Higher Multiplicity Two-Loop String Amplitudes

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String Amplitudes at finite α'
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Motivation

- **Structure of perturbative string amplitudes**
  - ⋆ kinematic structure
  - ⋆ relation to QFT amplitudes
  - ⋆ modular structure of integrands on (super-)moduli space
  - ⋆ matching low energy expansion with susy and S-duality predictions
- **Different approaches**
  - ⋆ Ramond-Neveu-Schwarz
  - ⋆ Pure spinor
  - ⋆ String field theory
  - ⋆ Flat space limit of AdS/CFT correlators
  - ⋆ Ambi-twistor string and QFT $\longrightarrow$ string amplitudes
  - ⋆ *Other fruitful approaches ?*
- **To what extent can one "bootstrap" string amplitudes ?**
- **Much is known for tree-level and one-loop amplitudes** [see also Oliver's talk]

This talk: Two loops

- **Amplitude for 5 massless strings** [ED, Mafra, Pioline, Schlotterer 2020]
  - ⋆ Using amalgam of chiral splitting and pure spinor methods
- **Amplitude for 5 massless NS strings** [ED, Schlotterer 2021]
  - ⋆ RNS first principles calculation
  - ⋆ Only even spin structure contribution
  - ⋆ remarkable simplicity of the final form
- **Towards amplitudes with $N > 5$ massless NS states in RNS**
  - ⋆ Summations over even spin structures for arbitrary N
    [ED, Hidding, Schlotterer 2022]
  - ⋆ Amplitude for 6 massless NS states (even spin structure only)
    [ED, Hidding, Schlotterer], in progress

- **Snowmass White Paper: String Perturbation Theory**
  [Berkovits, ED, Green, Johansson, Schlotterer, 2022]

Tree-level amplitudes

- **Virasoro-Shapiro Type II four graviton amplitude**

$$\mathcal{A}_4^{(0)} = \frac{1}{g_s^2} \frac{t_8 \, \tilde t_8}{stu} \frac{\Gamma(1-s)\Gamma(1-t)\Gamma(1-u)}{\Gamma(1+s)\Gamma(1+t)\Gamma(1+u)}$$

  - ⋆ $s_{ij} = -\frac{\alpha'}{4}(k_i + k_j)^2$ dimensionless with $s = s_{12}, t = s_{14}, u = s_{13}$
  - ⋆ basis of polarization tensors $\varepsilon_i^\mu \tilde\varepsilon_i^\nu$ with $f_i^{\mu\nu} = \varepsilon_i^\mu k_i^\nu - \varepsilon_i^\nu k_i^\mu$

$$t_8 = \mathrm{tr}(f_1 f_2 f_3 f_4) - \tfrac{1}{4}\mathrm{tr}(f_1 f_2)\mathrm{tr}(f_3 f_4) + \mathrm{cycl}(2,3,4)$$

  - ⋆ $t_8 \tilde t_8 \sim \mathcal{R}^4$ where $\mathcal{R}$ stands for the Riemann tensor

- α' **expansion is in terms of odd zeta-values only**

$$\mathcal{A}_4^{(0)} = \frac{1}{g_s^2} \frac{t_8 \, \tilde t_8}{stu} \exp\left\{ \sum_{m \text{ odd} \geq 3} \frac{2\zeta(m)}{m} (s^m + t^m + u^m) \right\}$$

  - ⋆ matches *single-valued map* $\mathbf{sv}$ applied to Veneziano amplitude

$$\mathbf{sv}\, \zeta(2n) = 0 \qquad\qquad \mathbf{sv}\, \zeta(2n+1) = 2\zeta(2n+1)$$

  - ⋆ Single-valued map derives from single-valued polylogarithms [Schnetz; Brown 2013]

Tree-level amplitudes (cont'd)

- **Kinematics of open string amplitudes** [Mafra, Schlotterer, Stieberger 2011]
  - ⋆ Color-ordered n-point function for the open superstring
    fix points $1, 2, 3$ and permute remaining points $\sigma = \sigma(4, 5, \cdots, n)$

$$\mathcal{A}_{\rm open}(1,2,3,\sigma) = \sum_{\rho,\tau \in \mathfrak{S}_{n-3}} Z_\sigma(1,2,3,\rho)\, S(\rho|\tau)\, A_{\rm SYM}(1,2,3,\tau)$$

where $S(\rho|\tau)$ is the KLT matrix of SYM [Broedel, Schlotterer, Stieberger 2013]

$$Z_\sigma(1,2,3,\rho) = \int_{z_{\sigma(i)} < z_{\sigma(i+1)}} \frac{dz_1 \cdots dz_n}{{\rm vol}\, SL(2,\mathbb{R})} \, \frac{\prod_{i<j} |z_{ij}|^{-4 s_{ij}}}{\rho(z_{12} z_{23} \cdots z_{n1})}$$

— see also the recent review [Mafra, Schlotterer 2022]

- **Kinematics of closed string amplitudes**
  - ⋆ single-valued projection of open string [Schlotterer, Stieberger 2012; Stieberger 2013]

$$\mathcal{A}_{\rm closed}(1,\cdots,n) = \sum_{\rho,\tau \in \mathfrak{S}_{n-3}} \tilde A_{\rm SYM}(1,2,3,\rho)\, S(\rho|\tau)\, \mathbf{sv}\, \mathcal{A}_{\rm open}(1,2,3,\tau)$$

  - ⋆ α' expansion in terms of single-valued multiple zeta values only

    [Brown, Dupont 2019; Vanhove, Zerbini 2020]

One-loop amplitudes

- **Type II one-loop four-graviton amplitude** [Green, Schwarz 1982]

$$\mathcal{A}_4^{(1)} = t_8 \tilde{t}_8 \int_{\mathcal{M}_1} \frac{d^2\tau}{\tau_2^2} \prod_{k=1}^{4} \int_\Sigma \frac{d^2 z_k}{\tau_2} \, \exp\Big\{ \sum_{1\leq i<j\leq 4} s_{ij}\, g(z_i - z_j|\tau) \Big\}$$

  ⋆ Torus $\Sigma = \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$; moduli space $\mathcal{M}_1 = \{\tau \in \mathbb{C},\ \mathrm{Im}\,\tau > 0\}/SL(2,\mathbb{Z})$
  ⋆ Scalar Green function on the torus

$$g(z|\tau) = -\ln |\vartheta_1(z|\tau)|^2 + \tfrac{2\pi}{\tau_2} (\mathrm{Im}\, z)^2$$

- **Analytic continuation and the $i\varepsilon$ prescription** [ED, Phong 1994]
  ⋆ the integral is absolutely convergent only for $\mathrm{Re}\,(s_{ij}) = 0$
  ⋆ analytic continuation in s_{ij} exists and produces poles and branch cuts
  ⋆ allows calculation of decay widths of massive states
  ⋆ double dispersion representation provides $i\varepsilon$ prescription

$$\mathcal{A}_4^{(1)} = \int_0^\infty d\sigma \int_0^\infty d\tau \, \frac{t_8 \tilde{t}_8\, \rho(s,t;\sigma,\tau)}{(s-\sigma+i\varepsilon)(t-\tau+i\varepsilon)} + \text{ analytic} + \text{cycl(s,t,u)}$$

  ⋆ Direct approach to the $i\varepsilon$ prescription [Berrera 1994; Witten 2015; Eberhardt, Mizera 2022]
  ⋆ Proposal for a KLT relation at one-loop [Stieberger 2022]

One-loop amplitudes (cont'd)

- **Loop momenta** [E&H Verlinde; ED, Phong 1988]
  - ⋆ Alternative presentation uses loop momentum $p \in \mathbb{R}^{10}$

$$\mathcal{A}_N^{(1)} = \int_{\mathcal{M}_1} d^2\tau \prod_i \int_\Sigma d^2 z_i \int_{\mathbb{R}^{10}} dp\, \mathcal{F}_N(\varepsilon, k, p|z,\tau) \overline{\mathcal{F}_N(\tilde\varepsilon^*, -k^*, p|z,\tau)}$$

where $\mathcal{F}_4(\varepsilon, k, p|z,\tau) = t_8\, \mathcal{I}_4(k,p|z,\tau)$ and the "chiral Koba-Nielsen factor"

$$\mathcal{I}_N(k,p|z,\tau) = \exp\Big\{ i\pi\tau p^2 + 2\pi i \sum_i z_i k_i \cdot p - \sum_{i \neq j} s_{ij} \ln \vartheta_1(z_i - z_j|\tau) \Big\}$$

  - ⋆ Integration over p produces τ_2^{-5} in measure and completes $g(z|\tau)$
  - ⋆ Chiral block $\mathcal{F}_4$ is locally holomorphic in z_i, τ

- **The five-point function** [Tsuchiya 1989; Green, Mafra, Schlotterer 2013, Mafra, Schlotterer 2018]
  - ⋆ Using the above notation for the chiral blocks for massless NS states

$$\mathcal{F}_5 = \mathcal{I}_5 \Big\{ \sum_i \mathfrak{t}_i\, \varepsilon_i \cdot \mathfrak{P}(z_i) - \sum_{i<j} \mathfrak{t}_{ij}\, g_{i,j} - \tfrac{\pi i}{8}\, \epsilon_{10}(p, \varepsilon_1, f_2, f_3, f_4, f_5) \Big\}$$

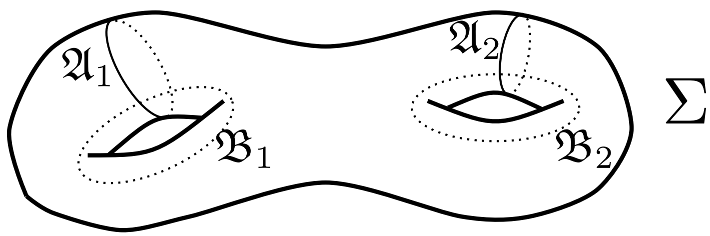
  - ⋆ where $\mathfrak{t}_1 = t_8(f_2, f_3, f_4, f_5)$ and $\mathfrak{t}_{12} = t_8([f_1, f_2], f_3, f_4, f_5)$ and where

$$\mathfrak{P}(z_i) = 2\pi i p + \sum_{j \neq i} g_{i,j} k_j \qquad\qquad g_{i,j} = \partial_{z_i} \ln \vartheta_1(z_i - z_j|\tau)$$

- **Higher point functions** see Oliver's talk

Compact genus-two Riemann surfaces

- **Homology** $H_1(\Sigma, \mathbb{Z}) \approx \mathbb{Z}^4$ **with intersection pairing** $\mathfrak{J}(\cdot,\cdot) \to \mathbb{Z}$
  - ⋆ Canonical basis $\mathfrak{J}(\mathfrak{A}_I, \mathfrak{A}_J) = \mathfrak{J}(\mathfrak{B}_I, \mathfrak{B}_J) = 0$, $\mathfrak{J}(\mathfrak{A}_I, \mathfrak{B}_J) = \delta_{IJ}$ for $I, J = 1, 2$



- **Modular group** $Sp(4, \mathbb{Z})$ **acts on** $H_1(\Sigma, \mathbb{Z})$ **leaving** $\mathfrak{J}(\cdot,\cdot)$ **invariant**

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad M^t \mathfrak{J} M = \mathfrak{J} \qquad \begin{pmatrix} \mathfrak{B} \\ \mathfrak{A} \end{pmatrix} \to M \begin{pmatrix} \mathfrak{B} \\ \mathfrak{A} \end{pmatrix}$$

- **Canonical basis of holomorphic one-forms** ω_I **in** $H^{(1,0)}(\Sigma)$

$$\oint_{\mathfrak{A}_I} \omega_J = \delta_{IJ} \qquad \oint_{\mathfrak{B}_I} \omega_J = \Omega_{IJ}$$

  - ⋆ Period matrix Ω obeys Riemann relations $\Omega^t = \Omega, \ \mathrm{Im}\,(\Omega) > 0$
  - ⋆ Moduli space $\mathcal{M}_2 = \{\Omega^t = \Omega, \ \mathrm{Im}\,(\Omega) > 0\}/Sp(4, \mathbb{Z})$ (minus the diagonal)

Two-loop four-graviton amplitude

- **Computed in the RNS formulation** [ED, Phong 2001-2005]

$$\mathcal{A}_4^{(2)} = g_s^2\, t_8 \tilde t_8 \int_{\mathcal{M}_2} \frac{|d\Omega^3|^2}{(\det \text{Im}\,\Omega)^3} \int_{\Sigma^4} \frac{\mathcal{Y} \wedge \bar{\mathcal{Y}}}{(\det \text{Im}\,\Omega)^2} \exp\left\{ \sum_{i<j} s_{ij}\, G(z_i, z_j|\Omega) \right\}$$

  ⋆ Measure on Σ^4 interlaces kinematic and worldsheet data

$$\begin{aligned} \mathcal{Y} &= t\Delta(z_1,z_2)\Delta(z_3,z_4) - s\Delta(z_1,z_4)\Delta(z_2,z_3) \\ \Delta(z,w) &= \omega_1(z)\omega_2(w) - \omega_2(z)\omega_1(w) \\ G(z,w|\Omega) &= -\ln|E(z,w|\Omega)|^2 + 2\pi(\text{Im}\,\Omega)^{-1}_{IJ}\, \text{Im}\int_w^z \omega_I\, \text{Im}\int_w^z \omega_J \end{aligned}$$

  ⋆ The prime form $E(z,w|\Omega)$ generalizes $\vartheta_1(z-w|\tau)$ of genus one

- **Computed in the pure spinor formulation** [Berkovits 2005; Berkovits, Mafra 2005]
  ⋆ generalized to include full supergravity multiplet of external states

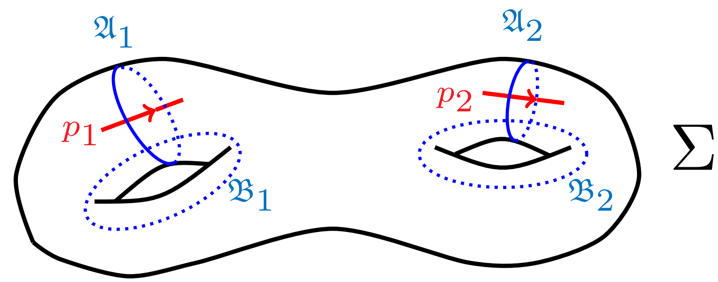
- α' **expansion matches S-duality predictions for BPS operators**
  ⋆ coefficient of $D^4\mathcal{R}^4$ [ED, Gutperle, Phong 2005; Gomez, Mafra 2010]
  ⋆ coefficient of $D^6\mathcal{R}^4$ [ED, Green, Pioline, Russo 2014]

- **Relation to ambi-twistor string** [Geyer, Monteiro, Tourkine 2016; Geyer, Monteiro 2018]

Chiral splitting and loop momenta

- **Loop momenta**
  - ⋆ Momentum flowing through a curve is the line integral of the $\partial_z x^\mu$
  - ⋆ A canonical homology basis gives a natural choice of loop momenta

$$p^I_\mu = \frac{1}{2\pi} \oint_{\mathfrak{A}_I} \partial_z x_\mu = -\frac{1}{2\pi} \oint_{\mathfrak{A}_I} \partial_{\bar z} x_\mu$$



- **Chiral splitting**
  - ⋆ Amplitude is given by pairing left and right chiral blocks

$$\mathcal{A}^{(2)}_N = \int_{\mathbb{R}^{20}} dp \int_{\mathcal{M}_2} d^6\Omega \int_{\Sigma^N} \mathcal{F}_N(\varepsilon, k, p|z, \Omega)\overline{\mathcal{F}_N(\tilde\varepsilon^*, -k^*, p|z, \Omega)}$$

  - ⋆ $\mathcal{F}_N$ is locally holomorphic in z and Ω and factorizes

$$\mathcal{F}_N = \mathcal{K}_N \, \mathcal{I}_N$$

  - ⋆ where $\mathcal{I}_N$ is the "chiral Koba-Nielsen factor"

$$\mathcal{I}_N = \exp\Big\{ i\pi\Omega_{IJ} p^I \cdot p^J + 2\pi i \sum_j k_j \cdot p^I \int_{z_0}^{z_j} \omega_I - \sum_{i\neq j} s_{ij} \ln E(z_i, z_j|\Omega)\Big\}$$

Properties of chiral blocks

- **Invariance under "homology shifts"** [ED, Phong 1988-89]
  - ⋆ Moving a vertex point z_j around an $\mathfrak{A}_J$ or $\mathfrak{B}_J$ cycle

$$\begin{aligned} \mathcal{F}_N(\varepsilon, k, p \vert z_i + \delta_{ij}\, \mathfrak{A}_J, \Omega) &= e^{2\pi i k_j \cdot p^J} \mathcal{F}_N(\varepsilon, k_i, p^I \vert z_i, \Omega) \\ \mathcal{F}_N(\varepsilon, k, p^I \vert z_i + \delta_{ij}\, \mathfrak{B}_J, \Omega) &= \mathcal{F}_N(\varepsilon, k, p^I + \delta^I_J k_j \vert z_i, \Omega) \end{aligned}$$

  - ⋆ Physical amplitude $\mathcal{A}_N^{(2)}$ is invariant by translation invariance of $\int dp$
  - ⋆ Relation to color-kinematic duality in QFT [Tourkine, Vanhove 2016]

- **The exponential factor $\mathcal{I}_N$ is universal** (bosonic, Type II, Heterotic)
  - ⋆ obeys the same homology shift relations as $\mathcal{F}_N$
  - ⋆ Measure $d\mu_\Sigma$ and Green function G recovered upon integrating out p^I and combining the prime form $E(z_i, z_j)$ with the Abelian integrals

- **The prefactor $\mathcal{K}_N$ depends on the specific string theory**
  - ⋆ meromorphic $(1,0)$-form in each vertex z_i
  - ⋆ invariant under homology shifts acting on both z_i and p

Obtaining $\mathcal{K}_N$

- **Indirect construction of $\mathcal{K}_5$ using pure spinors and chiral splitting**
  - ⋆ for the entire massless multiplet of string states
- **First principles calculation of $\mathcal{K}_5$ in the RNS formulation**
  - ⋆ for NS states and even spin structure contribution only
- **Carry out all even spin structure summations for $\mathcal{K}_N$ with $N \geq 6$**
  - ⋆ for NS states and even spin structure contribution only
  - ⋆ application to evaluating the $N = 6$ amplitude in progress
- **Applies to Type II or Heterotic strings**

Ingredients of the pure spinor formulation

- **The (non-minimal) pure spinor worldsheet fields** [Berkovits 2000-2008]
  - ⋆ space-time vector x^μ and spinors θ^α or d_α etc.
  - ⋆ $(0,0)$-forms: matter x^μ, θ^α; ghosts $\lambda^\alpha, \bar\lambda_\alpha, r_\alpha$
  - ⋆ $(1,0)$-forms: matter d_α; ghosts $w_\alpha, \bar w^\alpha, s^\alpha$
  - ⋆ pure spinor constraints $\lambda\gamma^\mu\lambda = \bar\lambda\gamma^\mu\bar\lambda = \bar\lambda\gamma^\mu r = 0$

- **Vertex operators**
  - ⋆ on-shell linearized SYM superfields $A_\alpha(x,\theta), A_\mu(x,\theta), W^\alpha(x,\theta), F_{\mu\nu}(x,\theta)$
  - ⋆ (integrated) vertex operator

$$U = \partial_z\theta^\alpha A_\alpha(x,\theta) + d_\alpha W^\alpha(x,\theta) + \cdots$$

  - ⋆ BRST $Q = \oint(\lambda^\alpha d_\alpha + \bar w^\alpha r_\alpha)$ invariance guarantees supersymmetry

- **Composite b-ghost** [Berkovits 2004]

$$b = s^\alpha \partial_z \bar\lambda_\alpha + \frac{\Pi_z^\mu(\bar\lambda\gamma_\mu d)}{2(\lambda\bar\lambda)} + \frac{(\bar\lambda\gamma^{\mu\nu\rho}r)}{192(\lambda\bar\lambda)^2}(d\gamma_{\mu\nu\rho}d) + \cdots$$

  - ⋆ where $\Pi_z^\mu = \partial_z x^\mu + \frac{1}{2}\theta\gamma^\mu\partial_z\theta$

Five-string amplitude via pure spinors & chiral splitting

- **Pure spinor calculations are manifestly supersymmetric**
  - ⋆ Zero modes facilitate calculations for low genus and few external states
    - – $(0,0)$-forms $\theta^\alpha, \lambda^\alpha, \bar\lambda_\alpha, r_\alpha$ have 16 zero modes (modulo constraints)
    - – $(1,0)$-forms $d_\alpha, w_\alpha, \bar w^\alpha, s^\alpha$ have 16×2 zero modes (modulo constraints)
  - ⋆ General Wick contractions with b-ghost exceedingly complicated
  - ⋆ Zero-mode integrals diverge for genus $\geq$ three [Aisaka, Berkovits 2009; Grassi, Vanhove 2009]

- **Consider the 5-point chiral amplitude factor** $\mathcal{K}_5$ [Gomez, Mafra, Schlotterer 2015]
  - ⋆ Counting zero modes shows $\mathcal{K}_5 = \mathcal{K}_5^V + \mathcal{K}_5^S$
    where $\mathcal{K}_5^V$ is linear in p and $\mathcal{K}_5^S$ is independent of p
  - ⋆ $\mathcal{K}_5^V$ may be evaluated using zero mode integrations and BRST closure

$$\mathcal{K}_5^V = 2\pi i\, T^\mu_{1,2,3|4,5}\, p^I_\mu\, \omega_I(2)\Delta(3,4)\Delta(5,1) + \text{cycl}(1,2,3,4,5)$$

  - ⋆ $\mathcal{K}_5^S$ involves Wick contractions that are not available with pure spinors
  - ⋆ $\mathcal{K}_5^V$ and singular part of $\mathcal{K}_5^S$ give leading α' contribution

Five-string amplitude: chiral splitting & pure spinors (cont'd)

- **While $\mathcal{K}_5^V$ may be completed into a BRST invariant candidate $\mathcal{K}'_5$**
  - ⋆ the resulting $\mathcal{K}'_5$ is not invariant under homology shifts
  - ⋆ The following combination includes $2\pi i p^I \omega_I(z_i)$

$$\mathcal{P}(z_i) = 2\pi i p^I \omega_I(z_i) + \sum_{j\neq i} k_j \, \partial_i \ln E(z_i, z_j)$$

  and is invariant under all homology shifts

- **Promote $2\pi i p^I_\mu \omega_I(z_i) \to \mathcal{P}_\mu(z_i)$ in the expression for $\mathcal{K}_5^V$**

$$\mathcal{K}_5^V \to \tilde{\mathcal{K}}_5^V = T^\mu_{1,2,3|4,5} \mathcal{P}_\mu(z_2) \Delta(3,4)\Delta(5,1) + \mathrm{cycl}(1,2,3,4,5)$$

  - ⋆ While $\tilde{\mathcal{K}}_5^V$ is homology shift invariant, its BRST properties are modified
  - ⋆ Determine $\mathcal{K}_5$ by requiring BRST invariance of

$$\mathcal{K}_5 = \tilde{\mathcal{K}}_5^V + \tilde{\mathcal{K}}_5^S$$

  and $\tilde{\mathcal{K}}_5^S$ independent of p and polynomial in s_{ij}

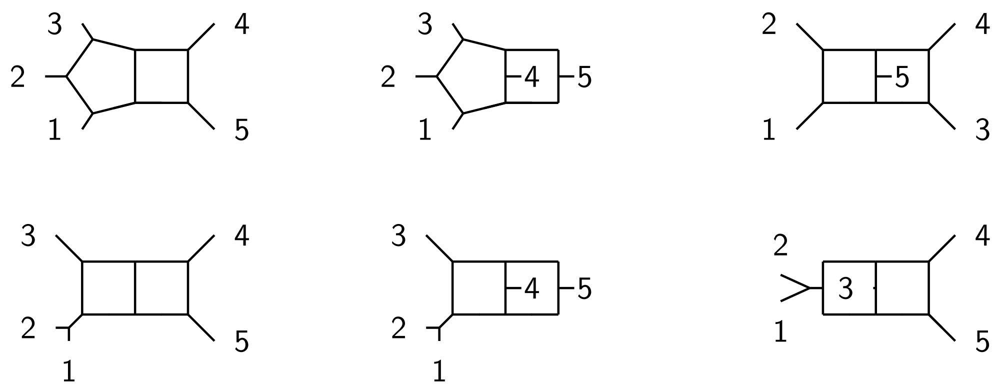
  $\Longrightarrow$ **unique solution** [ED, Mafra, Pioline, Schlotterer 2020]

- **Playing the same game for the 3-loop four-graviton amplitude ?**
  - ⋆ fails because there is no unique $\tilde{\mathcal{K}}_4^S$ solution for genus 3

    [ED, Mafra, Pioline, Schlotterer 2020] unpublished

The supergravity limit of the five-string amplitude

- **Tropical limit** [Tourkine 2016]
  - ⋆ perfectly matches $\mathcal{N}=4$ SYM and $\mathcal{N}=8$ supergravity [Carrasco, Johansson 2011]
    - – via double copy and color kinematic duality [Bern, Carrasco, Johansson 2008, 2010]
    - – via pure spinor bootstrap methods [Mafra, Schlotterer 2015]



- **Type IIB string kinematics via SYM field theory kinematics**
  - ⋆ for $U(1)_R$-preserving amplitudes [ED, Mafra, Pioline, Schlotterer 2020]

$$\mathcal{A}_5^{(2)}\Big|_{U(1)_R} = \begin{pmatrix} \tilde{A}_{\rm SYM}(1,2,3,5,4) \\ \tilde{A}_{\rm SYM}(1,3,2,5,4) \end{pmatrix}^t \mathcal{M} \begin{pmatrix} A_{\rm SYM}(1,2,3,4,5) \\ A_{\rm SYM}(1,3,2,4,5) \end{pmatrix}$$

Low energy expansion in Type IIB

- **Match with predictions from S-duality and supersymmetry**
  - ⋆ compare tree-level and one-loop coefficients
  - ⋆ compare genus-two modular integrands
  - ⋆ decompose into $U(1)_R$ preserving and violating amplitudes

- $U(1)_R$ **preserving: (non-linear) supersymmetry suggests pairing**
  - ⋆ $D^4\mathcal{R}^4$ and $D^2\mathcal{R}^5$ tree-level $\zeta(5)$; genus-two constant measure $d\mu_2$
  - ⋆ $D^6\mathcal{R}^4$ and $D^4\mathcal{R}^5$ tree-level $\zeta(3)^2$; genus-two Kawazumi-Zhang
  - ⋆ $D^8\mathcal{R}^4$ and $D^6\mathcal{R}^5$ same genus-two modular graph function
  - ⋆ $(D^6\mathcal{R}^5)'$ modular graph function absent from the 4-point amplitude

- $U(1)_R$ **violating: S-duality suggests pairing with** $U(1)_R$ **preserving**
  - ⋆ $\phi D^4\mathcal{R}^4$ and $D^2\mathcal{R}^5$ same genus-two modular graph function
  - ⋆ $\phi D^6\mathcal{R}^4$ and $D^4\mathcal{R}^5$ "
  - ⋆ $\phi D^8\mathcal{R}^4$ and $D^6\mathcal{R}^5$ "
  - ⋆ $(\phi D^8\mathcal{R}^4)'$ and $(D^6\mathcal{R}^5)'$ "

Five-string amplitude: RNS formulation

- **Massless NS states; even spin structures** [ED, Schlotterer 2021]
  - ⋆ First principles calculation gives

$$\mathcal{F}_5 = \mathcal{I}_5 \sum_i \Big\{ \mathfrak{P}^I(z_i) \cdot \big(\varepsilon_i \, \mathfrak{t}_i \, \mathcal{Y}_I + k_i \, \mathfrak{T}_{iI} \big) - \sum_{j \neq i} \mathcal{Y}_I \, \mathfrak{t}_{ij} \, g^I_{i,j} \Big\}$$

  - ⋆ Ingredients uplift expressions familiar from the one-loop 5-point amplitude

$\mathfrak{t}_1 = t_8(f_2, f_3, f_4, f_5)$ and $\mathfrak{t}_{12} = t_8([f_1, f_2], f_3, f_4, f_5)$

$$\mathfrak{P}^I(z_i) = 2\pi i p^I + \sum_{j \neq i} g^I_{i,j} k_j \qquad\qquad g^I_{i,j} = \partial^I \ln \vartheta[\nu](z_j - z_i|\Omega)$$

  – and from the two-loop four-point amplitude

$$\mathcal{Y}_I = 4 s_{12} \, \omega_I(4) \Delta(5,1) \Delta(2,3) + \text{ cycl}(1,2,3,4,5)$$

  – but also contains new ingredients

$$\mathfrak{T}_{1I} = (\mathfrak{t}_{12} - 2\mathfrak{t}_1 \varepsilon_1 \cdot k_2) \Big\{ \omega_I(3) \Delta(1,5) \Delta(2,4) + \text{ cycl}(3,4,5) \Big\} + \text{ cycl}(2,3,4,5)$$

  - ⋆ $\mathcal{F}_5$ is independent of the odd spin structure ν

- **Major effort in RNS goes into evaluating sums over spin structures**
  - ⋆ spin structure sums GSO project onto supersymmetric amplitudes

Spin structure sums for higher multiplicity

- **Restrict to even spin structures and NS external states**
  - ⋆ Correlator of chiral fermions for spin structure δ is given by the Szegö kernel

$$\langle \psi(z)\psi(w)\rangle = S_\delta(z,w) = \frac{\vartheta[\delta](\int_w^z \omega|\Omega)}{\vartheta[\delta](0|\Omega)E(z,w)}$$

where the Riemann ϑ-function for $\delta = [\delta'|\delta'']$ is defined by

$$\vartheta[\delta](\zeta|\Omega) = \sum_{n\in\mathbb{Z}^2} \exp\left\{ i\pi(n+\delta')^t\Omega(n+\delta') + 2\pi i(n+\delta')^t(\zeta+\delta'')\right\}$$

  - ⋆ String amplitude integrands involve cyclic products of Szegö kernels

$$C_\delta(z_1,\cdots,z_n) = S_\delta(z_1,z_2)S_\delta(z_2,z_3)\cdots S_\delta(z_{n-1},z_n)S_\delta(z_n,z_1)$$

(they also involve other products that may be treated similarly)

  - ⋆ Evaluating the spin structure sums for $n=4,5$ point amplitudes involved
    - – the Fay trisecant identity (cfr bosonization)
    - – the Riemann identities
    - – and every other trick we could think of

$\Rightarrow$ those methods show no promising generalization to $n \geq 6$

- – the problem was also considered in [Tsuchiya 2012; 2017; 2022]

Spin structure sums for higher multiplicity (cont'd)

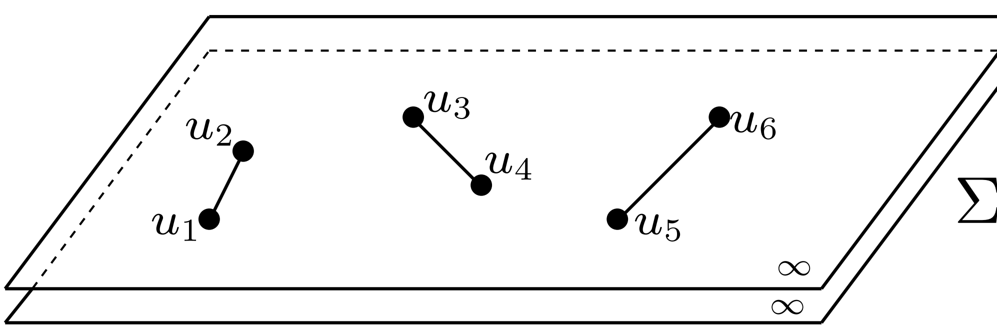
- **Theorem** [ED, Hidding, Schlotterer 2022]

  The spin structure sum of $C_\delta(z_1, \cdots, z_n)$ for arbitrary n reduces to the spin structure sums for the cases $n = 0, 2, 3, 4$

  ⋆ The proof is constructive and formulated in the hyper-elliptic formulation
  ⋆ The result will be translated into the ϑ-function formulation

- **Every genus two surface Σ is hyper-elliptic**

  ⋆ namely a double cover of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$
  ⋆ ramified over 6 branch points $u_1, \cdots, u_6$
  ⋆ points $z \in \Sigma$ parametrized by $z = (x, s)$ where $s^2 = (x - u_1) \cdots (x - u_6)$
  ⋆ Moduli space $\mathcal{M}_2$ isomorphic to $\{u_1, \cdots, u_6\}/SL(2, \mathbb{C}) \times \mathfrak{S}_6$



Sketch of proof of the Theorem

⋆ An even spin structure δ is isomorphic to a $3+3$ partition of branch points

$$\{u_1, \cdots, u_6\} = A \cup B \qquad A \cap B = \emptyset \qquad |A| = |B| = 3$$

(an odd spin structure is isomorphic to a 1+5 partition)

⋆ The Szegö kernel is given in terms of this partition by

$$S_\delta(z_1, z_2) = \frac{s_A(x_1)s_B(x_2) + s_B(x_1)s_A(x_2)}{2(x_1 - x_2)} \left[\frac{dx_1\, dx_2}{s(x_1)\, s(x_2)} \right]^{\frac{1}{2}}$$

where $s_A(x)s_B(x) = s(x)$ and $s_A(x)^2$ and $s_B(x)^2$ are polynomials given by

$$s_A(x)^2 = \prod_{r \in A} (x - u_r) \qquad s_B(x)^2 = \prod_{r \in B} (x - u_r)$$

⋆ The cyclic product of Szegö kernels is thus given by (using $x_{n+1} = x_1$)

$$C_\delta(z_1, \cdots z_n) = \frac{\prod_{i=1}^n (s_A(x_i)s_B(x_{i+1}) + s_B(x_i)s_A(x_{i+1}))}{2^n\, x_{12}x_{23} \cdots x_{n1}} \frac{dx_1 \cdots dx_n}{s(x_1) \cdots s(x_n)}$$

⋆ All spin structure dependence is contained in polynomials with $2m \leq n$

$$Q_\delta(i_1, \cdots, i_m | j_1, \cdots, j_m) = \prod_{\alpha=1}^m s_A(x_{i_\alpha})^2 s_B(x_{j_\alpha})^2 + (A \leftrightarrow B)$$

Sketch of proof of the Theorem (cont'd)

- **Lemma 1**

  All spin structure dependence of Q_δ *is polynomial in* $\ell_\delta^{11}, \ell_\delta^{12} = \ell_\delta^{21}, \ell_\delta^{22}$

$$\ell_\delta^{11} = \tfrac{1}{4}\alpha_2\beta_2 - \tfrac{3}{20}\mu_4 \qquad s_A(x)^2 = x^3 - \alpha_1 x^2 + \alpha_2 x - \alpha_3$$
$$\ell_\delta^{12} = \tfrac{1}{4}(\alpha_1\beta_2 + \alpha_2\beta_1) - \tfrac{9}{40}\mu_3 \qquad s_B(x)^2 = x^3 - \beta_1 x^2 + \beta_2 x - \beta_3$$
$$\ell_\delta^{22} = \tfrac{1}{4}\alpha_1\beta_1 - \tfrac{3}{20}\mu_2 \qquad s(x)^2 = x^6 - \mu_1 x^5 + \cdots - \mu_5 x + \mu_6$$

- **Lemma 2: The trilinear relations**

  Every trilinear $\ell_\delta^{a_1a_2}\,\ell_\delta^{a_3a_4}\,\ell_\delta^{a_5a_6}$ *may be expressed as a polynomial of total degree two in the combinations* $\ell_\delta^{11}, \ell_\delta^{12}$ *and* ℓ_δ^{22} *whose coefficients are polynomials in* $\mu_1, \cdots, \mu_6$

- *Combining Lemmas 1 and 2 implies that all spin structure dependence* of Q_δ and C_δ is given by a quadratic polynomial in $\ell_\delta^{11}, \ell_\delta^{12}, \ell_\delta^{22}$ with coefficients that depend only on μ_i.

- *The spin structure sums of the linears* $\ell_\delta^{a_1a_2}$ *and of the bilinears* $\ell_\delta^{a_1a_2}\ell_\delta^{a_3a_4}$ *are determined by* N*-point functions with* $N \leq 4$, *which concludes the proof of the Theorem.*

$SL(2,\mathbb{C})$ tensorial structure of the trilinear relations

- **Component form of the trilinear relations** e.g.

$$(\boldsymbol{\ell}_\delta^{11})^3 = \frac{\mu_4(\boldsymbol{\ell}_\delta^{11})^2}{20} - \frac{\mu_5\boldsymbol{\ell}_\delta^{11}\boldsymbol{\ell}_\delta^{12}}{4} + \mu_6\boldsymbol{\ell}_\delta^{11}\boldsymbol{\ell}_\delta^{22} - \frac{\mu_6(\boldsymbol{\ell}_\delta^{12})^2}{4} + \frac{\mu_4^2\boldsymbol{\ell}_\delta^{11}}{50} - \frac{9\mu_3\mu_5\boldsymbol{\ell}_\delta^{11}}{160} + \frac{3\mu_2\mu_6\boldsymbol{\ell}_\delta^{11}}{20} + \frac{\mu_4\mu_5\boldsymbol{\ell}_\delta^{12}}{40}$$
$$- \frac{9\mu_3\mu_6\boldsymbol{\ell}_\delta^{12}}{80} - \frac{\mu_5^2\boldsymbol{\ell}_\delta^{22}}{16} + \frac{3\mu_4\mu_6\boldsymbol{\ell}_\delta^{22}}{20} - \frac{3\mu_4^3}{2000} + \frac{9\mu_3\mu_4\mu_5}{1600} - \frac{3\mu_2\mu_5^2}{320} - \frac{81\mu_3^2\mu_6}{6400} + \frac{9\mu_2\mu_4\mu_6}{400}$$

- **The $\boldsymbol{\ell}_\delta^{ab}$ transform under the 3-dimensional irrep of $SL(2,\mathbb{C})$ by**

$$\boldsymbol{\ell}_\delta^{ab} \to J\, g^a_c\, g^b_d\, \boldsymbol{\ell}_\delta^{cd} \qquad g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{C}) \qquad J = \prod_{j=1}^{6} (\gamma u_j + \delta)^{-1}$$

- **The trilinear relations in $SL(2,\mathbb{C})$ tensorial form**

$$\mathbf{7} \qquad \boldsymbol{\ell}_\delta^{(a_1a_2}\,\boldsymbol{\ell}_\delta^{a_3a_4}\,\boldsymbol{\ell}_\delta^{a_5a_6)} = \mathrm{M}_1^{b_1b_2(a_1\cdots a_4}\,\boldsymbol{\ell}_\delta^{a_5a_6)}\,\boldsymbol{\ell}_\delta^{c_1c_2}\,\varepsilon_{b_1c_1}\,\varepsilon_{b_2c_2} + \cdots$$

$$\mathbf{3} \qquad (\det \boldsymbol{\ell}_\delta)\,\boldsymbol{\ell}_\delta^{a_1a_2} = \tfrac{3}{2}\,\mathrm{M}_1^{a_1a_2b_1\cdots b_4}\,\boldsymbol{\ell}_\delta^{c_1c_2}\boldsymbol{\ell}_\delta^{c_3c_4}\varepsilon_{b_1c_1}\cdots\varepsilon_{b_4c_4} + \cdots$$

⋆ where $\mathbf{M}_1$ is the symmetric rank 6 tensor under $SL(2,\mathbb{C})$ with components

$$\mathrm{M}_1^{111111} = \mu_6 \qquad \mathrm{M}_1^{111112} = \tfrac{\mu_5}{6} \qquad \mathrm{M}_1^{111122} = \tfrac{\mu_4}{15} \cdots$$

$Sp(4,\mathbb{Z})$ tensorial structure of the trilinear relations

- **Correspondence between hyper-elliptic and ϑ-function formulations**
  - ⋆ via standard Thomae formulas and holomorphic 1-forms ω_I

$$\varpi_1 = \frac{dx}{s(x)} \qquad \varpi_2 = -\frac{x\,dx}{s(x)} \qquad \omega_I(z) = \varpi_a(z)\,\sigma^a{}_I$$

  - ⋆ we obtain the *modular tensors* $\mathfrak{L}_\delta$ and $\mathfrak{M}_1$ (δ transforms)

$$\boldsymbol{\ell}_\delta^{ab} = \sigma^a{}_I\,\sigma^b{}_J\,\mathfrak{L}_\delta^{IJ} \qquad \mathrm{M}_1^{a_1\cdots a_6} = (\det\sigma)^{-2}\,\sigma^{a_1}{}_{I_1}\cdots\sigma^{a_6}{}_{I_6}\,\mathfrak{M}_1^{I_1\cdots I_6}$$

$$\mathfrak{L}_\delta^{IJ} = \frac{\pi}{5i}\partial^{IJ}\ln\left\{\frac{\vartheta[\delta](0)^{20}}{\Psi_{10}}\right\} \qquad \mathfrak{M}_1^{I_1\cdots I_6} = \Psi_{10}^{-\frac{1}{2}}\partial^{(I_1}\vartheta[\nu_1](0)\cdots\partial^{I_6)}\vartheta[\nu_6](0)$$

  - ⋆ where $\nu_1,\cdots,\nu_6$ are the six (distinct) odd spin structures

- **Trilinear relations are between $Sp(4,\mathbb{Z})$ modular tensors**

$$\mathfrak{L}_\delta^{(I_1I_2}\,\mathfrak{L}_\delta^{I_3I_4}\,\mathfrak{L}_\delta^{I_5I_6)} = \mathfrak{M}_1^{J_1J_2(I_1\cdots I_4}\,\mathfrak{L}_\delta^{I_5I_6)}\,\mathfrak{L}_\delta^{K_1K_2}\,\varepsilon_{J_1K_1}\,\varepsilon_{J_2K_2} + \cdots$$

$$(\det\mathfrak{L}_\delta)\,\mathfrak{L}_\delta^{I_1I_2} = \tfrac{3}{2}\,\mathfrak{M}_1^{I_1I_2J_1\cdots J_4}\,\mathfrak{L}_\delta^{K_1K_2}\,\mathfrak{L}_\delta^{K_3K_4}\,\varepsilon_{J_1K_1}\cdots\varepsilon_{J_4K_4} + \cdots$$

Modular tensors

- **Structure of (locally holomorphic) modular tensors under $Sp(4,\mathbb{Z})$**
  - ⋆ totally symmetric tensor $\mathfrak{T}$ of rank r and weight w
  - ⋆ transforms under $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp(4,\mathbb{Z})$ by

$$\mathfrak{T}^{I_1 \cdots I_r} \to \det (C\Omega + D)^w \, (C\Omega + D)^{I_1}{}_{J_1} \cdots (C\Omega + D)^{I_r}{}_{J_r} \, \mathfrak{T}^{J_1 \cdots J_r}$$

- **Classic Siegel modular forms of weight w correspond to rank $r = 0$**
  - ⋆ $Sp(4,\mathbb{Z})$: polynomial ring generated by $\Psi_4, \Psi_6, \Psi_{10}, \Psi_{12}, \Psi_{35}$ [Igusa]

- **Ring structure for modular tensors of arbitrary rank and weight**
  - ⋆ what are the generators ?
  - ⋆ which subspace is needed for string amplitudes ?

- **Non-holomorphic modular tensors arise in the α' expansion**
  - ⋆ related to the Kawazumi-Zhang invariant [Kawazumi 2008;Zhang 2008; Kawazumi 2016]
  - ⋆ higher genus modular graph functions [ED, Green, Pioline, Schlotterer 2020; ED, Schlotterer 2021]

Summary and outlook

- **Five-point 2-loop amplitude**
  - ⋆ direct calculation in RNS for NS states and even spin structure contribution
  - ⋆ indirect calculation via pure spinors and chiral splitting
  - ⋆ remarkable simplicity of the final amplitude

- **Higher point 2-loop amplitudes**
  - ⋆ all even spin structures sums available in RNS via modular tensors
  - ⋆ constraints from indirect methods ?
  - ⋆ relate kinematics to QFT amplitudes ?

- **Original question: what is the structure of string amplitudes ?**
  - ⋆ the expansion in α' showed extensive structure
    - – modular graph functions and relation to polylogarithms
  - ⋆ finite α' involves modular tensors and intertwined kinematic dependence
    - – **can one build an efficient library for these structures** ?