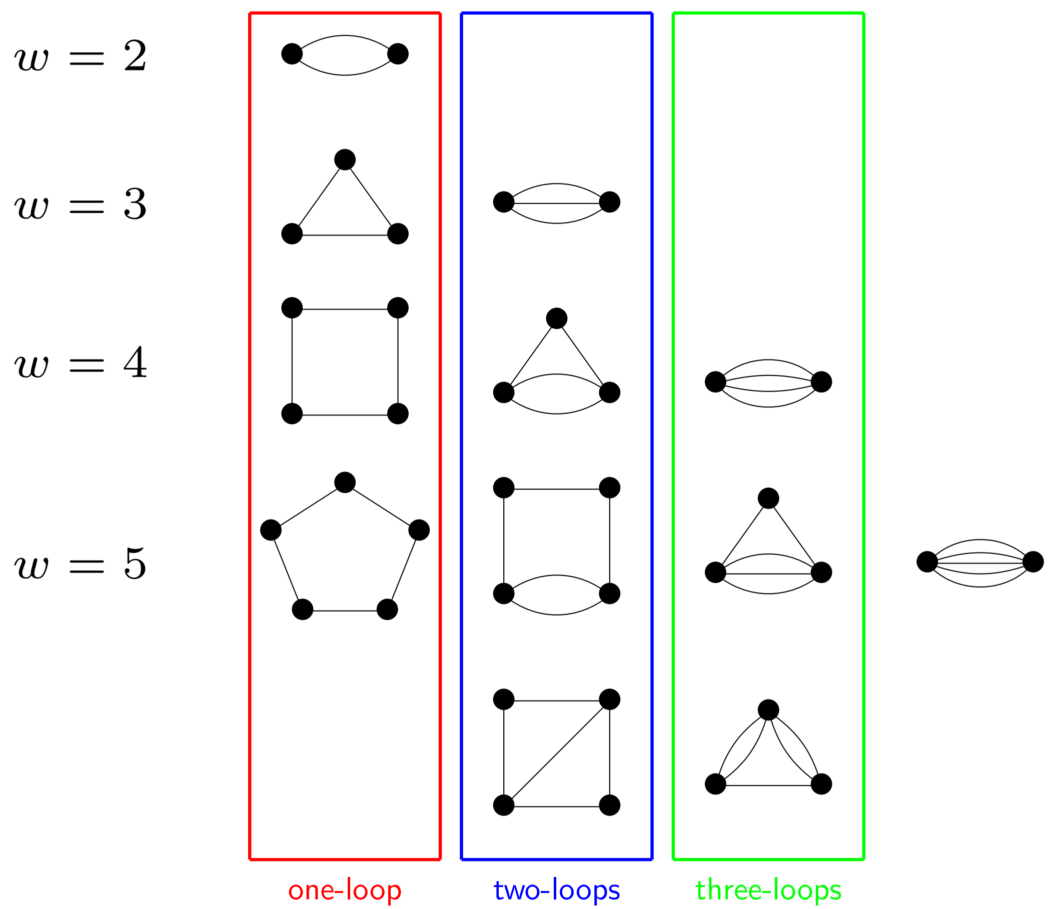
$$= \quad g(z_i - z_j|\tau)$$

  - ⋆ Each vertex z_i is integrated with $\int_\Sigma d\mu_{z_i}$
  - ⋆ To a graph with w edges we assign *weight* w

- **Reducibility** : A graph which becomes disconnected upon
  - ⋆ cutting one edge vanishes by $\int_\Sigma g = 0$
  - ⋆ removing one vertex factorizes into the product of its components

Modular graph functions

- **To each graph is associated a non-holomorphic modular function**
  since $\mathcal{B}_N^{(1)}(s_{ij}|\tau)$ is $SL(2,\mathbb{Z})$-invariant, so are its Taylor coefficients
  (ED, Green, Vanhove 2015; Zerbini (2015); ED, Green, Gurdogan, Vanhove 2015)



One-loop modular graph functions = Eisenstein series

- **One-loop graph of weight** w

$$\prod_{i=1}^{w} \int_\Sigma \frac{d^2 z_i}{\tau_2} g(z_i - z_{i+1}|\tau) = \sum_{p \in \Lambda'} \frac{\tau_2^w}{\pi^w |p|^{2w}} = E_w(\tau)$$

  ⋆ $\Lambda = \mathbb{Z}\tau + \mathbb{Z}$ is the lattice of "momenta" p and we set $\Lambda' = \Lambda \setminus \{0\}$

- **Non-holomorphic Eisenstein series** $E_s(\tau)$ **for** $s \in \mathbb{C}$

  ⋆ Invariant under $SL(2,\mathbb{Z})$

  ⋆ Eigenfunction of the Laplace-Beltrami operator $\Delta = 4\tau_2^2 \partial_\tau \partial_{\bar\tau}$ on $\mathcal{H}_1$

$$\Delta E_s = s(s-1) E_s$$

  ⋆ Expansion near the cusp $\tau \to i\infty$ (constant Fourier mode)

$$E_s = \frac{2\zeta(2s)}{\pi^s} \tau_2^s + \frac{2\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)\, \pi^{s-\frac{1}{2}}} \tau_2^{1-s} + \mathcal{O}(e^{-4\pi\tau_2})$$

  ⋆ Fourier series, Poincaré series, eigenfunctions of Hecke operators, etc

Two-loop modular graph functions

- **Two-loop graphs evaluate to a multiple Kronecker-Eisenstein series**

$$C_{a_1,a_2,a_3}(\tau) = \sum_{p_1,p_2,p_3 \in \Lambda'} \delta\Big(\sum_{r=1}^3 p_r\Big) \prod_{r=1}^3 \left(\frac{\tau_2}{\pi |p_r|^2}\right)^{a_r}$$

  - ⋆ "momentum conservation" δ-function results from translation invariance
  - ⋆ absolutely convergent for $a_r \in \mathbb{N}$, of weight $w = a_1 + a_2 + a_3$
  - ⋆ invariant under $SL(2,\mathbb{Z})$

- **Theorem** (ED & Bill Duke 2017)

  Expansion as $\tau \to i\infty$: Laurent polynomial in τ_2 of degree $(w, 1-w)$

$$C_{a_1,a_2,a_3}(\tau) = c_w(-4\pi\tau_2)^w + \frac{c_{2-w}}{(4\pi\tau_2)^{w-2}} + \sum_{k=1}^{w-1} \frac{c_{w-2k-1}\,\zeta(2k+1)}{(4\pi\tau_2)^{2k+1-w}} + \mathcal{O}(e^{-2\pi\tau_2})$$

  ⋆ $c_w, c_{w-2k-1} \in \mathbb{Q}$; c_{2-w} *bilinear in odd zeta values with coeffs in* $\mathbb{Q}$

- **Fourier and Poincaré series** (ED & Kaidi 2019; Dorigoni & Kleinschmidt 2019)

System of differential and algebraic identities

- **Theorem** (ED, Green, Vanhove 2015)
  All modular graph functions $C_{a,b,c}$ *for* $a,b,c \in \mathbb{N}$ *of weight* $w = a+b+c$ *are linear combinations over* $\mathbb{Q}$ *of modular functions* $\mathfrak{C}_{w;s;n}$ *satisfying*

$$\big(\Delta - s(s-1)\big)\mathfrak{C}_{w;s;n} = \mathfrak{F}_{w;s;n}$$

  ⋆ *where* $s = w-2, w-4, \cdots, 1$ *or* 2 *and* $n = 0, \cdots, [\frac{1}{3}(s-1)]$
  ⋆ $\mathfrak{F}_{w;s;\mathfrak{p}}$ *is a polynomial of degree 2 and weight* w *in* $E_{s'}$ *with* $2 \leq s' \leq w$.

- **Certain differential identities imply algebraic identities of uniform weight**

$$\Delta C_{1,1,1} = 6E_3 \qquad \Delta(9C_{4,4,1} + 18C_{4,3,2} + 4C_{3,3,3}) = 288E_9$$

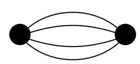
$$\Longrightarrow \; C_{1,1,1} = E_3 + \zeta(3) \qquad \Longrightarrow \; 9C_{4,4,1} + 18C_{4,3,2} + 4C_{3,3,3} = 4E_9 + \frac{\zeta(9)}{240}$$

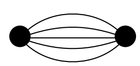
  ⋆ obtained by integrating using $\Delta E_s = s(s-1)E_s$
  and fixing the integration constant via asymptotics at cusp
  ⋆ $C_{1,1,1} = E_3 + \zeta(3)$ proven by direct summation (Zagier, unpublished)

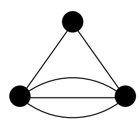
Modular graph functions at higher loop order

- **Expansion near the cusp** $\tau \to i\infty$
  - ⋆ Laurent polynomial in τ_2 of degree $(w, 1-w)$ $+\mathcal{O}(e^{-4\pi\tau_2})$
  - ⋆ coefficients include "single-valued" zeta-values and multiple zeta-values (Zerbini 2017; ED, Green 2019; Zagier, Zerbini 2019)

- **Modular graph functions satisfy algebraic identities of uniform weight**

e.g.  $= 24C_{2,1,1} + 3E_2^2 - 18E_4$

 $= 60C_{3,1,1} + 10E_2C_{1,1,1} - 48E_5 + 16\zeta(5)$

 $= \frac{15}{2}C_{3,1,1} + 3E_2E_3 - \frac{69}{10}E_5 + \frac{7}{40}\zeta(5)$

- **Laplace-Beltrami operator acting on 3 loops and higher**
  - ⋆ no longer maps the space of modular graph functions to itself

- **Amplitudes with** $N > 4$ **involve factors of** $\partial_z g(z|\tau)$ **in addition to** g
  - ⋆ produce unequal exponents of p_r and $\bar{p}_r$ (Green, Mafra, Schlotterer 2013)

$\Longrightarrow$ **requires generalization**

Arbitrary number of loops and exponents

- **Modular graph forms** (ED & Green 2016)

A decorated graph (Γ, A, B) *with* V *vertices and* R *edges*

- ⋆ *connectivity matrix* Γ_{vr}, $v = 1, \cdots, V$, $r = 1, \cdots, R$
- ⋆ *decoration of the edges by integer "exponents"* a_r, b_r

$A = [a_1, \cdots, a_R]$ and $B = [b_1, \cdots, b_R]$

To the decorated graph (Γ, A, B) *we associate a function on* $\mathcal{H}_1$

$$\mathcal{C}_\Gamma \begin{bmatrix} A \\ B \end{bmatrix} (\tau) = \sum_{p_1, \cdots, p_R \in \Lambda'} \left(\prod_{r=1}^{R} \frac{(\tau_2/\pi)^{a_r}}{(p_r)^{a_r} \, (\bar p_r)^{b_r}} \right) \prod_{v=1}^{V} \delta \left(\sum_{r=1}^{R} \Gamma_{vr} p_r \right)$$

- **Transformation under** $SL(2,\mathbb{Z})$

$$\mathcal{C}_\Gamma \begin{bmatrix} A \\ B \end{bmatrix} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta} \right) = (\gamma\bar\tau + \delta)^\mu \, \mathcal{C}_\Gamma \begin{bmatrix} A \\ B \end{bmatrix} (\tau) \qquad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{Z})$$

- ⋆ modular weight $(0, \mu)$ with $\mu = \sum_r (b_r - a_r)$: *"modular graph form"*
- ⋆ when $\mu \neq 0$ there is no canonical normalization for powers of τ_2
- ⋆ $A = B \Rightarrow \mu = 0$ recover modular graph functions

Identities between modular graph forms

- **Momentum conservation at each vertex v provides algebraic identities**

$$\sum_{r=1}^{R} \Gamma_{v\,r}\, \mathcal{C}\left[\begin{smallmatrix} A-S_r \\ B \end{smallmatrix}\right] = \sum_{r=1}^{R} \Gamma_{v\,r}\, \mathcal{C}\left[\begin{smallmatrix} A \\ B-S_r \end{smallmatrix}\right] = 0 \qquad\qquad S_r = [0_{r-1}\ 1\ 0_{R-r}]$$

- **The Maass operator $\nabla = 2i\tau_2^2 \partial_\tau$ provides differential identities**

$$\nabla\, \mathcal{C}\left[\begin{smallmatrix} A \\ B \end{smallmatrix}\right] = \sum_{r=1}^{R} a_r\, \mathcal{C}\left[\begin{smallmatrix} A+S_r \\ B-S_r \end{smallmatrix}\right]$$

- **Algorithm for modular graph function identities** (ED & Green 2016)

  If a linear combination F of weight w modular graph functions with $A = B$ satisfies $\nabla^n F = 0$ for an integer $n \geq 1$ then F is independent of τ.

  $\star$ Relations between holomorphic modular forms are used to establish $\nabla^n F = 0$

$\Longrightarrow$ **All identities between modular functions of weight $w \leq 6$**

(ED, Green 2016; ED, Kaidi 2016; Broedel, Schlotterer, Zerbini 2018)

– available in Mathematic package (Gerken 2020)

– special cases discussed in (Basu 2015-19; Kleinschmidt, Verschinnin 2017)

Relation to iterated elliptic integrals

- **Powers of ∇ relate to holomorphic modular forms**

$$\nabla^n E_n \sim \tau_2^{2n} G_{2n} \qquad \nabla^3 C_{2,1,1} \sim \tau_2^4 G_4 \nabla E_2 \qquad G_{2n} = \sum_{p \in \Lambda'} \frac{1}{p^{2n}}$$

⋆ where G_{2n} is a holomorphic modular form of weight $2n$ for $n \geq 2$.
⋆ Differential eqs are a non-holomorphic (or "single-valued") version of

$$\partial_\tau \mathcal{E}\left[\begin{smallmatrix} j_1, \cdots, j_r \\ k_1, \cdots, k_r \end{smallmatrix}\right](\tau) = \tau^{j_r} G_{k_r}(\tau)\, \mathcal{E}\left[\begin{smallmatrix} j_1, \cdots, j_{r-1} \\ k_1, \cdots, k_{r-1} \end{smallmatrix}\right](\tau)$$

⋆ Admit a (regularized) iterated integral representation

$$\mathcal{E}\left[\begin{smallmatrix} j_1, \cdots, j_r \\ k_1, \cdots, k_r \end{smallmatrix}\right](\tau) = \int_{i\infty}^{\tau} d\sigma\, \sigma^{j_r} G_{k_r}(\sigma)\, \mathcal{E}\left[\begin{smallmatrix} j_1, \cdots, j_{r-1} \\ k_1, \cdots, k_{r-1} \end{smallmatrix}\right](\sigma)$$

⋆ which satisfy shuffle relations

- **Modular graph forms may be expressed via iterated elliptic integrals**

(Broedel, Mafra, Matthes, Schlotterer 2014; Brown 2017; Broedel, Mafra, Schlotterer 2018; Gerken, Kleinschmidt, Schlotterer 2020; Gerken, Kleinschmidt, Mafra, Schlotterer, Verbeek 2020)

Elliptic Modular Graph functions and forms

- **Non-holomorphic single-valued elliptic functions**
  - ⋆ equivalently modular graph functions with character of the Abelian group Σ
  - ⋆ prototype example: the Siegel-Zagier single-valued elliptic polylogarithm

    for $a-b \in \mathbb{Z}$ and $z \in \Sigma$

$$D_{a,b}(z|\tau) = \frac{(2i\tau_2)^{a+b-1}}{2\pi i} \sum_{p \in \Lambda'} \frac{\chi_p(z|\tau)}{p^a \bar{p}^b} \qquad\qquad \chi_p(z|\tau) = e^{\pi(\bar{p}z - p\bar{z})/\tau_2}$$

- **Elliptic modular graph forms with arbitrary number of loops and exponents**

  A decorated graph (Γ, A, B, Z) *with* V *vertices and* R *edges with decorations*

  $A=[a_1,\cdots,a_R]$, $B=[b_1,\cdots,b_R]$, $Z=[z_1,\cdots z_R]$ *for* $a_r - b_r \in \mathbb{Z}$, $z_r \in \Sigma$ *maps to*

$$\mathcal{C}_\Gamma \left[\begin{smallmatrix} A \\ B \\ Z \end{smallmatrix} \right](\tau) = \sum_{p_1,\cdots,p_R \in \Lambda'} \left(\prod_{r=1}^R \frac{(\tau_2/\pi)^{a_r} \chi_{p_r}(z_r|\tau)}{(p_r)^{a_r}\,(\bar{p}_r)^{b_r}} \right) \prod_{v=1}^V \delta \left(\sum_{r=1}^R \Gamma_{vr} p_r \right)$$

  - ⋆ differential and algebraic identities (Basu 2020; ED, Kleinschmidt, Schlotterer 2020)
  - ⋆ relation to iterated integrals (ED, Hidding, Kleinschmidt, Schlotterer, Verbeek 2021)
  - ⋆ provides modular graph forms for congruence subgroups of $SL(2,\mathbb{Z})$

    e.g. for $\Gamma(n)$ with z_i torsion points $nz_i \equiv 0 \pmod{\Lambda}$

Higher genus

- **How to generalize the genus-one construction to higher genus ?**
  - ⋆ recall the genus-one generating function

$$\mathcal{B}_N^{(1)}(s_{ij}|\tau) = \prod_{i=1}^N \int_\Sigma d\mu_{z_i} \, \exp\left\{ \sum_{1\leq i<j\leq N} s_{ij}\, g(z_i - z_j|\tau) \right\}$$

- **Compact Riemann surface Σ of genus $h \geq 2$ without boundary**
  - ⋆ we need a scalar Green function $G(z_i, z_j|\Sigma)$
  - ⋆ and a measure $d\mu_N$ on Σ^N

$$\mathcal{B}_N^{(h)}(s_{ij}|\Sigma) = \int_{\Sigma^N} d\mu_N \, \exp\left\{ \sum_{1\leq i<j\leq N} s_{ij}\, G(z_i, z_j|\Sigma) \right\}$$

Compact Riemann surfaces Σ of genus h

- **Homology and cohomology**
  - ⋆ One-cycles $H_1(\Sigma,\mathbb{Z}) \approx \mathbb{Z}^{2h}$ with intersection pairing $\mathfrak{J}(\cdot,\cdot) \to \mathbb{Z}$
  - ⋆ Canonical basis $\mathfrak{J}(\mathfrak{A}_I,\mathfrak{A}_J) = \mathfrak{J}(\mathfrak{B}_I,\mathfrak{B}_J) = 0$, $\mathfrak{J}(\mathfrak{A}_I,\mathfrak{B}_J) = \delta_{IJ}$ for $1 \leq I, J \leq h$
  - ⋆ Canonical dual basis of holomorphic one-forms ω_I in $H^{(1,0)}(\Sigma)$

$$\oint_{\mathfrak{A}_I} \omega_J = \delta_{IJ} \qquad \qquad \oint_{\mathfrak{B}_I} \omega_J = \Omega_{IJ}$$

  - ⋆ Period matrix Ω obeys Riemann relations $\Omega^t = \Omega, \ \mathrm{Im}\,(\Omega) > 0$
  - ⋆ Jacobian variety $J(\Sigma) = \mathbb{C}^h/(\mathbb{Z}^h\Omega + \mathbb{Z}^h)$

- **Modular group** $Sp(2h,\mathbb{Z}) : H_1(\Sigma,\mathbb{Z}) \to H_1(\Sigma,\mathbb{Z})$ **leaves** $\mathfrak{J}(\cdot,\cdot)$ **invariant**

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad \qquad M^t \mathfrak{J} M = \mathfrak{J} \qquad \qquad \begin{pmatrix} \mathfrak{B} \\ \mathfrak{A} \end{pmatrix} \to M \begin{pmatrix} \mathfrak{B} \\ \mathfrak{A} \end{pmatrix}$$

- **Siegel half space** $\mathcal{H}_h = \{\Omega \in \mathbb{C}^{h^2}, \, \Omega^t = \Omega, \, \mathrm{Im}\,(\Omega) > 0\} = Sp(2h,\mathbb{R})/U(h)$
  - ⋆ Moduli space of genus-two surfaces $\mathcal{M}_2 = \mathcal{H}_2/Sp(4,\mathbb{Z})$ (minus diagonal Ω)

Modular graph functions for arbitrary genus

- **Canonical metric on Σ = pull-back of flat metric on Jacobian $J(\Sigma)$**
  - ⋆ conformal and modular invariant canonical volume form on Σ

$$\kappa = \frac{i}{2h} \sum_{I,J} Y^{-1}_{IJ} \, \omega_I \wedge \overline{\omega_J} \qquad Y = \operatorname{Im} \Omega \qquad \int_\Sigma \kappa = 1$$

- **The Arakelov Green function $G(z, w|\Sigma)$ is defined by**

$$\partial_{\bar w} \partial_w G(w, z|\Sigma) = -\pi \delta(w, z) + \pi \kappa(w) \qquad \int_\Sigma \kappa \, G = 0$$

- **"Natural" generating function for higher genus modular graph functions**

$$\mathcal{C}^{(h)}_N (s_{ij}|\Sigma) = \int_{\Sigma^N} \prod_{i=1}^N \kappa(z_i) \, \exp \left\{ \sum_{1 \leq i < j \leq N} s_{ij} \, G(z_i, z_j|\Sigma) \right\}$$

  - ⋆ Integrals absolutely convergent for $|s_{ij}| < 1$; analytic near $s_{ij} = 0$;
  - ⋆ Taylor coeffs in s_{ij} define *higher genus modular graph functions*
    (ED, Green, Pioline 2017, 2018)

Genus-two four-graviton amplitude

- **The genus 2 string amplitude does NOT correspond to** $\mathcal{C}_4^{(2)}(s_{ij}|\Sigma)$
  - ⋆ volume form κ is unique on Σ
  - ⋆ but κ^N is not unique on Σ^N for $N \geq 2$

- **Genus-two four-graviton string amplitude given by** (ED & Phong 2005)

$$\mathcal{B}_4^{(2)}(s_{ij}|\Omega) = \int_{\Sigma^4} \frac{\mathcal{Y} \wedge \bar{\mathcal{Y}}}{(\det Y)^2} \exp\left\{ \sum_{1\leq i<j\leq 4} s_{ij}\, G(z_i, z_j|\Omega) \right\}$$

  - ⋆ Measure given by a holomorphic $(1,0)^{\otimes 4}$ form $\mathcal{Y}$ on Σ^4

$$\mathcal{Y} = s_{14}\Delta(z_1, z_2) \wedge \Delta(z_3, z_4) - s_{12}\Delta(z_1, z_4) \wedge \Delta(z_2, z_3)$$

  - ⋆ where Δ is a holomorphic $(1,0)^{\otimes 2}$ form on Σ^2

$$\Delta(z_i, z_j) = \omega_1(z_i) \wedge \omega_2(z_j) - \omega_2(z_i) \wedge \omega_1(z_j)$$

  - ⋆ $\mathcal{B}_4^{(2)}(s_{ij}|\Omega)$ is $Sp(4,\mathbb{Z})$-invariant and thus produces $\mathcal{B}_4^{(2)}(s_{ij}|\Sigma)$ on $\mathcal{M}_2$
  - ⋆ Extended to include external fermions (Berkovits 2005; Berkovits, Mafra 2005)

Taylor expansion of the amplitude

- **Low energy expansion of the genus-two four graviton $\mathcal{B}_4^{(2)}$**

$$\mathcal{B}_4^{(2)}(s_{ij}|\Sigma) = 32\sigma_2 + 64\sigma_3\varphi(\Sigma) + 32\sigma_4\psi(\Sigma) + \mathcal{O}(s_{ij}^5) \qquad \sigma_k = s_{12}^k + s_{13}^k + s_{14}^k$$

  ⋆ $\varphi(\Sigma)$ is the Kawazumi-Zhang invariant (Kawazumi 2008; Zhang 2008)

$$\varphi(\Sigma) = -\frac{1}{4}\sum_{I,J,K,L} Y^{-1}_{IL} Y^{-1}_{JK} \int_{\Sigma^2} G(x,y|\Sigma)\, \omega_I(x)\overline{\omega_J(x)}\omega_K(y)\overline{\omega_L(y)}$$

  ⋆ related to the genus-two Faltings invariant (De Jong 2010)

- **Theorem** $\varphi(\Sigma)$ *satisfies an inhomogeneous eigenvalue equation on* $\overline{\mathcal{M}_2}$
  (ED, Green, Pioline, Russo 2014; see also Kawazumi 2008)

$$(\Delta - 5)\varphi = -2\pi\delta_{SN}$$

  ⋆ Δ is the Laplace-Beltrami operator on $\mathcal{H}_2$ and $\mathcal{M}_2$
  ⋆ δ_{SN} has support on separating node (into two genus-one surfaces)
  ⋆ Integrating φ over $\mathcal{M}_2$ gives genus-two $D^6\mathcal{R}^4$ effective interaction
  ⋆ Admits a ϑ-lift (Pioline 2014)

- **Expansion of $\mathcal{B}_4^{(2)}$ produces new invariants at every order in s_{ij}**

Genus-two five-graviton amplitude

- **Evaluated recently**
  - ⋆ using ingredients from pure spinor and RNS formulations
    (ED, Mafra, Pioline, Schlotterer 2020) see also (Gomez, Mafra, Schlotterer 2015)
  - ⋆ NS-NS sector from first principles RNS (ED, Schlotterer 2021) in preparation

- **Many kinematic invariants** (as opposed to a single one for four gravitons)
  - ⋆ generating 17 new integrals on Σ^5

e.g.

$$\int_{\Sigma^5} \sum_{I,J} \frac{\omega_I(z_1) Y^{-1}_{IJ} \bar\omega_J(z_2)}{(\det Y)^2} \Delta(z_2,z_3) \overline{\Delta(z_1,z_2)} |\Delta(z_4,z_5)|^2 \, e^{sG}$$

$$\int_{\Sigma^5} \partial_{z_1} G(z_1,z_2) \bar\partial_{z_3} G(z_1,z_3) \Delta(z_3,z_4) \overline{\Delta(z_1,z_4)} |\Delta(z_2,z_5)|^2 \, e^{sG}$$

  - ⋆ where the Koba-Nielsen factor is abbreviated by

$$e^{sG} = \exp\Big\{ \sum_{1 \leq i < j \leq 5} s_{ij} \, G(z_i, z_j | \Omega) \Big\}$$

  - ⋆ Taylor series in s_{ij} of each integral gives new modular graph functions

- **Non-trivial identities ?**

Modular graph tensors for arbitrary genus

- **Genus-one modular graph forms $\Rightarrow$ Higher-genus modular graph tensors**
  (Kawazumi 2010-11; ED, Schlotterer 2020)
- **Construction for arbitrary genus h in the special case without $\partial_z G(z,w)$**
  - ⋆ The matrix of $(1,1)$ forms on Σ

$$\mu_I^J(z) = \sum_K \omega_I(z)\, Y^{-1}_{JK}\, \overline{\omega}_K(z)$$

  - ⋆ transforms as a tensor under the modular group $Sp(2h,\mathbb{Z})$

$$\mu_I^J(z) \to \sum_{K,L} (C\Omega + D)^{-1}_{IK}\, (C\Omega + D)_{JL}\, \mu_K^L(z)$$

- **Open chain modular graph tensors** (no loops)

$$\mathcal{A}^{J_1\cdots J_n}_{I_1\cdots I_n} = \int_{\Sigma^n} \mu_{I_1}^{J_1}(z_1)\, G(z_1,z_2)\, \mu_{I_2}^{J_2}(z_2) \cdots G(z_{n-1},z_n)\, \mu_{I_n}^{J_n}(z_n)$$

  - ⋆ Kawazumi-Zhang invariant is a special case $\varphi = -\frac{1}{4}\sum_{I,J} \mathcal{A}^{JI}_{IJ}$
- **Closed loop modular graph tensors** (one loop)

$$\mathcal{B}^{J_1\cdots J_n}_{I_1\cdots I_n} = \int_{\Sigma^n} \mu_{I_1}^{J_1}(z_1)\, G(z_1,z_2)\, \mu_{I_2}^{J_2}(z_2) \cdots G(z_{n-1},z_n)\, \mu_{I_n}^{J_n}(z_n)\, G(z_n,z_1)$$

Modular graph tensors cont'd

- **How to obtain identities without translation symmetry of the torus ?**
  - ⋆ Translation symmetry and momentum conservation is crucial for genus one
  - ⋆ Not available at genus $h \geq 2$ but ... instead

- **Interchange Lemma** (ED, Schlotterer 2020)
  - ⋆ key to moving derivatives across the graph

$$\partial_x G(x,y)\omega_I(y) + \partial_y G(x,y)\omega_I(x) = \sum_J \Big(\partial_y \Phi_I^J(y)\omega_J(x) + \partial_x \Phi_I^J(x)\omega_J(y)\Big)$$

$$\Phi_I^J(z) = \int_\Sigma G(z,x)\mu_I^J(x)$$

- **Construction of identities at arbitrary genus h and rank n**
  - ⋆ e.g. arbitrary h and $n=2$

$$\mathcal{B}_{IJ}^{KL} - \mathcal{B}_{IJ}^{LK} = \sum_M \left(\mathcal{A}_{MIJ}^{LKM} + \mathcal{A}_{MJI}^{KLM}\right) + \sum_{M,N} \mathcal{A}_{MJ}^{LN}\mathcal{A}_{NI}^{KM} - (I \leftrightarrow J)$$

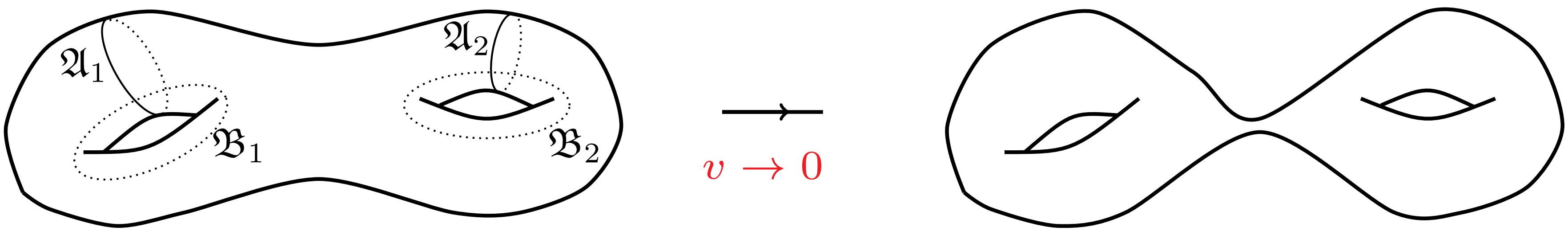
  - ⋆ left side: one-loop, right side: zero loops
  - ⋆ generalizes a genus 2 rank 2 identity discovered earlier via
    - – a complicated differential identity (Basu 2018)
    - – asymptotics of five-graviton amplitude (ED, Mafra, Pioline, Schlotterer 2020)

Degenerations of genus-two Riemann surfaces

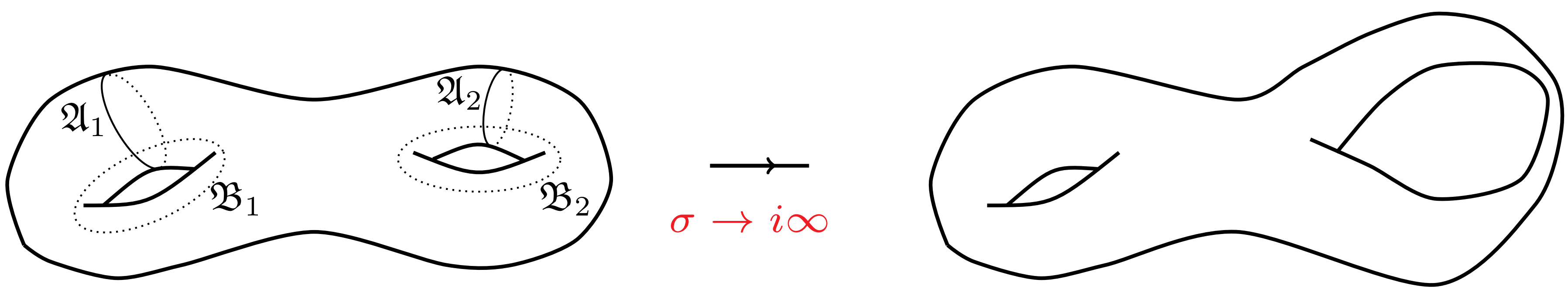
- **Locally parametrize** $\mathcal{M}_2 = \mathcal{H}_2 / Sp(4, \mathbb{Z})$ **by the period matrix**

$$\Omega = \begin{pmatrix} \tau & v \\ v & \sigma \end{pmatrix} \qquad \tau, \sigma, v \in \mathbb{C} \qquad \det(\mathrm{Im}\,\Omega) > 0$$

⋆ *Separating degeneration*



⋆ *Non-separating degeneration*



Non-separating degeneration

- Σ **degenerates to torus** Σ_1 **of modulus** τ **with punctures** p_a, p_b
  - $\star$ keep the cycles $\mathfrak{A}_1, \mathfrak{B}_1, \mathfrak{A}_2$ fixed, and let $\mathfrak{B}_2 \to \infty$ as $\text{Im}\,(\sigma) \to \infty$

- **Modular group** $Sp(4,\mathbb{Z})$ **reduces to** $SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^3$ **Fourier-Jacobi group**
  - $=$ the subgroup that leaves $\mathfrak{B}_2$ invariant

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{Z}) : \qquad \Omega = \begin{pmatrix} \tau & v \\ v & \sigma \end{pmatrix} \begin{cases} \tau & \to & (\alpha\tau+\beta)/(\gamma\tau+\delta) \\ v & \to & v/(\gamma\tau+\delta) \\ \sigma & \to & \sigma - \gamma v^2/(\gamma\tau+\delta) \end{cases}$$

  - $\star$ The degeneration parameter σ is not invariant under $SL(2,\mathbb{Z})$
  - $\star$ Siegel modular forms degenerate to Jacobi forms (Eichler & Zagier 1985)

- **There exists a real-valued** $SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^3$ **invariant parameter** $t > 0$

$$t = \frac{\det\,(\text{Im}\ \Omega)}{\text{Im}\ \tau} = \text{Im}\,\sigma - \frac{(\text{Im}\,v)^2}{\text{Im}\,\tau} \qquad \Omega = \begin{pmatrix} \tau & v \\ v & \sigma \end{pmatrix}$$

  - $\star$ the non-separating node is characterized by $t \to \infty$

    (an analogous invariant parameter exists for arbitrary genus)

Degeneration of modular graph functions

- **Recall the generating function of genus-two string invariants**

$$\mathcal{B}(s_{ij}|\Sigma) = \int_{\Sigma^4} \frac{\mathcal{Y} \wedge \overline{\mathcal{Y}}}{(\det Y)^2} \exp \left\{ \sum_{i<j} s_{ij} G(z_i, z_j) \right\} = \sum_{w=0}^{\infty} \frac{1}{w!} \mathcal{B}_w(s_{ij}|\Sigma)$$

$\star$ Taylor series in s_{ij} produces modular graph functions of weight w

$$\mathcal{B}_w(s_{ij}|\Sigma) = \int_{\Sigma^4} \frac{\mathcal{Y} \wedge \overline{\mathcal{Y}}}{(\det Y)^2} \left(\sum_{i<j} s_{ij} G(z_i, z_j) \right)^w$$

- **Theorem** (ED, Green, Pioline 2017)

The expansion of $\mathcal{B}_w(s_{ij}|\Sigma)$ near the non-separating node is given by a Laurent polynomial of **finite** *degree $(w, -w)$ in t*

$$\mathcal{B}_w(s_{ij}|\Omega) = \sum_{k=-w}^{w} \mathcal{B}_w^{(k)}(s_{ij}|v, \tau)\, t^k + \mathcal{O}(e^{-2\pi t})$$

The coefficients are invariant under $SL(2, \mathbb{Z}) \ltimes \mathbb{Z}^2 \subset Sp(4, \mathbb{Z})$

$$\mathcal{B}_w^{(k)} \left(s_{ij} \middle| \frac{v + m\tau + n}{\gamma\tau + \delta}, \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \right) = \mathcal{B}_w^{(k)}(s_{ij}|v, \rho)$$

These are "elliptic modular graph functions".

Summary and outlook

- **Low energy expansion of string theory reveals a rich structure of**
  - ⋆ Genus one modular graph functions and forms
  - ⋆ Higher genus modular graph functions and tensors
- **Further directions not covered here**
  - ⋆ Integration over genus 1 moduli (Zagier 1981; ED, Green 2019; ED, Geiser 2021)
  - ⋆ Uniform transcendental weight for genus 1 amplitude (ED, Green 2019)
  - ⋆ Integration over genus 2 moduli (ED, Green, Pioline, Russo 2014)
  - ⋆ Matching S-duality and susy predictions in Type IIB (many papers !)
  - ⋆ Relations with super-Yang-Mills and supergravity amplitudes (many papers !)
- **Arithmetic significance ?**