Lectures on Modular Graph Functions

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Bibliography

Main papers on which this talk is based

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Introduction

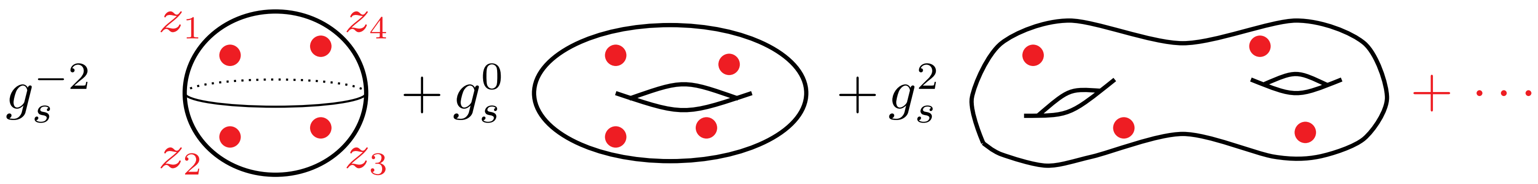
- **Modular graph functions**
  - ⋆ map graphs to $SL(2,\mathbb{Z})$-invariant functions on half-plane $\mathcal{H}_1$
  - ⋆ generalize non-holomorphic Eisenstein series
  - ⋆ generalize multiple zeta values
  - ⋆ related to single-valued elliptic polylogarithms

- **Higher genus modular graph functions**
  - ⋆ map graphs to functions on the moduli space of Riemann surfaces
  - ⋆ generalize invariants of Kawazumi and Zhang and Faltings

- **Modular graph functions naturally emerge from string theory**
  - ⋆ string perturbation theory is a genus expansion
  - ⋆ low energy expansion in terms of modular graph functions
  - ⋆ matches predictions of $SL(2,\mathbb{Z})$ symmetry of Type IIB string theory

Sums over Riemann surfaces

- **Probability amplitude in string theory**



$$g_s^{-2} \quad + g_s^0 \quad + g_s^2 \quad + \cdots$$

⋆ topological expansion in the string coupling $g_s \in \mathbb{R}$ parameter

⋆ each marked point z_i represents an incoming or outgoing string
for a give physical process, the number of marked points N is fixed

⋆ for each genus h
- integrate over N marked points
- integrate over the moduli space $\mathcal{M}_h$ of Riemann surfaces

Genus zero

- **Genus-zero graviton amplitudes are integrals of the type**

$$\prod_{i=1}^{N-3} \int_{\mathbb{C}} d^2 z_i \, |z_i|^{-2-2s_{i,N-1}} |1-z_i|^{-2s_{iN}} \prod_{j \neq i}^{N-3} |z_i - z_j|^{-2s_{ij}}$$

  - ⋆ three points chosen at $0, 1, \infty$ by conformal invariance
  - ⋆ parameters $s_{ij} = s_{ji}$ are kinematic variables

  - ⋆ satisfy $s_{ii} = 0$ and $\sum_{i=1}^N s_{ij} = 0$ for all $i, j = 1, \cdots, N$;
  - ⋆ Meromorphic in s_{ij} with simple poles at non-negative integers

- **Genus-zero four-graviton amplitude**

$$\mathcal{A}_4^{(0)}(s_{ij}) = \frac{1}{s_{12}\, s_{13}\, s_{14}} \, \frac{\Gamma(1-s_{12})\,\Gamma(1-s_{13})\,\Gamma(1-s_{14})}{\Gamma(1+s_{12})\,\Gamma(1+s_{13})\,\Gamma(1+s_{14})}$$

Genus-one

- **The surface is a torus $\Sigma = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ with modulus $\tau \in \mathcal{H}_1$**
  - ⋆ Integral over N marked points $z_i \in \Sigma$

$$\mathcal{B}_N^{(1)}(s_{ij}|\tau) = \prod_{k=1}^{N} \int_\Sigma \frac{d^2 z_k}{\operatorname{Im}\tau} \exp\Big\{ \sum_{1 \leq i < j \leq N} s_{ij}\, g(z_i - z_j|\tau) \Big\}$$

  - ⋆ Scalar Green function (in terms of $z = u + v\tau$, $u, v \in \mathbb{R}$)

$$g(z|\tau) = \sum_{m,n \in \mathbb{Z}}{}' \frac{\operatorname{Im}\tau}{\pi |m\tau + n|^2} e^{2\pi i(mu - nv)}$$

  - ⋆ $\mathcal{B}_N^{(1)}(s_{ij}|\tau)$ is invariant under the modular group $SL(2,\mathbb{Z})$
    holomorphic in s_{ij} for $|s_{ij}| < 1$; poles at $s_{ij} = 1, 2, 3, \cdots$

- **Genus-one four graviton amplitude is given by** (Green, Schwarz, 1982)

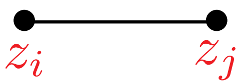
$$\mathcal{A}_4^{(1)}(s_{ij}) = \int_{\mathcal{M}_1} \frac{d^2\tau}{(\operatorname{Im}\tau)^2} \mathcal{B}_4^{(1)}(s_{ij}|\tau) \qquad \qquad \mathcal{M}_1 = PSL(2,\mathbb{Z}) \backslash \mathcal{H}_1$$

  - ⋆ absolutely convergent only for $\operatorname{Re}(s_{ij}) = 0$;
  - ⋆ analytic continuation was proven to exist (ED & Phong 1994)

Graphical Representation of Taylor series of $\mathcal{B}_N^{(1)}(s_{ij}|\tau)$

- **Absolute convergence of the integral $\mathcal{B}_N^{(1)}(s_{ij}|\tau)$ for $|s_{ij}| < 1$ and fixed τ**
  - ⋆ allows for Taylor expansion in the variables s_{ij}
    - $=$ physically corresponds to the "low energy expansion"

- **Represented by Feynman graphs** (Green, Russo, Vanhove 2008)
  - ⋆ Each integration point z_i on Σ is represented by a vertex •
  - ⋆ Each Green function by an edge between vertices z_i and z_j



$$= \quad g(z_i - z_j|\tau)$$

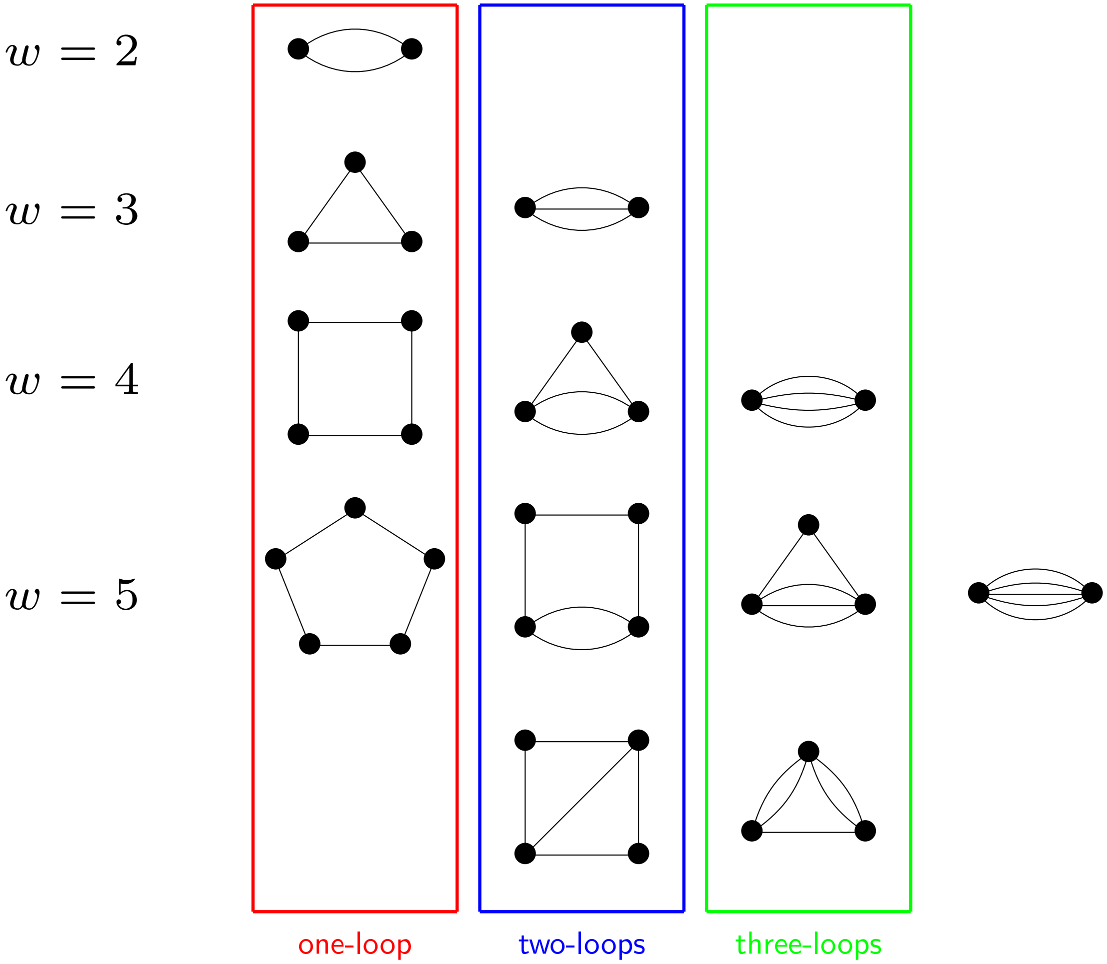
  - ⋆ Each vertex is integrated over Σ
  - ⋆ To a graph with w edges we assign *weight* w

- **Reducibility** : A graph which becomes disconnected
  - ⋆ upon cutting one edge vanishes by $\int_\Sigma g = 0$
  - ⋆ upon removing one vertex factorizes into the product of its components

Modular graph functions

- **To each graph is associated a non-holomorphic modular function**

  since $\mathcal{B}_N^{(1)}(s_{ij}|\tau)$ is a modular function, so are its Taylor coefficients



One-loop modular graph functions = Eisenstein series

- **One-loop weight w graph has w edges and w vertices**

$$\prod_{i=1}^{w} \int_\Sigma \frac{d^2 z_i}{\tau_2} g(z_i - z_{i+1}|\tau) = \sum_{p\,\in\Lambda'} \frac{\tau_2^w}{\pi^w |p|^{2w}} = E_w(\tau)$$

  - ⋆ with $\Lambda = \mathbb{Z} + \tau\mathbb{Z}, \ \Lambda' = \Lambda \setminus \{0\}$ and $\tau_2 = \mathrm{Im}\,\tau$

- **Non-holomorphic Eisenstein series** $E_s(\tau)$ for $s \in \mathbb{C}$
  - ⋆ Invariant under the modular group $\Gamma = PSL(2,\mathbb{Z})$
  - ⋆ Expansion near the cusp $\tau \to i\infty$ (constant Fourier mode)

$$E_s = \frac{2\zeta(2s)}{\pi^s}\tau_2^s + \frac{2\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)\,\pi^{s-\frac{1}{2}}}\,\tau_2^{1-s} + \mathcal{O}(e^{-4\pi\tau_2})$$

  - ⋆ Poincaré series

$$E_s = \frac{2\zeta(2s)}{\pi^s} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} (\mathrm{Im}\,\gamma\tau)^s$$

  - ⋆ Eigenfunction of the Laplace-Beltrami operator Δ on $\mathcal{H}_1$

$$\Delta E_s = s(s-1)E_s \qquad\qquad \Delta = 4\tau_2^2 \partial_\tau \partial_{\bar\tau}$$

  - ⋆ Eigenfunction of the Hecke operators

Two-loop modular graph functions

- **Two-loop graphs evaluate to a multiple Kronecker-Eisenstein series**

$$C_{a_1,a_2,a_3}(\tau) = \sum_{p_1,p_2,p_3 \in \Lambda'} \delta\Big(\sum_{r=1}^{3} p_r \Big) \prod_{r=1}^{3} \left(\frac{\tau_2}{\pi |p_r|^2} \right)^{a_r}$$

  ⋆ Absolutely convergent for $a_r \in \mathbb{N}$, of weight $w = a_1 + a_2 + a_3$
  ⋆ invariant under $SL(2,\mathbb{Z})$
  ⋆ generalizes Eisenstein series and multiple zeta values

- **Theorem** (ED & Bill Duke 2017)
  Expansion as $\tau \to i\infty$: Laurent polynomial in τ_2 of degree $(w, 1-w)$

$$C_{a_1,a_2,a_3}(\tau) = c_w (-4\pi\tau_2)^w + \frac{c_{2-w}}{(4\pi\tau_2)^{w-2}} + \sum_{k=1}^{w-1} \frac{c_{w-2k-1}\, \zeta(2k+1)}{(4\pi\tau_2)^{2k+1-w}} + \mathcal{O}(e^{-2\pi\tau_2})$$

  ⋆ $c_w, c_{w-2k-1} \in \mathbb{Q}$ *explicitly known*
  ⋆ c_{2-w} *bilinear in odd zeta values over* $\mathbb{Z}$

- **Fourier series** (ED & Kaidi 2019)
- **Poincaré series** (ED & Kaidi 2019; Dorigoni & Kleinschmidt 2019)

System of differential identities

Theorem (ED, Green, Vanhove 2015)

- ⋆ *Two-loop modular graph functions* $C_{a,b,c}$ *of weight* $w = a+b+c,\ a,b,c \in \mathbb{N}$ *obey a finite system of differential equations of uniform weight* w

$$\begin{aligned} 2\Delta C_{a,b,c} \quad &= \quad 2ab\, C_{a+1,b-1,c} + ab\, C_{a+1,b+1,c-2} - 4ab\, C_{a+1,b,c-1} \\ &\quad +a(a-1)\, C_{a,b,c} + \text{ 5 permutations of } (a,b,c) \end{aligned}$$

where $\Delta = 4\tau_2^2\partial_\tau\partial_{\bar\tau}$ *and "boundary values" given by Eisenstein series*

$$C_{a,b,0} = E_a E_b - E_{a+b} \qquad\qquad C_{a,b,-1} = E_{a-1}E_b + E_a E_{b-1}$$

- ⋆ *All weight* w *modular graph functions* $C_{a,b,c}$ *are linear combinations of modular functions* $\mathfrak{C}_{w;s;n}$ *satisfying*

$$\big(\Delta - s(s-1)\big)\mathfrak{C}_{w;s;n} = \mathfrak{F}_{w;s;n} \qquad\qquad s = w - 2m$$

$$m = 1, \cdots, \left[\tfrac{1}{2}(w-1)\right] \qquad\qquad n = 0, \cdots, \left[\tfrac{1}{3}(s-1)\right]$$

$\mathfrak{F}_{w;s;\mathfrak{p}}$ *is a polynomial of degree 2 and weight* w *in* $E_{s'}$ *with* $2 \le s' \le w$.

System of algebraic identities

- **Differential identities for odd weight w and zero eigenvalue**
  - ⋆ the inhomogeneous term is linear in E_w for all odd w, e.g.

$$\Delta C_{1,1,1} = 6E_3 \qquad \Delta(C_{3,3,1} + C_{3,2,2}) = 18E_7$$

$$\Delta C_{2,2,1} = 8E_5 \qquad \Delta(9C_{4,4,1} + 18C_{4,3,2} + 4C_{3,3,3}) = 288E_9$$

  - ⋆ Integrate using $\Delta E_s = s(s-1)E_s$
  - ⋆ fix integration constant using asymptotics at cusp

- **Algebraic identities at weight w mixing loop orders**, e.g.

$$C_{1,1,1} = E_3 + \zeta(3) \qquad C_{3,3,1} + C_{3,2,2} = \frac{3}{7}E_7 + \frac{\zeta(7)}{252}$$

$$C_{2,2,1} = \frac{2}{5}E_5 + \frac{\zeta(5)}{30} \qquad 9C_{4,4,1} + 18C_{4,3,2} + 4C_{3,3,3} = 4E_9 + \frac{\zeta(9)}{240}$$

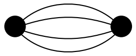
  - ⋆ $C_{1,1,1} = E_3 + \zeta(3)$ proven by direct summation (Zagier, unpublished)
  - ⋆ One algebraic identity for each odd weight. (ED, Green, Vanhove 2015)

Modular graph functions at higher loop order

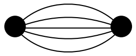
- **Expansion near the cusp** $\tau \to i\infty$
  - ⋆ Laurent polynomial in τ_2 of degree $(w, 1-w)$ $+\mathcal{O}(e^{-4\pi\tau_2})$
  - ⋆ coefficients include "irreducible" multiple zeta-values (Zerbini 2017)
  - ⋆ coeffs are "single-valued" multiple zeta-values (ED, Green, Gurdogan, Vanhove 2015)

- **Modular graph functions satisfy algebraic identities of uniform weight**

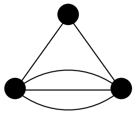
e.g.



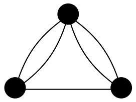
$$= 24C_{2,1,1} + 3E_2^2 - 18E_4$$



$$= 60C_{3,1,1} + 10E_2C_{1,1,1} - 48E_5 + 16\zeta(5)$$



$$= \frac{15}{2}C_{3,1,1} + 3E_2E_3 - \frac{69}{10}E_5 + \frac{7}{40}\zeta(5)$$



$$= 2C_{3,1,1} - \frac{2}{5}E_5 + \frac{3}{10}\zeta(5)$$

- **Laplace-Beltrami operator for 3 loops and higher**
  - ⋆ generically no longer maps the space of modular graph functions to itself

Arbitrary number of loops and exponents

- **Modular graph forms** (ED & Green 2016)

  A decorated graph (Γ, A, B) *with* V *vertices and* R *edges consists of*

  ⋆ *connectivity matrix with components* $\Gamma_{v\,r}$, $v = 1, \cdots, V$, $r = 1, \cdots, R$

  ⋆ *decoration of the edges by "exponents"* $a_r, b_r \in \mathbb{N}$ *for* $r = 1, \cdots, R$

  $A = [a_1, \cdots, a_R]$ and $B = [b_1, \cdots, b_R]$

  To the decorated graph (Γ, A, B) *we associate a function on* $\mathcal{H}_1$

$$\mathcal{C}_\Gamma \begin{bmatrix} A \\ B \end{bmatrix} (\tau) = \sum_{p_1, \cdots, p_R \in \Lambda'} \left(\prod_{r=1}^R \frac{(\tau_2/\pi)^{a_r}}{(p_r)^{a_r} \, (\bar p_r)^{b_r}} \right) \prod_{v=1}^V \delta \left(\sum_{r=1}^R \Gamma_{v\,r} p_r \right)$$

- **Transformation under** $\Gamma = PSL(2, \mathbb{Z})$

$$\mathcal{C}_\Gamma \begin{bmatrix} A \\ B \end{bmatrix} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta} \right) = (\gamma\bar\tau + \delta)^\mu \, \mathcal{C}_\Gamma \begin{bmatrix} A \\ B \end{bmatrix} (\tau) \qquad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma$$

⋆ when $\mu \neq 0$ there is no canonical normalization for powers of τ_2

⋆ modular weight $(0, \mu)$ with $\mu = \sum_r (b_r - a_r)$: "modular graph form"

⋆ $A = B \Rightarrow \mu = 0$ recover modular graph functions for Green function g

Identities on modular graph forms

- **Momentum conservation at each vertex v provides algebraic identities**

$$\sum_{r=1}^{R} \Gamma_{v\,r}\,\mathcal{C}\left[\begin{smallmatrix} A-S_r \\ B \end{smallmatrix}\right] = \sum_{r=1}^{R} \Gamma_{v\,r}\,\mathcal{C}\left[\begin{smallmatrix} A \\ B-S_r \end{smallmatrix}\right] = 0 \qquad\qquad S_r = [0_{r-1}\ 1\ 0_{R-r}]$$

- **The Maass operator $\nabla = 2i\tau_2^2\partial_\tau$ provides differential identities**

$$\nabla\,\mathcal{C}\left[\begin{smallmatrix} A \\ B \end{smallmatrix}\right] = \sum_{r=1}^{R} a_r\,\mathcal{C}\left[\begin{smallmatrix} A+S_r \\ B-S_r \end{smallmatrix}\right]$$

- **Algorithm for modular graph function identities** (ED & Green 2016)

  To search for and prove an identity $F=0$ of weight w for the case $A=B$:

  ⋆ take a general linear combination F of weight w modular graph functions,

  ⋆ solve the weaker condition $\nabla^w F = 0$ for general F,

  ⋆ using relations between holomorphic modular forms to simplify derivatives

  " holomorphic subgraph reduction"

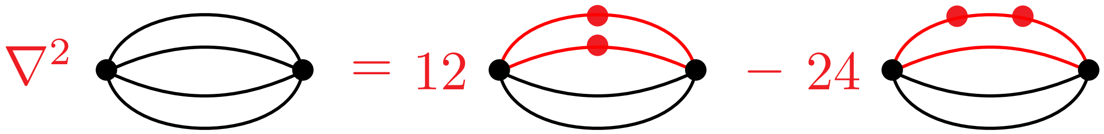
  ⋆ **Lemma** *If* $\nabla^n F = 0$ *for an integer* $n \geq 1$ *then* F *is independent of* τ*.*

Holomorphic subgraph reduction

- **The operator ∇ increases upper exponents, lowers lower exponents**

$$\text{e.g.} \qquad \nabla^2 \mathcal{C}\begin{bmatrix}1&1&1&1\\1&1&1&1\end{bmatrix} = 12\,\mathcal{C}\begin{bmatrix}2&2&1&1\\0&0&1&1\end{bmatrix} - 24\,\mathcal{C}\begin{bmatrix}3&1&1&1\\0&0&1&1\end{bmatrix}$$

⋆ subgraph of first two columns produces a holomorphic modular form in τ



⋆ Summation over $p \in \Lambda'$ in red lines lowers the number of loops

$$\mathcal{C}\begin{bmatrix}2&2&1&1\\0&0&1&1\end{bmatrix} - 2\,\mathcal{C}\begin{bmatrix}3&1&1&1\\0&0&1&1\end{bmatrix} = 2\,\mathcal{C}\begin{bmatrix}4&1&1\\0&1&1\end{bmatrix} + \frac{1}{2}E_2\nabla^2 E_2$$

⋆ All holomorphic closed subgraphs evaluate to holomorphic modular forms

$\Longrightarrow$ **All identities between modular functions of weight $w \leq 6$**

(ED, Green 2016; ED, Kaidi 2016; Broedel, Schlotterer, Zerbini 2018)

– available in Mathematic package (Gerken 2020)

– special cases discussed in (Basu 2015-19; Kleinschmidt, Verschinnin 2017)

Relation to iterated elliptic integrals

- **Powers of ∇ relate to holomorphic modular forms**

$$\nabla^n E_n \sim \tau_2^{2n} G_{2n} \qquad \nabla^3 C_{2,1,1} \sim \tau_2^4 G_4 \nabla E_2 \qquad G_{2n} = \sum_{p \in \Lambda'} \frac{1}{p^{2n}}$$

  - where G_{2n} is the holomorphic modular form of weight $2n$ for $n \geq 2$.
  - Differential eqs are a non-holomorphic (or "single-valued") version of

$$\partial_\tau \mathcal{E}\left[{j_1, \cdots, j_r \atop k_1, \cdots, k_r} \right](\tau) = \tau^{j_r} \, G_{2k_r}(\tau) \, \mathcal{E}\left[{j_1, \cdots, j_{r-1} \atop k_1, \cdots, k_{r-1}} \right](\tau)$$

  - Admit a (regularized) iterated integral representation

$$\mathcal{E}\left[{j_1, \cdots, j_r \atop k_1, \cdots, k_r} \right](\tau) = \int_{i\infty}^{\tau} d\sigma \, \sigma^{j_r} G_{2k_r}(\sigma) \, \mathcal{E}\left[{j_1, \cdots, j_{r-1} \atop k_1, \cdots, k_{r-1}} \right](\sigma)$$

  - which satisfy shuffle relations
  - may be collected into a generating function

- **Modular graph forms may be expressed via iterated elliptic integrals**

(Broedel, Mafra, Matthes, Schlotterer 2014; Brown 2017; Broedel, Mafra, Schlotterer 2018; Gerken, Kleinschmidt, Schlotterer 2020; Gerken, Kleinschmidt, Mafra, Schlotterer, Verbeek 2020)

Higher genus

- **How to generalize the genus-one formula to higher genus ?**
  - ⋆ recall the genus-one generating function

$$\mathcal{B}_N^{(1)}(s_{ij}|\tau) = \prod_{i=1}^N \int_{\Sigma_1} \frac{d^2 z_i}{\tau_2} \, \exp\left\{ \sum_{1 \leq i < j \leq N} s_{ij} \, g(z_i - z_j) \right\}$$

- **Compact Riemann surface Σ of genus $h \geq 2$ without boundary**
  - ⋆ we need a scalar Green function $G(z_i, z_j)$
  - ⋆ and a measure $d\mu_N$ on Σ^N

$$\mathcal{B}_N^{(h)}(s_{ij}|\Sigma) = \int_{\Sigma^N} d\mu_N \, \exp\left\{ \sum_{1 \leq i < j \leq N} s_{ij} \, G(z_i, z_j) \right\}$$

Compact Riemann surfaces Σ of genus h

- **Homology and cohomology**
  - ⋆ One-cycles $H_1(\Sigma,\mathbb{Z}) \approx \mathbb{Z}^{2h}$ with intersection pairing $\mathfrak{J}(\cdot,\cdot) \to \mathbb{Z}$
  - ⋆ Canonical basis $\mathfrak{J}(\mathfrak{A}_I,\mathfrak{A}_J) = \mathfrak{J}(\mathfrak{B}_I,\mathfrak{B}_J) = 0$, $\mathfrak{J}(\mathfrak{A}_I,\mathfrak{B}_J) = \delta_{IJ}$ for $1 \leq I, J \leq h$
  - ⋆ Canonical dual basis of holomorphic one-forms ω_I in $H^{(1,0)}(\Sigma)$

$$\oint_{\mathfrak{A}_I} \omega_J = \delta_{IJ} \qquad\qquad \oint_{\mathfrak{B}_I} \omega_J = \Omega_{IJ}$$

  - ⋆ Period matrix Ω obeys Riemann relations $\Omega^t = \Omega,\ \mathrm{Im}\,(\Omega) > 0$
  - ⋆ Jacobian variety $J(\Sigma) = \mathbb{C}^h/(\mathbb{Z}^h + \Omega\mathbb{Z}^h)$

- **Modular group** $Sp(2h,\mathbb{Z}) : H_1(\Sigma,\mathbb{Z}) \to H_1(\Sigma,\mathbb{Z})$ **leaves** $\mathfrak{J}(\cdot,\cdot)$ **invariant**

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad\qquad M^t \mathfrak{J} M = \mathfrak{J} \qquad\qquad \begin{pmatrix} \mathfrak{B} \\ \mathfrak{A} \end{pmatrix} \to M \begin{pmatrix} \mathfrak{B} \\ \mathfrak{A} \end{pmatrix}$$

  – action on 1-forms ω and period matrix Ω given by

$$\begin{aligned} \omega &\to \omega\,(C\Omega + D)^{-1} \\ \Omega &\to (A\Omega + B)\,(C\Omega + D)^{-1} \end{aligned}$$

Modular graph functions for arbitrary genus

- **Canonical metric on Σ = pull-back of flat metric on Jacobian $J(\Sigma)$**
  - ⋆ conformal and modular invariant canonical volume form on Σ

$$\kappa = \frac{i}{2h} \sum_{I,J} Y^{-1}_{IJ} \, \omega_I \wedge \overline{\omega_J} \qquad \int_\Sigma \kappa = 1 \qquad Y = \operatorname{Im} \Omega$$

- **The Arakelov Green function $G(z,w)$ is defined by**

$$\partial_{\bar w} \partial_w G(w,z) = -\pi \delta(w,z) + \pi \kappa(w) \qquad \int_\Sigma \kappa \, G = 0$$

- **"Natural" generating function for higher genus modular graph functions**

$$\mathcal{C}^{(h)}_N (s_{ij}|\Omega) = \int_{\Sigma^N} \prod_{i=1}^N \kappa(z_i) \ \exp \left\{ \sum_{1 \leq i < j \leq N} s_{ij} \, G(z_i, z_j) \right\}$$

  - ⋆ Integrals absolutely convergent for $|s_{ij}| < 1$; analytic near $s_{ij} = 0$;
  - ⋆ Invariant under $Sp(2h, \mathbb{Z})$ modular group acting on Ω;
  - ⋆ Taylor coeffs in s_{ij} give *modular graph functions* (ED, Green, Pioline 2017)

Genus-two string amplitude

- **Actually ... genus 2 string amplitude does NOT correspond to** $\mathcal{C}_4^{(2)}(s_{ij}|\Omega)$
  - ⋆ volume form κ is unique on Σ
  - ⋆ but κ^N is not unique on Σ^N for $N \geq 2$

- **Genus-two four-graviton string amplitude corresponds to** (ED & Phong 2005)

$$\mathcal{B}_4^{(2)}(s_{ij}|\Omega) = \int_{\Sigma^4} \frac{\mathcal{Y} \wedge \bar{\mathcal{Y}}}{(\det Y)^2} \exp\Big\{ \sum_{1 \leq i < j \leq 4} s_{ij}\, G(z_i, z_j) \Big\}$$

  - ⋆ Measure given by a holomorphic $(1,0)^{\otimes 4}$ form $\mathcal{Y}$ on Σ^4

$$\mathcal{Y} = (s_{14} - s_{13}) \Delta(z_1, z_2) \wedge \Delta(z_3, z_4) + 2 \text{ cycl perms of } (2,3,4)$$

  - ⋆ where Δ is a holomorphic $(1,0)^{\otimes 2}$ form on Σ^2

$$\Delta(z_i, z_j) = \omega_1(z_i) \wedge \omega_2(z_j) - \omega_2(z_i) \wedge \omega_1(z_j)$$

  - ⋆ Volume form $\mathcal{Y} \wedge \bar{\mathcal{Y}}/(\det Y)^2$ and thus $\mathcal{B}_4^{(2)}(s_{ij}|\Omega)$ are $Sp(4,\mathbb{Z})$-invariant.

- **Genus-two five-graviton string amplitude** (ED, Mafra, Pioline, Schlotterer 2020)

Taylor expansion of genus two amplitude

- **Low energy expansion of the genus-two four graviton $\mathcal{B}_4^{(2)}$**

$$\mathcal{B}_4^{(2)}(s_{ij}|\Sigma) = 32\sigma_2 + 64\sigma_3\varphi(\Sigma) + 32\sigma_4\psi(\Sigma) + \mathcal{O}(s_{ij}^5)$$

  ⋆ where $\sigma_k = s_{12}^k + s_{13}^k + s_{14}^k$.

- **Theorem** (ED & Green 2013)
  The coefficient $\varphi(\Sigma)$ is the Kawazumi-Zhang invariant for genus two

$$\varphi(\Sigma) = -\frac{1}{4} \sum_{I,J,K,L} Y^{-1}_{IL} Y^{-1}_{JK} \int_{\Sigma^2} G(x,y)\, \omega_I(x) \overline{\omega_J(x)} \omega_K(y) \overline{\omega_L(y)}$$

  ⋆ introduced as a spectral invariant (Kawazumi 2008; Zhang 2008)
  ⋆ related to the genus-two Faltings invariant (De Jong 2010)

- **New higher invariants at every order in s_{ij} , e.g.**

$$\psi(\Sigma) = \int_{\Sigma^4} \frac{|\Delta(1,2)\Delta(3,4)|^2}{(\det Y)^2} \Big(G(1,4) + G(2,3) - G(1,3) - G(2,4) \Big)^2$$

Modular geometry and differential equation

- **Siegel half space** $\mathcal{H}_h = \{\Omega \in \mathbb{C}^{h^2}, \, \Omega^t = \Omega, \, Y = \mathrm{Im}\,(\Omega) > 0\} = \frac{Sp(2h,\mathbb{R})}{U(h)}$
  - ⋆ with $Sp(2h,\mathbb{R})$ invariant Kähler metric

$$ds^2 = \sum_{I,J,K,L=1,\cdots,h} Y^{-1}_{IJ}\, d\bar\Omega_{JK}\, Y^{-1}_{KL}\, d\Omega_{LI}$$

  - ⋆ and Laplace-Beltrami operator on $\mathcal{H}_h$

$$\Delta = \sum_{I,J,K,L} 4\, Y_{IK}\, Y_{JL}\, \bar\partial^{IJ}\, \partial^{KL} \qquad \partial^{IJ} = \frac{1}{2}(1+\delta^{IJ})\frac{\partial}{\partial \Omega_{IJ}}$$

- **Moduli space of genus-two surfaces** $\mathcal{M}_2 = \mathcal{H}_2 / Sp(4,\mathbb{Z})$ (minus diagonal Ω)

- **Theorem** (ED, Green, Pioline, Russo 2014; see also Kawazumi 2008)
  $\varphi(\Sigma)$ *satisfies the following inhomogeneous eigenvalue equation on* $\overline{\mathcal{M}_2}$

$$(\Delta - 5)\varphi = -2\pi \delta_{SN}$$

  - ⋆ δ_{SN} has support on separating node (into two genus-one surfaces)
  - ⋆ Integrating φ over $\mathcal{M}_2$ gives genus-two $D^6\mathcal{R}^4$ effective interaction

Modular tensors at arbitrary genus

- **Modular forms $\Rightarrow$ Modular tensors** (Kawazumi 2010-11; ED, Schlotterer 2020)
  - ⋆ Define the tensor-valued $(1,1)$ form on Σ

$$\mu_I^J(z) = \omega_I(z)\, Y^{JK}\, \overline{\omega}_K(z)$$

- **Open chain modular tensors**

$$\mathcal{A}^{J_1\cdots J_n}_{I_1\cdots I_n} = \int_{\Sigma^n} \mu^{J_1}_{I_1}(z_1)\, G(z_1,z_2)\, \mu^{J_2}_{I_2}(z_2) \cdots G(z_{n-1},z_n)\, \mu^{J_n}_{I_n}(z_n)$$

- **Closed loop modular tensors**

$$\mathcal{B}^{J_1\cdots J_n}_{I_1\cdots I_n} = \int_{\Sigma^n} \mu^{J_1}_{I_1}(z_1) G(z_1,z_2) \mu^{J_2}_{I_2}(z_2) \cdots G(z_{n-1},z_n) \mu^{J_n}_{I_n}(z_n) G(z_n,z_1)$$

- **At genus $h \geq 2$ the interchange Lemma** (instead of momentum conservation)

$$\partial_x \Big(G(x,y)\omega_I(y) - \Phi_I^J(x)\omega_J(y) \Big) = -\partial_y \Big(G(x,y)\omega_I(x) - \Phi_I^J(y)\omega_J(x) \Big)$$

- **Allows for the derivation of identities at all genera and all rank n, e.g.**

$$\mathcal{B}^{KL}_{IJ} - \mathcal{B}^{LK}_{IJ} = \mathcal{A}^{LKM}_{MIJ} + \mathcal{A}^{KLM}_{MJI} - \mathcal{A}^{KLM}_{MIJ} - \mathcal{A}^{LKM}_{MJI} - \mathcal{A}^{KN}_{MJ}\mathcal{A}^{LM}_{NI} + \mathcal{A}^{LN}_{MJ}\mathcal{A}^{KM}_{NI}$$

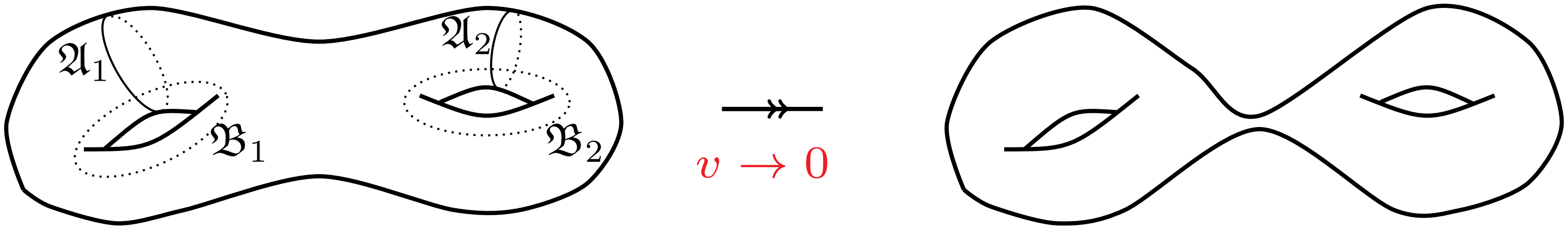
⋆ generalizes rank 2 genus 2 identity (Basu 2018; ED, Mafra, Pioline, Schlotterer 2020)

Degenerations of genus-two Riemann surfaces

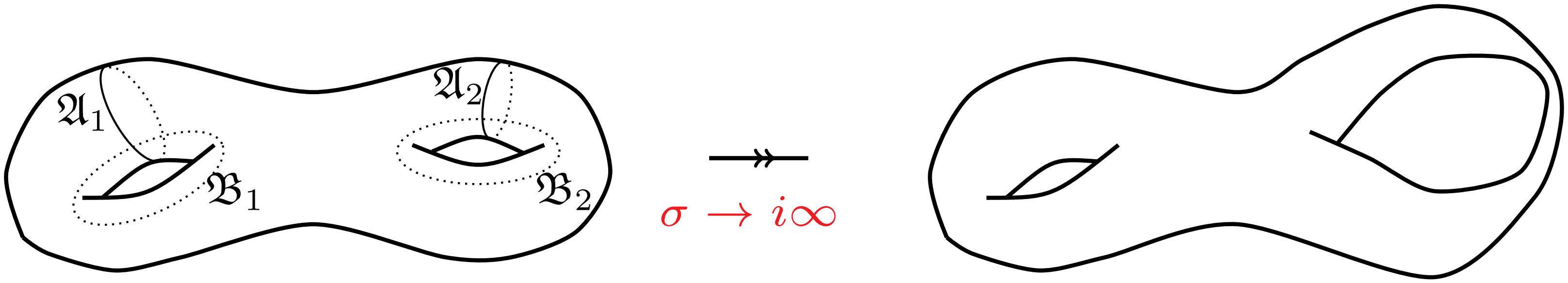
- **Locally parametrize $\mathcal{M}_2 = \mathcal{H}_2 / Sp(4,\mathbb{Z})$ by the period matrix**

$$\Omega = \begin{pmatrix} \tau & v \\ v & \sigma \end{pmatrix} \qquad \tau, \sigma, v \in \mathbb{C} \qquad \det(\mathrm{Im}\,\Omega) > 0$$

⋆ *Separating degeneration*



⋆ *Non-separating degeneration*



Non-separating degeneration

- **Σ degenerates to torus Σ_1 of modulus τ with punctures p_a, p_b**
  ⋆ keep the cycles $\mathfrak{A}_1, \mathfrak{B}_1, \mathfrak{A}_2$ fixed, and let $\mathfrak{B}_2 \to \infty$ as $\mathrm{Im}\,(\sigma) \to \infty$

- **Modular group $Sp(4,\mathbb{Z})$ reduces to $SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^3$ Fourier-Jacobi group**
  $=$ the subgroup that leaves $\mathfrak{B}_2$ invariant

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) : \qquad \Omega = \begin{pmatrix} \tau & v \\ v & \sigma \end{pmatrix} \begin{cases} \tau & \to & (a\tau+b)/(c\tau+d) \\ v & \to & v/(c\tau+d) \\ \sigma & \to & \sigma - cv^2/(c\tau+d) \end{cases}$$

  ⋆ The degeneration parameter σ is not invariant under $SL(2,\mathbb{Z})$
  ⋆ Siegel modular forms degenerate to Jacobi forms (Eichler & Zagier 1985)

- **There exists a real-valued $SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^3$ invariant parameter $t > 0$**

$$t = \frac{\det\,(\mathrm{Im}\ \Omega)}{\mathrm{Im}\ \tau} = \mathrm{Im}\,\sigma - \frac{(\mathrm{Im}\,v)^2}{\mathrm{Im}\,\tau} \qquad \Omega = \begin{pmatrix} \tau & v \\ v & \sigma \end{pmatrix}$$

  ⋆ the non-separating node is characterized by $t \to \infty$
  ⋆ a corresponding invariant parameter exists for arbitrary genus.

Degeneration of genus-two string invariants

- **Recall the generating function of genus-two string invariants**

$$\mathcal{B}(s_{ij}|\Sigma) = \int_{\Sigma^4} \frac{\mathcal{Y}\wedge\overline{\mathcal{Y}}}{(\det Y)^2} \exp\left\{ \sum_{i<j} s_{ij} G(z_i,z_j) \right\} = \sum_{w=0}^{\infty} \frac{1}{w!} \mathcal{B}_w(s_{ij}|\Sigma)$$

  ⋆ Taylor series in s_{ij} produces modular graph functions of weight w

$$\mathcal{B}_w(s_{ij}|\Sigma) = \int_{\Sigma^4} \frac{\mathcal{Y}\wedge\overline{\mathcal{Y}}}{(\det Y)^2} \left(\sum_{i<j} s_{ij} G(z_i,z_j) \right)^w$$

- **Theorem** (ED, Green, Pioline 2017)

  The expansion of $\mathcal{B}_w(s_{ij}|\Sigma)$ near the non-separating node is given by a Laurent polynomial of **finite** *degree $(w,-w)$ in t*

$$\mathcal{B}_w(s_{ij}|\Omega) = \sum_{k=-w}^{w} \mathcal{B}_w^{(k)}(s_{ij}|v,\tau)\, t^k + \mathcal{O}(e^{-2\pi t})$$

  where the coefficients are invariant under $SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^2 \subset Sp(4,\mathbb{Z})$

$$\mathcal{B}_w^{(k)}\left(s_{ij} \middle| \frac{v+m\tau+n}{c\tau+d}, \frac{a\tau+b}{c\tau+d} \right) = \mathcal{B}_w^{(k)}(s_{ij}|v,\rho)$$

  These are "elliptic modular graph functions" or single-valued elliptic polylogs.

Summary and outlook

- **Low energy expansion of string theory reveals a rich structure of**
  - ⋆ Identities between modular graph functions at genus one;
  - ⋆ Identities and degeneration of higher genus modular graph functions.
- **Further directions not covered here**
  - ⋆ Matching S-duality and susy predictions in Type IIB (many papers !)
  - ⋆ Systematic integration over moduli (Zagier 1981; ED, Green 2019; ED, Geiser 2020)
  - ⋆ Uniform transcendental weight for genus-one amplitude (ED, Green 2019)
- **Arithmetic significance ?**