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## SUPERSTRINGS, SUPER - RIEMANN SURFACES, AND SUPERMODULI SPACE

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## 1. INTRODUCTION

Superstring theories [1] are at present the only candidates for a unified theory of all interactions which can incorporate a consistent theory of quantum gravity. All evidence so far points to their being anomaly free and finite to one-loop order. They are however still far from being well-understood. For example, we know now that their classical vacua correspond to two-dimensional conformal field theories. But we still do not know the classical equations of motion to which these vacua are solutions. Nor do we have any clue about the role of non-perturbative effects in lifting the vacuum degeneracy. It is important at this point to probe more deeply into the structure of the theory. Some fundamental questions which have attracted attention recently [2] are:

a) String perturbation theory;

b) Field theory and non-perturbative formulations of string theory;

c) Compactification and classification of conformal field theories;

d) Symmetries of the theory at the Planck scale;

e) String loop corrections to the equations of motion.

In this article we shall describe recent progress on the first issue. We shall restrict our discussion to theories of closed oriented strings. Strings moving in a fixed background sweep out a two-dimensional surface, which is called the world-sheet. Strings interact by joining and splitting. Thus handles on the world-sheet indicate

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creation and annihilation of virtual pairs, and the loop order of string perturbation theory is just the genus of the world-sheet. Quantization requires a sum over histories, which becomes then a sum over surfaces. A key principle is conformal invariance, which says that the contribution to quantum amplitudes of each surface should depend only on the conformal structure of the surface, and not on more detailed information such as the world-sheet metric. Thus quantum amplitudes at each loop order $h = 0, 1, 2, \ldots, \infty$ should be expressible as integrals over the moduli space of Riemann surfaces of genus $h$.

The main problem in string perturbation theory is to formulate practical rules for evaluating these amplitudes, which should play a role analogous to that of Feynman rules in quantum field theory. Once the correct rules have been laid down, we can hope to establish the finiteness and unitarity of string amplitudes, order by order in the loop expansion. For the bosonic string, a reasonably simple set of such rules is known (see [3], Chapters 2,8, and references therein). An important feature of these rules is the independence of left and right movers on the world-sheet, which translates into holomorphicity and antiholomorphicity of the integrand on moduli space [4]. This of course is a powerful constraint, which allows us to exploit the machinery of algebraic geometry and the theory of modular forms. These theories have allowed us to gain a very good understanding in this case. The bosonic string is however an unrealistic model since it carries no fermionic degrees of freedom, and its spectrum contains a tachyon. In superstring theories, these problems are expected to be cured respectively by introducing fermionic superpartners to the bosonic degrees of freedom, and by carrying out the so-called GSO projection (c.f. Section 3) which eliminates the tachyon mode. These very remedies are the source of new serious difficulties in evaluating superstring amplitudes. In particular they will force us to probe deeply into supergeometry, the theory of super Riemann surfaces, and their relation with more classical algebraic geometry and modular forms. We shall now describe the main issues briefly.

In bosonic string theory, the action is invariant under reparametrization of the world-sheet and Weyl scalings of the metric. These symmetries are crucial to insure decoupling of the ghosts and unitarity. The string equations for the background in which the string propagates are actually the conditions for conformal invariance in the two-dimensional quantum theory. Reparametrization invariance can be maintained by writing down only manifestly reparametrization invariant measures. A first difficulty peculiar to superstrings is that in this case, we need two additional symmetries to decouple the ghosts, namely local $N = 1$ supersymmetry and super Weyl symmetry. Unlike for reparametrization invariance, we do not have at our disposal functional measures on spaces of tensors and spinors which are manifestly supersymmetric. A way around this is to rely on the superspace formalism, and express superstring scattering amplitudes in terms of integrals over the «supermoduli space» of super Riemann surfaces. The metrics and measures that arise this way are however no longer positive definite. Also the global theory of supermanifolds and super Riemann surfaces is still in its infancy. The two fundamental problems from this point of view would be to develop this theory to the level of the modern theory of Riemann surfaces, and/or to integrate out the odd parameters and reduce integrals over supermoduli space to the more familiar integrals over moduli space. Various Ansatze for the latter procedure have been suggested in the literature [5], each with its own problems. We shall return to them shortly. A second difficulty of superstrings is a correct prescription for the GSO projection. In the sum over histories approach, this should correspond to factoring superstring integrands into forms holomorphic and antiholomorphic on moduli or supermoduli space, assigning independent spin structures to each, and summing over them. It is by no means clear how to carry this out in practice, due to the fact that the fields describing propagation of superstrings cannot be split, and $N = 1$ supersymmetry forces the inclusion of many terms mixing intricately left and right chiralities. Upont quantization we are led to expressions involving functional determinants and correlation functions of fields of various spins. The combined determinants are known to factor, thanks to the Belavin-Knizhnik theorem which already made its appearance in the bosonic string. Thus the issue here is to factor the correlation functions.

We shall derive rules for factoring correlation functions, enforcing the GSO projection, and reducing integrals over supermoduli space to integrals over moduli space. It turns out that although we initially separated the two problems about measures and holomorphic factoring in superstring amplitudes somewhat artificially, their resolution is intimately linked. We develop the geometry of super Riemann surfaces to the point where these rules can be enunciated in an especially simple way. They are given in the supergravity formalism for super Riemann surfaces, which is the one naturally resulting form the Polyakov [6] covariant formulation of superstring theory. It is likely that the superanalytic objects emerging in this way will prove to be key ingredients in the understanding of the super algebraic geometry of supermoduli space. It would of course be very valuable to make contact with other approaches to super Riemann surfaces, in particular the light-cone approach of Mandelstam [7]. The equivalence of the light-cone and covariant formalisms would establish unitarity of the covariant approach. Efforts in this direction are in [8] [9].

We expect that our simple chiral amplitudes in terms of which all perturbative amplitudes are formulated will provide a solid starting ground for a complete proof of finiteness. Nevertheless such a proof of finiteness is still lacking. Of course arguments of a rather general nature in favor of finiteness at the perturbative level of superstring theories have been advanced by many authors, but certainly no proof from a precise prescription (say, comparable to one-loop formulas for scattering of massless bosons) is available at this point. On the other hand, by the fall of 1987, many apparent inconsistencies and ambiguities had been uncovered in the various prescriptions suggested to date [10], [11], [12], including the one based on $BRST$ invariance [11] [12] [14]. Some inconsistencies seemed to even suggest a catastrophic breakdown of gauge invariance [13] [15] [16] [17]. We shall provide later (see Section 8) a fuller account of them. An essential criterion of any new prescription should certainly be that it offer a way out of such problems. We shall argue that our prescription does, providing strong evidence for the internal consistency of string perturbation theory.

This paper is based on lectures given by the authors at the Rome Conference on String Theory in July 1988. Or goal is provide a useful complement to the survey article [3]. We have attempted to provide a quick path to the most urgent problems in perturbation theory for superstrings. For this we have omitted many full derivations. On the other hand we have attempted to reach a wide audience by providing some informal discussions of basic material which may not be familiar to non-specialists. Also some more recent work on function theory on super Riemann surfaces has been included, which was not available in [3].

## 2. THE BOSONIC STRING

We shall begin with a brief discussion of some key features of the bosonic string which will serve as a guide to the superstring.

At the order $h$ of perturbation theory the world-sheet is a surface with $h$ handles imbedded in space time. If space-time is a Riemannian manifold of dimension $d$ with coordinates $x^1, \dots, x^d$, we can view such a surface as the image of a fixed surface with $h$ handles $M$ by $d$ scalar functions $x^1, \dots, x^d$ defined on $M$. The Nambu-Goto action for string propagation is just the area of the imbedded surface. In local coordinates $\xi^1, \xi^2$ on $M$, it can be written as

$$I_{NG}(x^\mu) = \frac{T}{4\pi} \int_M d^2\xi \sqrt{h} \tag{2.1}$$

where

$$h = \det h_{mn}$$

$$h_{mn} = \partial_m x^\mu \partial_n x^\nu G_{\mu\nu}(x) \; , \tag{2.2}$$

$G_{\mu\nu}$ is the metric of space-time, and $T$ is the string tension. Another action is that of the two-dimensional sigma model, where in addition to the scalar fields $x^\mu$ we also introduce a two-dimensional metric $g_{mn}$ on the world-sheet

$$I(x^\mu, g_{mn}) = \frac{T}{8\pi}\int_M d^2\xi\sqrt{g}g^{mn}\partial_m x^\mu \partial_n a x^\nu G_{\mu\nu}(x) \; . \tag{2.3}$$

In the mathematics literature, it is rather called the energy integral, and its critical points harmonic maps. Classically the dynamics of the two actions are the same, since the variational derivative of the sigma model action with repect to the metric $g_{mn}$, namely the stress tensor

$$\begin{aligned} T_{mn} &= -\frac{4\pi}{\sqrt{g}}\frac{\delta I}{\delta g^{mn}} \\ &= -\frac{T}{2}(h_{mn} - \frac{1}{2}g_{mn}g^{pq}h_{pq}) \end{aligned}$$

vanishes exactly when $g_{mn}$ is conformal to $h_{mn}$, in which case the two actions coincide. In the Polyakov formulation we quantize strings using the sigma model action, since it has the advantage of being quadratic in $x^\mu$ when space-time is flat Minkowski space:

$$I(x^\mu, g_{mn}) = \frac{T}{8\pi}\int_M d^2\xi\sqrt{g}g^{mn}\partial_m x^\mu \partial_n x^\mu \; . \tag{2.4}$$

Henceforth we shall only deal with Minkowski space-time and set the string tension $T$ to $1$. The sum over histories becomes the sum over all fields $x^\mu$ and $g_{mn}$, so that quantization leads to functional integrals with respect to the measure

$$Dg_{mn}Dx^\mu e^{-I(x^\mu, g_{mn})} \; . \tag{2.5}$$

In particular the partition function for the bosonic string is given by

$$Z_{BOS} = \int D g_{mn} D x^\mu e^{-I(x^\mu, g_{mn})} \tag{2.6}$$

and scattering amplitued on $N$ particles of momenta $k_i^\mu$ and masses $m^2 = -k_i^\mu k_{i\mu}$ are given by

$$\left\langle \prod_{i=1}^N V_i(k_i^\mu) \right\rangle = \int D g_{mn} D x^\mu e^{-I(x^\mu, g_{mn})} \prod_{i=1}^N V_i(k_i^\mu) \; . \tag{2.7}$$

Here the $V_i(k_i^\mu)$ can be viewed mathematically as various moments of this measure. The simplest ones which correspond respectively to the tachyon and the graviton multiplet are

$$V_{-1}k^\mu = \int d^2\xi\sqrt{g}e^{ik^\mu x_\mu}$$

$$V_0(k^\mu) = \epsilon_{\mu\nu} \int d^2\xi \sqrt{g} \partial_m x^\mu \partial^m x^\nu e^{ik^\rho x_\rho} . \tag{2.8}$$

We can now have an idea of why the theory is conformally invariant, in the sense that the contribution of each surface to the amplitudes (2.4) depends only on its complex structure. The action (2.3) is invariant under reparametrizations of the world-sheet and Weyl scalings of the metric $g_{mn}$. For later reference we list here the infinitesimal forms of these two gauge symmetries. Their infinitesimal generators are respectively a vector field $\delta v^m$ and a scalar field $\delta\sigma$, and the corresponding infinitesimal changes of the metrics will be

$$\mathrm{Diff}(M) : \delta g_{mn} = \nabla_m \delta v_n + \nabla_n \delta v_m$$

$$\mathrm{Weyl}(M) : \delta g_{mn} = 2\,\delta\sigma g_{mn} . \tag{2.9}$$

In the absence of anomalies, we should be able to factor out the volumes of the gauge groups and the functional integrals (2.5) and (2.6) should reduce to integrals over the moduli space $\mathcal{M}_h$ of Riemann surfaces of genus $h$

$$\mathcal{M}_h = \{\text{metrics } g_{mn}\}/\mathrm{Diff}(M) \times \mathrm{Weyl}(M) . \tag{2.10}$$

Of course Weyl symmetry is in general anomalous, but it is by now well-known that the anomalies of matter fields $x^\mu$ and reparametrization ghosts $b_{m,n} c^m$ cancel for the critical dimension $d = 26$. We shall discuss this cancellation in greater detail below. This anomalous behavior is also responsible for conformal invariance of the scattering amplitudes (2.7), which naively seem to depend on the scale of the world-sheet metric $g_{mn}$. In the critical dimension, scattering amplitudes do reduce then to integrals over moduli space, and we shall now discuss the exact form of the integrand for the simplest case of the partition function. It is instructive to exhibit it in several forms, each with its own advantages.

For this we recall some basic facts about the geometry of moduli space. It is convenient to introduce local isothermal coordinates $z$ and $\bar z$, under which the metric becomes $ds^2 = 2 g_{z\bar z} dz d\bar z$. A variation of complex structure is parametrized by a Beltrami differential $\mu_{\bar z}{}^z$, and local isothermal coordinates $w, \bar w$ for the deformed structure can be obtained by solving the Beltrami equation

$$\partial_{\bar z} w = \mu_{\bar z}{}^z \partial_z w . \tag{2.11}$$

Now new complex structures obtained this way from two Beltrami differentials could still be equivalent by a reparametrization. This will happen when the Beltrami differentials differ by a term of the form $\partial_{\bar{z}}\delta v^z$ (in view of (2.9)). Thus the tangent space to moduli space can be identified with

$$T(M_h) = \{\text{Beltrami differentials } \mu_{\bar{z}}{}^z\}/\{\text{Range } \partial_{\bar{z}} \text{ on vector fields}\} . \tag{2.12}$$

Since Beltrami differentials can be paired off with quadratic differentials $\phi_{zz}$ by

$$\langle\mu|\phi\rangle = \int d^2 z \mu_{\bar{z}}{}^z \phi_{zz} \tag{2.13}$$

we deduce that the cotangent space to moduli space is the space of holomorphic quadratic differentials. The index theorem applied to the $\partial_{\bar{z}}$ operator on rank 2 tensors shows that the dimension of the space of quadratic differentials is 0 for genus 0, 1, for genus 1, and $3h-3$ for genus $h \geq 2$, the cases of genera 0 and 1 being different because of the presence of continuous families of holomorphic automorphisms of dimension 3 and 1 respectively. To fix the notation, we shall put ourselves in the genus $h \geq 2$ case, the other cases following with simple modifications. We can now return to the gauge-fixed forms for the bosonic string partition function.

## The general gauge-fixed form

We begin by the most general form. We parametrize moduli space by $3h-3$ dimensional slices $S$ of metrics $g_{z\bar{z}}(\tau)$, where $\tau_1, \ldots, \tau_{3h-3}$ are local coordinates for moduli space. Let $\mu_i, i = 1, \ldots, 3h-3$ be the corresponding Beltrami differentials

$$\delta g_{\bar{z}\bar{z}} = \sum_{i=1}^{3h-3} \delta\tau_i g_{z\bar{z}}(\mu_{\bar{z}}{}^z)_i . \tag{2.14}$$

At each metric $g_{z\bar{z}}(\tau)$, let $\phi_{zz,j}, j = 1, \ldots, 3h-3$ be any basis of quadratic differentials, holomorphic with respect to the complex structure defined by $g_{z\bar{z}}(\tau)$. We shall denote by $\bar{\partial}_n$, the operator on rank $n$ tensors given by

$$\bar{\partial}_n(\psi_{z\ldots z}dz^n) = (\partial_{\bar{z}}\psi_{z\ldots z})d\bar{z}dz^n . \tag{2.15}$$

All inner products and adjoints on spaces of tensor at $\tau$ will be taken using the metric $g_{z\bar{z}}(\tau)$. Then the expression in local coordinates for the bosonic partition function is

$$Z_{BOS} = \int d^{3h-3}\tau_i d^{3h-3}\bar{\tau}_i \cdot \tag{2.16}$$
$$\cdot \frac{|\det < \mu_i|\phi_j > |^2}{\det < \phi_i|\phi_j >} \det \bar{\partial}_2^\dagger \bar{\partial}_2 \left( \frac{\det \bar{\partial}_0^\dagger \bar{\partial}_0}{\int d^2 z g_{z\bar{z}}} \right)^{-13} .$$

It is not difficult to give a simple motivation for all the ingredients appearing in the basic formula (2.16). The expression involving the determinant of $\bar{\partial}_0$ is just the gaussian integrals over the free scalar field $x^\mu, \mu = 1, \ldots, 26$ . The reason for the area factor is the constant zero mode of the Laplacian $\Delta = \bar{\partial}_0^\dagger \bar{\partial}_0$ , which has to be factored out in order to produce a finite answer. For the partition function, this means we have omitted the infinite volume of Minkowski space-time. For scattering amplitudes the integration over the constant zero mode will produce a Dirac $\delta$ factor guaranteeing conservation of overall momentum. To understand the remaining ingredients, we note that locally the space of two-dimensional metrics can be parametrized by orbits of gauge transformations $\delta\sigma, \delta v^z$ , and the moduli parameters $\tau_i$ . The jacobian of the change of variables from $\delta g = (\delta g_{\bar{z}\bar{z}}, \delta g_{zz}, \delta g_{z\bar{z}})$ to

$$\left( 2\nabla_{\bar{z}} \delta v_{\bar{z}} + \sum_{i=1}^{3h-3} \delta\tau_i g_{z\bar{z}} \mu_i + c.c., \nabla_z v_{\bar{z}} + \delta\sigma \right) \tag{2.17}$$

must involve the determinant of the operator $\nabla_z^\dagger \nabla_z$ acting on vector fields $v^z$ , as well as the angle that the slice $S$ makes with the orbits of the symmetry groups. But the adjoint of this operator is just the operator $\bar{\partial}_2^\dagger, \bar{\partial}_2$ , and we have accounted for this determinant in (2.16). Finally we note that the space of holomorphic quadratic differentials $\phi_\alpha$ is orthogonal to the orbits of the symmetry groups, so that the ratio of finite determinants occurring in (2.16) can be interpreted as the sin of the angle between $S$ and the orbits, as it should be.

The cases of genera 1 and 0 require modifications because there are holomorphic automorphisms of the surface. These produce the same effect as Weyl scalings, so there is redundancy in the above evaluation of jacobians. The way to handle this is to restrict the gauge-fixing operator $\bar{\partial}_2$ only to the orthogonal complement of the space of conformal Killing vectors. The net result is that we have to divide now the integrand in (2.16) by the volume $\mathrm{Vol}(\mathrm{Ker}\,\bar{\partial}_2)$ of the group of holomorphic automorphism. For genus 0 , this volume is infinite, so that the partition function vanishes. For genus 1 it is finite. In this case we can evaluate all relevant determinants by zeta function regularization and arrive at the formula

$$Z_{BOS} = \int_{\mathcal{M}_1} \frac{d^2\tau}{8\pi^2\tau_2^2} (4\pi\tau_2)^{-12} |\eta(\tau)|^{-48} \tag{2.17}$$

where $\eta(\tau) = e^{i\pi\tau/12}\prod_{i=1}^{\infty}(1 - e^{2\pi i n\tau})$ is the Dedekind eta function.

The formula (2.16) for the bosonic partition function is completely unambiguous, and manifestly independent of the choice of slice $S$. Although we have discussed only the factoring out of the volumes of the continuous symmetry groups, invariance under the discrete reparametrizations of the world-sheet, the so-called mapping class group of surfaces of genus $h$, requires that we restrict to a fundamental domain, in other words, to moduli space. It should be pointed out that there are no possible anomalies for these symmetries in the present situation, since all infinite-dimensional determinants appearing in (2.16) are determinants of self-adjoint Laplacians. These can be regularized in a manifestly reparametrization invariant way, for example by the zeta function method. The above one-loop formula provides a concrete example: its being unambiguous means that $|\eta(\tau)|^4\tau_2$ is a scalar under $SL(2,Z)$, which is essentially equivalent to the well-known transformation law for the Dedekind eta function.

### Formulation with ghosts

Another way of representing the string measure is to introduce ghost fields $b = b_{zz}dz^2$ and $c = c^z(dz)^{-1}$, with action

$$I_{gh}(b,c) = \frac{1}{2\pi}\int d^2z(b_{zz}\partial_{\bar z}c^z + \bar b_{\bar z\bar z}\partial_z\bar c^{\bar z}) . \tag{2.18}$$

Naively the integral over $b\bar b c\bar c$ of $\exp(-I_{gh})$ should give us the Faddeev-Popov determinant $\det\bar\partial_2^{\dagger}\bar\partial_2$ of (2.16). This would have been the case if the operator $\bar\partial_2$ had no zero mode, and the modes of the fields $b$ and $c$ were in one-to-one correspondence. However there are $3h-3$ $b-$zero modes, and the index theorem asserts that the asymmetry between $b$ and $c$ modes is exactly $3h-3$. Thus to get a non-zero answer, we should insert at least $3h-3$ $b$ terms, and the number of $b$ insertions should always exceed the number of $c$ insertions by $3h-3$. In particular

$$\int D(b\bar b c\bar c)\,e^{-I_{gh}(b,c)}|\prod_{i=1}^{3h-3} < \mu_i|b> |^2 = \det{}'\bar\partial_2^{\dagger}\bar\partial_2\frac{\det < \mu_i|\phi_j > |^2}{\det < \phi_i|\phi_j >} . \tag{2.19}$$

The right hand side of (2.19) is precisely the full gauge-fixing factor in (2.16). Thus thanks to the ghost formalism, the difficulty due to global topology of zero modes for the gauge-fixing operator has provided its own cure: absorbing the zero modes gives the correct string measure. The final answer is very simple

$$Z_{\rm BOS} = \int D(b\bar b c\bar c x^\mu)|\prod_{i=1}^{3h-3} < \mu_i|b> |^2 e^{-(I(g_{mn},x)+I_{gh}(b,c))} . \tag{2.20}$$

One especially powerful feature of the ghost formulation is that it exhibits Becchi-Rouet-Stora-Tyutin or $BRST$ invariance, which provides a check on gauge-fixed amplitudes. It has also proved crucial in certain attempts at a field theory of strings [26], but we shall have no occasion to discuss these aspects here. The infinitesimal parameter for $BRST$ symmetry is an anti-commuting number $\lambda$ , and it acts as follows:

$$
\begin{aligned}
\delta x^\mu &= \lambda c^z \partial_z x^\mu \\
\delta c^z &= -\lambda c^z \nabla_z c^z \\
\delta b_{zz} &= -\lambda \left[ -\frac{1}{2} \partial_z x^\mu \partial_z x^\mu + c^z \nabla_z b_{zz} + 2(\nabla_z c^z) b_{zz} \right] .
\end{aligned} \tag{2.21}
$$

Note that the variation for $b_{zz}$ is just the total stress tensor.

**Conformal anomalies and critical dimension**

A way of motivating the critical dimension $d = 26$ is by requiring that the expression (2.16) be indeed independent of the choice of metric $g_{mn}$ within each conformal class. For this we need the dependence on the conformal factor of the determinants normalized by their zero modes. A heat kernel computation gives

$$
\begin{aligned}
\delta_\sigma \log \frac{\det \bar\partial_n^\dagger \bar\partial_n}{\det < \phi_i^{(n)} | \phi_j^{(n)} > \det < \phi_i^{(1-n)} | \phi_j^{(1-n)} >} &= \\
- \frac{6n^2 - 6n + 1}{6\pi} \int_M d^2 z \sqrt{g} R \delta\sigma - \frac{1}{2\pi\epsilon} \int_M d^2 z \sqrt{g} \delta\sigma
\end{aligned} \tag{2.22}
$$

where $\epsilon$ is a short time cut-off, and $\phi_\alpha^{(n)}$ denotes holomorphic forms of rank $n$ when they exist. If we represent determinants by functional integrals (as for the scalar Laplacian and the ghost determinant above), the actions are Weyl invariant, and this non-trivial dependence on the conformal factor can be attributed to the measures $Dx$ and $D(b\bar{b}c\bar{c})$ . For scalars the coefficient $c_n = 6n^2 - 6n + 1$ is $1$ while it is $13$ for $n = 2$ . The full expression (2.16) is only conformally invariant when there are 26 scalar fields $x^\mu$ , whence the dimension of space-time.

There is a faster way of deriving the coefficient $c^n$ . From Ward identities for reparametrization invariance, it is readily seen that the coefficient of the conformal anomaly $c_n$ must be $\pm 1/2$ the central charge of the Virasoro algebra, depending on whether the fields are commuting or anti-commuting. In other words, if $T_{zz}$ is the stress tensor of the corresponding theory, we must have

$$
T_{zz} T_{\omega\omega} \sim \frac{\pm c_n}{(z - \omega)^4} + \dots
$$

For chiral scalars $x$, the stress tensor is $T_{zz} = -\partial_z x \partial_z x/2$, the propagator $< x(z)x(\omega) >$ has singularity $\sim \log(z-\omega)$, and we find $T_{zz}T_{\omega\omega} \sim \frac{1/2}{(z-\omega)^4}$. This agrees with the previous result, since $x$ can be formally split into its two chiral halves. For general rank $n$ the corresponding fields are $b(dz)^n$ and $c(dz)^{1-n}$, the action is similar to (2.30), and the stress tensor can be worked out to be $T_{zz} = -nb\partial_z c + (1-n)(\partial_z b)c$. Since the propagator $< b(z)c(\omega) >$ behaves as $1/(z-\omega)$ at short distances, a simple calculation by Wick contractions gives $T_{zz}T_{\omega\omega} \sim -(6n^2-6n+1)/(z-\omega)^4$, if $b,c$ are chosen to be anti-commuting as was the case for ghosts. This gives again the coefficient of the conformal anomaly.

**Hyperbolic geometry formulation**

We can obtain a very simple expression for the bosonic string partition function by representing a complex structure by a metric $\hat{g}$ og constant negative curvature $-1$. Quadratic differentials can be given a norm

$$(2.23) \qquad ||\phi||^2 = \int d^2 z \hat{g}^{z\bar{z}} \phi_{zz} \bar{\phi}_{\bar{z}\bar{z}}$$

which provides moduli space with a Kahler metric, called the Weil-Petersson metric. Denoting the corresponding Kahler form by $\omega_{W,P}$, we can rewrite the partition function as

$$(2.24) \qquad Z_{\mathrm{BOS}} = c_h \int_{M_h} (\omega_{WP})^{3h-3} Z(2) Z'(1)^{-13}$$

where for each hyperbolic geometry $Z(s)$ is the corresponding Selberg zeta function

$$(2.25) \qquad Z(s) = \prod_{\text{closed geodesicsl}} \prod_{k=1}^{\infty} (1 - e^{-(k+s)l}) .$$

The reason (2.16) can be put in this form is because the expression

$$\prod d\tau_i d\bar{\tau}_i \frac{|\det < \mu|\phi > |^2}{\det < \phi|\phi >}$$

can be recognized as the coset measure on moduli space viewed as

$$\mathcal{M}_h = \{\hat{g}_{mn} \text{with } \hat{R} = -1\}/\mathrm{Diff}(M)$$

and that determinants of Laplacians on rank $-n$ tensors on hyperbolic surfaces are given by values of Selberg zeta functions at $n+1$, or of their derivatives if there are zero modes. Thus the derivative of $Z(s)$ at 1 is due to the constant zero mode of the laplacian on scalars.

### Algebraic geometry formulation

A formulation which is especially important is the one which exhibits the holomorphic structure of the integrand. Physically this holomorphic structure reflects the independence of the left and right movers that lies at the core of the construction of superstring theories. The geometric fact which allows to state the formula in the most economical way is that the line bundle

$$\kappa \otimes \lambda^{-13} \tag{2.26}$$

over moduli space is holomorphically trivial [19]. Here $\kappa$ is the canonical bundle of moduli space, in other words the highest wedge power of its cotangent bundle. Since a cotangent vector on moduli space is a quadratic differential $\phi_i$, $\kappa$ is the line bundle whose local sections are of the form $\phi_1 \wedge \ldots \wedge \phi_{3h-3}$. Similarly $\lambda$ is the Hodge bundle over moduli space, namely the highest wedge power of the bundle of abelian differentials (=holomorphic $1-$ forms on the surface). Recall that the space of abelian differentials is always of dimension $h$, so that a local section of $\lambda$ is of the form $\omega_1 \wedge \ldots \wedge \omega_h$, where $\omega_I$ is a basis of abelian differentials. We shall shortly give an explanation for the triviality of (2.26). Presently we note that up to a constant independent of the complex structure (but depending on the genus), there is a unique holomorphic nowhere vanishing section $s$ of $\kappa \otimes \lambda^{-13}$. Thus given say, any basis of abelian differentials $\omega_I$, we can select a basis $\phi_i$ of quadratic differentials so that $\wedge\phi_i \otimes (\wedge\omega_I)^{-13} = s$. The desired expression for the bosonic partition function is then the following

$$Z_{\mathrm{BOS}} = c_h \int \phi_1 \wedge \ldots \wedge \phi_{3h-3} \wedge \bar{\phi}_1 \wedge \ldots \wedge \bar{\phi}_{3h-3} \det < \omega_I | \omega_J >^{-13} \tag{2.27}$$

which is independent of the choices of abelian and quadratic differentials. It is often convenient to fix a homology basis $A_I, B_I$ with intersection numbers

$$\begin{aligned} \sharp(A_I, A_J) &= \sharp(B_I, B_J) = 0 \\ \sharp(A_I, B_J) &= \delta_{IJ} \end{aligned} \tag{2.28}$$

and define the abelian differentials $\omega_I$ to be the dual basis to the $A_I$ cycles. The integrals of the abelian differentials around the $B_I$ cycles form then the period matrix $\Omega_{IJ}$, which in fact characterizes the complex structure of the surface:

$$\begin{aligned} \int_{A_I} \omega_J &= \delta_{IJ} \\ \int_{B_I} \omega_J &= \Omega_{IJ} . \end{aligned} \tag{2.29}$$

With this choice the matrix of inner products of the abelian differentials becomes the imaginary part of $\Omega_{IJ}$, and we can rewrite (2.27) as

$$Z_{\rm BOS} = c_h \int \phi_1 \wedge \ldots \phi_{3h-3} \wedge \bar{\phi}_1 \wedge \ldots \wedge \bar{\phi}_{3h-3} (\det Im\Omega)^{-13} . \tag{2.30}$$

The constant depending on the genus can be explicitly derived from its value at $h = 1$ by letting the surfaces degenerate.

We provide now an explanation for this approach. The underlying principle is that in any reasonable sense, the operators $\bar{\partial}$ can be viewed as depending holomorphically on moduli parameters. For example if we let $z$ denote isothermal coordinates for a reference complex structure and its operator $\partial_{\bar{z}}$, the operator corresponding to a deformation by a Beltrami differential $\mu = \mu_{\bar{z}}{}^{z}$ is $\partial_{\bar{z}} - \mu\partial_z$, which depends only on $\mu$ and not its complex conjugate. Furthemore bases of zero modes can be chosen locally to depend holomorphically on moduli parameters. We have already referred implicitly to this latter fact above, in constructing holomorphic local sections of the bundles $\kappa$ and $\lambda$ on moduli space. This suggests that the determinants of $\bar{\partial}$ should also depend holomorphically on moduli. Defining determinants of non self-adjoint operators such as $\bar{\partial}$ is however problematic, and requires some arbitrary choices. In this context, we note that only determinants of Laplacians such as $\bar{\partial}^\dagger\bar{\partial}$ intervene, so that the zeta function provides a determinant which is a scalar, invariant under reparametrizations of the world-sheet. The extraneous data in this case is the metric, by opposition to just the complex structure. It is then natural to ask whether the determinant of the Laplacian is the square of a holomorphic function on moduli space. The answer is provided by the Belavin-Knizhnik theorem, which says that there is a «holomorphic» anomaly, which is very similar to chiral anomalies in gauge theories:

$$\begin{aligned} \delta_{\bar{\mu}}\delta_{\mu} \log \frac{\det \bar{\partial}_n^\dagger \bar{\partial} n}{\det < \phi_i^{(n)} | \phi_j^{(n)} > \det < \phi_i^{(1-n)} | \phi_j^{(1-n)} >} &= \\ - \frac{6n^2 - 6n + 1}{12\pi} \int d^2 z \nabla_{\bar{z}} \mu \nabla_z \bar{\mu} . \end{aligned} \tag{2.31}$$

Here we have demormed the metric to second order by

$$2 g_{z\bar{z}} dz d\bar{z} \rightarrow 2 g_{z\bar{z}} |dz + \mu_{\bar{z}}{}^{z} d\bar{z}|^2 .$$

The upper index $n$ for the holomorphic forms $\phi_i^{(n)}$ refers to their rank. The important property of the holomorphic anomaly is that it has the same coefficient as the conformal anomaly, and the contributions of the determinants on 2 forms and on scalars will again cancel in dimension 26 . This means that the expression

$$\left[\frac{\det \bar{\partial}_2^\dagger \bar{\partial}_2}{\det < \phi_i^{(2)} | \phi_j^{(2)} >}\right] \left[\frac{\det \bar{\partial}_0^\dagger \bar{\partial}_0}{\int d^2 z g_{z\bar{z}} \det < \omega_I | \omega_J >}\right]^{-13} \tag{2.32}$$

is the square of a holomorphic function on moduli space. No global choice of bases $\phi_i$ and $\omega_I$ exist globally, but (2.32) depends only on the combined section $s = (\wedge_i \phi_i) \otimes (\wedge_I \omega_I)^{-13}$ , which does exist globally. Thus (2.32) is actually a constant independent of moduli. Since $\prod d\tau_i d\bar{\tau}_i | \det < \mu_i | \phi_j > |^2$ is just the coordinate expression of $\wedge_\alpha \phi_\alpha \otimes \wedge_\alpha \bar{\phi}_\alpha$ , the formula (2.27) for the bosonic partition function follows from the general gauge fixed formula (2.16).

The triviality of $\kappa \otimes \lambda^{-13}$ was noticed a while ago by Mumford [19]. Actually the Belavin-Knizhnik formula can be viewed as a curvature version of the original characteristic class arguments, which made use rather of the Grothendieck-Riemann-Roch theorem. The proper setting is the determinant line bundles of Atiyah and Singer [20], which had provided a geometric interpretation for chiral anomalies. In this case, associated to each complex structure and each $\bar{\partial}$ operator on rank $n$ tensors is a one-dimensional vector space

$$\left(\bigwedge^{\max} \mathrm{KER}\, \bar{\partial}_n\right)^{-1} \otimes \left(\bigwedge^{\max} \mathrm{KER}\, \bar{\partial}_{1-n}\right)^{-1} . \tag{2.33}$$

Together they form a holomorphic line bundle over moduli space, denoted by $\mathrm{DET}\, \bar{\partial}_n$ . A local section $f$ of this bundle is of the form

$$f = (\wedge_i \phi_i^{(n)} \otimes \wedge_j \phi_j^{(1-n)})^{-1} . \tag{2.34}$$

We can define a metric, the Quillen metric [21], on $\mathrm{DET}\, \bar{\partial}_n$ by

$$||f||_Q^2 = \frac{\det \bar{\partial}_n^\dagger \bar{\partial}_n}{\det < \phi_i^{(n)} | \phi_j^{(n)} > \det < \phi_i^{(1-n)} | \phi_j^{(1-n)} >} . \tag{2.35}$$

The Belavin-Knizhnik formula (2.31) gives then the curvature of $\mathrm{DET}\, \bar{\partial}_n$ with respect to the Quillen metric. Cancellation of holomorphic anomalies means that the tensor products of the corresponding determinant bundles are flat. For the bosonic string in genus $h \geq 2$ the relevant bundles are $\mathrm{DET}\, \bar{\partial}_2 = \kappa$ , and $(\mathrm{DET}\, \bar{\partial}_0)^{-13} = (\lambda \otimes 1)^{-13} = \lambda^{-13}$ . Strictly speaking a complete argument requires cancellation of global anomalies as well. As pointed out by Witten, this means vanishing holonomy. The full treatment of these last issues can be found in [22], and we shall not pursue them further here.

The net outcome is that we have at this point a complete understanding of how to gauge fix and compute scattering amplitudes for the bosonic string. Physically the bosonic string moving in flat euclidian space-time is an ill-behaved model, due to the presence in the spectrum of a tachyon. This particle can be detected by inspecting the degeneration behavior of the partition function. From the point of view of hyperbolic geometry, one can approach the boundary of moduli space by shrinking to zero the lenght of a closed geodesic $l$. Values of the Serlberg zeta function and its derivatives at $s$ will vanish at essentially the exponential rate $e^{-\pi^2/2l}$. Combining this with known asymptotics for the Weil-Petersson measure shows immediately the divergence of the partition function. Alternatively, the line bundle $\kappa \otimes \lambda^{-13}$ is trival over moduli space, but not over the compactified moduli space where one includes surfaces with nodes. In fact denoting the divisor of surfaces with nodes by $\Delta$, one can show that its divisor is $-2\Delta$, so that the string integrand has a pole of order two at the boundary. Clearly these contributions are due to a tachyon.

As an example we can easily derive an explicit expression for the two-loop partition function in terms of theta functions, extending the well-known one-loop formula (2.17) [25]. Rewriting the formula (2.27) as an integral over period matrices $\Omega$ gives

$$Z_{\text{BOS}} = c_h \int |\prod_{I \le J} d\,\Omega_{IJ}|^2 (\det \operatorname{Im} \Omega)^{-13} |\Phi(\Omega)|^{-2} \tag{2.36}$$

where $\Phi(\Omega)$ is a modular form. Since the modular invariant measure is

$$|\prod_{I \le J} d\,\Omega_{IJ}|^2 (\det \operatorname{Im} \Omega)^{-(h+1)}$$

and the determinant of the imaginary part of the period matrix transforms as

$$\det \operatorname{Im} \Omega \to (\det \operatorname{Im} \Omega) |\det(C\Omega + D)|^{-2}$$

under modular transformations $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, modular invariance of the full partition function implies that $\Phi(\Omega)$ be a modular form of weight $12 - h$, that is,

$$\Phi(\Omega) \to \Phi(\Omega)(\det(C\Omega + D))^{12-h}.$$

In genus 2 the weight of $\Phi(\Omega)$ is then 10. Now the ring of modular forms in genus 2 has been completely identified by Igusa [26], as a polynomial ring with generators $\Psi_4, \Psi_6, \Psi_{10}, \Psi_{12}$ of weights $4, 6, 10, 12$. The only candidates for $\Psi(\Omega)$ must be multiples of $\Psi_4 \Psi_6$ or $\Psi_{10}$. To decide which one, we compare boundary behaviors. The components of the boundary of the space of period matrices is given by $\Delta_0 = \{\Omega_{11} \to \infty\}$, corresponding to the case when a handle degenerates, and $\Delta_1 = \{\Omega_{IJ} = 0\}$ corresponding to the surface being separated in two by a zero homology cycle shrinking to zero. Recall that a holomorphic transversal coordinate to the boundary of moduli space is provided by the parameter $t$ of the plumbing fixture $zw = t$ that models the formation of a node. In terms of $t$ the asymptotic form of the entries of the period matrix behave as $\Omega_{11} \sim \log t$ near $\Delta_0$, and $\Omega_{12} \sim t$ near $\Delta_1$. Thus the form $\Psi(\Omega)$ must vanish of first order along $\Delta_0$ and of second order along $\Delta_1$. Since the form $\Psi_4\Psi_6$ does not vanish along $\Delta_1$, this identifies $\Phi(\Omega)$ with $\Psi_{10}$, which is given by

$$\Psi_{10} = \prod_{\delta \text{ even}} \vartheta[\delta](0,\Omega) .$$

Thus for genus 2

$$Z_{\text{BOS}} = c_2 \int |\prod_{I\leq J} d\,\Omega_{IJ}|^2 | \prod_{\delta \text{ even}} \vartheta[\delta](0,\Omega)|^2 (\det \operatorname{Im}\Omega)^{-13} . \tag{2.37}$$

A more detailed analysis for genus $h = 3$ gives a similar answer

$$Z_{\text{BOS}} = c_3 \int |\prod_{I\leq J} d\,\Omega_{IJ}|^2 | \prod_{\delta \text{ even}} \vartheta[\delta](0,\Omega)|^{-1} (\det \operatorname{Im}\Omega)^{-13} . \tag{2.38}$$

## 3. FERMIONIC STRINGS

We begin by introducing the fields necessary to describe propagation of fermionic strings. A guiding symmetry is $N = 1$ world-sheet supersymmetry, which is the supersymmetric partner of reparametrization invariance. Just as reparametrization invariance led to the Einstein-Hilbert action in 4-dimensional space-time, the action of fermionic string propagation will be dictated by reparametrization invariance and supersymmetry. The superpartners of the scalar fields $x^\mu$ will be Majorana spinors $\psi^\mu$, while the superpartner of the world-sheet metric will be a two-dimensional gravitino field $\chi_m^\alpha$. Since these new anti-commuting fields are spinors, and the world-sheet has non-trivial topology, we need to specify a spin structure. On a surface of genus $h$, there are $2^{2h}$ spin structures, corresponding to the sign ambiguities of parallel transport of spinors around the $2h$ cycles of the homology basis. We shall temporarily fix such a spin structure $\delta$. Spinors can now be viewed as forms of half-integral orders, and we can freely use the symbols

$(dz)^{1/2}$ and $(d\bar{z})^{1/2}$. It will be convenient to introduce the following indices for spinors:

$$\psi = \psi_+(dz)^{1/2} \qquad \bar{\psi} = \psi_-(d\bar{z})^{1/2}$$
$$\chi = \chi_{\bar{z}}^+ d\bar{z}(dz)^{-1/2} \quad \bar{\chi} = \chi_z^- dz(d\bar{z})^{-1/2} \tag{3.1}$$

which means that an index $+$ (respectively $-$) is half of an index $z$ (respectively $\bar{z}$). The action for the Type II superstring [26] is the supersymmetric extension of the sigma model action describing propagation of the bosonic string:

$$\begin{aligned} I_\delta = \frac{1}{4\pi}\int d^2z(&\partial_z x^\mu \partial_{\bar{z}} x^\mu - \psi_+^\mu \partial_{\bar{z}} \psi_+^\mu - \psi_-^\mu \partial_z \psi_-^\mu \\ &+ \chi_{\bar{z}}^+ \psi_+^\mu \partial_z x^\mu + \chi_z^- \chi_z^- \psi_-^\mu \partial_{\bar{z}} x^\mu \partial_{\bar{z}} x^\mu - \frac{1}{2}\chi_{\bar{z}}^+ \chi_z^- \psi_+^\mu \psi_-^\mu) . \end{aligned} \tag{3.2}$$

The fact that the action can be written in terms of the complex structure alone and does not involve the component $\chi_{\bar{z}}^-$ of the gravitino means that it is Weyl and super-Weyl invariant. Besides reparametrization invariance, whose infinitesimal generator is a vector field $\delta v^m$, the crucial additional symmetry is supersymmetry, whose infinitesimal generator is a spinor $\zeta$. More explicitly the fields in the theory transform as follows

$$\begin{aligned} \delta x^\mu &= \zeta^+ \psi_+^\mu + \zeta^- \psi_-^\mu \\ \delta \psi_+^\mu &= -\zeta^-(\partial_z x^\mu + \frac{1}{2}\chi_z^- \psi_-^\mu) \\ \delta e_{\bar{z}}{}^z &= -\zeta^+ \chi_{\bar{z}}^+ \\ \delta \chi_{\bar{z}}{}^+ &= -2\nabla_{\bar{z}} \zeta^+ . \end{aligned} \tag{3.3}$$

We turn now to quantization. Before embarking onto a detailed treatment of gauge fixing, it is useful to recall the principles underlying string scattering amplitudes.

(A) CHIRAL SPLITTING. In Minkowski signature for the world-sheet, the world-sheet spinors $\psi_+^\mu$ and $\psi_-^\mu$ are independent Majorana-Weyl fermions. In two dimensions these do not exist in Euclidan signature. The way around this in Euclidian signature is to view $\psi_+^\mu$ and $\psi_-^\mu$ as complex conjugate components of a Majorana fermion, and try to separate in the functional integrals what can be viewed as their respective contributions. They should correspond to factors which are respectively holomorphic and anti-holomorphic in external parameters such as moduli parameters, polarization tensors, and insertion points of vertex operators. For example, in the bosonic string, the fields $x^\mu$ are real and do not admit an evident decomposition into left and right movers in the functional formalism. Nevertheless, in the gauge-fixed form (2.30), the integrand for the bosonic partition function has effectively been split into holomorphic and anti-holomorphic factors, up to the harmless term $(\det \operatorname{Im} \Omega)^{13}$ . The expectation then is that something similar should take place for the superstring. In mathematical terms, the problem here is to factor into holomorphic and anti-holomorphic factors complicated expressions built out of determinants and Green's functions of Laplacians acting on tensors and spinors of various weights.

(B) GSO PROJECTION. Chiral splitting as prescribed above is done with an arbitrary fixed spin structure $\delta$ . The factors one gets at this stage are not yet of physical interest. Physical string scattering amplitudes are obtained by pairing a holomorphic factor of one spin structure with an antiholomorphic one of another independent spin structure, and sum over all possible choices of spin structures. The physical reason behind this prescription can be easily understood in the case of a torus. Let a torus be represented as a parallelogram in $(\sigma, \tau)$ space. Recall that the Ramond sector of the string corresponds to $\psi$ which are periodic in $\sigma$ , while the Neveu-Schwarz sector corresponds to the anti-periodic $\psi$'s . In functional integrals integrating with respect to fields periodic with respect to time $\tau$ gives the trace of $(-1)^F e^{-\beta H}$ . Anti-periodic fields produce instead the trace of $e^{-\beta H}$ . Thus to project on states of even chirality requires summing over both types of time boundary conditions. Altogether the contributions of the Ramond and the Neveu-Schwarz sectors are obtained by summing over all 4 spin structures for the torus [27].

(C) SUPERSTRING MEASURE. Closely related with the splitting into left and right movers required by the GSO projection is the problem of gauge-fixing and reducing the evaluation of superstring scattering amplitudes so that of finite-dimensional integrals. In the absence of anomalies, gauge-fixing should lead to integrals over the space of supergeometries on a surface of genus $h$ , namely supermoduli space:

$$(3.4) \qquad s\mathcal{M}_h = \{(g_{mn}, \chi_{\bar{z}}^{+})\}/\mathrm{Weyl} \times s\mathrm{Weyl} \times \mathrm{Diff}(M) \times \mathrm{SUSY} \ .$$

Gauge-fixing in the superstring case is more difficult that in the case of the bosonic string, if only because in this formalism we do not known of manifestly supersymmetry invariant measures on spaces of various tensor and spinor fields. This can be avoided by working in the superfield formalism, where the concept of supergeometry can be given many equivalent reformulations which let us follow more closely the methods of the theory of Riemann surfaces. Once the correct gauge-fixing procedures are available, we can analyze the corresponding expressions for superstring scattering amplitudes directly as integrals on supermoduli space, or by reducing them to integrals over moduli space by integrating out the odd variables in supermoduli space. In the latter case, one may hope to rely then on the theory of modular forms. This reduction will have to be done with care, as we shall explain in Section 8. In the other case, it would of course be necessary to develop further function theory over supermoduli space, which would be interesting in its own right. It is likely that ultimately the superspace formulation throughout will provide us with the most economical way for evaluating scattering amplitudes. However it is well-recognized to be especially treacherous, and should certainly be checked against componentwise prescriptions.

## 4. SUPER RIEMANN SURFACES

Before proceeding with superstrings proper, we pause to discuss the notion of supergeometry and super Riemann surfaces in some greater detail.

### Supermanifolds in the sense of DeWitt

In field theory, fermions are represented by anti-commuting fields $\psi$. More precisely let $C$ be the usual space of complex numbers, and let $\zeta_a, a = 1, 2, \ldots, N$ be a set of Grassmann generators satisfying

$$\{\zeta_a, \zeta_b\} = \zeta_a\zeta_b + \zeta_b\zeta_a = 0 \ . \tag{4.1}$$

In practice $N$ is so large that it can be viewed as infinite. We can build a graded Grassmann algebra by considering expressions of the form

$$x = x_B + x_S \tag{4.2}$$

where the «body» part of $x$ is an element of $C$, and the soul part $x_S$ is of the form

$$x_S = \sum_{n=1}^{\infty} \frac{x_{a_1,\ldots,a_n}}{n!}\zeta^{a_1}\ldots\zeta^{a_n} \tag{4.3}$$

with the coefficients being standard complex numbers. A grading into even and odd elements is obtained by viewing $x$ as the sum of terms with an even number of generators with terms with an odd number of generators. In particular it should be pointed out that the even part in general will contain soul terms in addition to the body. The space of all even elements is usually denoted by $C_c$, and the space of odd elements by $C_a$, where the subscripts $c$ and $a$ stand for commuting and anti-commuting. Their elements will be referred to respectively as $c-$ numbers and $a-$ numbers.

We can now discuss functions of these generalized numbers, taking values also in generalized numbers. Henceforth $z$ will denote a $c-$ number, while $\theta$ will denote an $a-$ number. A function $f(z,\theta)$ can be expanded as

$$f(z,\theta) = f_0(z) + f_+(z)\theta \tag{4.4}$$

with coefficients functions of $c-$ numbers alone. These in turn can be viewed as characterized by their restrictions to the body part, since we can expand in a Taylor series in the soul par. To integrate we need a notion of measure $dzd\theta$. The measure $dz$ on functions of $c-$ numbers should be viewed as a line integral measure on a one-dimensional path within the space of $c-$ numbers $C$. It depends only on the end points and is independent of the path chosen. In particular the symbol

$$\int \mathrm{d}\, z g(z)$$

stands for integration along any path having the same end points as $C$. In evaluating these integrals we can thus restrict the integrand to the body part and carry out the integration as with regular numbers. The integration with respect to $a-$ numbers on the other hand requires rather different rules. We need specify them only on the functions 1 and $\theta$

$$\begin{aligned} &\int \mathrm{d}\,\theta 1 = 0 \\ &\int \mathrm{d}\,\theta\theta = 1 \,. \end{aligned} \tag{4.5}$$

A (complex) supermanifold is described by local coordinate patches $(z_\alpha, \theta_\alpha)$ which transform by regular functions on their overlaps:

$$\begin{aligned} z_\alpha &= z_\alpha(z_\beta, \theta_\beta) \\ \theta_\alpha &= \theta_\alpha(z_\beta, \theta_\beta) \,. \end{aligned} \tag{5.6}$$

The transformation is regular in the sense that the superdeterminant

$$\det \begin{pmatrix} \partial_{z_\alpha} z_\beta & \partial_{z_\alpha} \theta^\beta \\ \partial_{\theta_\alpha} z_\beta & \partial_{\theta_\alpha} \theta_\beta \end{pmatrix}$$

is never 0 . A comprehensive treatment is in [28].

### Super Riemann Surfaces from transition functions

The notion of super Riemann surface is more restrictive than that of a one-dimensional complex supermanifold. To motivate it, we consider the model case of the one-dimensional supermanifold $C^{1|1} = C_c \times C_a$. The key new ingredient is a square root of the $\partial_z$ operator

$$D_\theta = \partial_\theta + \theta \partial_z \quad (4.7)$$
$$D_\theta^2 = \partial_z .$$

A complex supermanifold becomes a super Riemann surface if the operators $D_\theta$ in each coordinate patch transform homogeneously into one another

$$D_{\theta_\alpha} = (D_{\theta_\alpha} \theta^\beta) D_{\theta_\beta} . \quad (4.8)$$

This means that the new coordinates $(z_\beta, \theta_\beta)$ must satisfy

$$D_{\theta_\alpha} z^\beta = \theta^\beta D_{\theta_\alpha} \theta^\beta . \quad (4.9)$$

From the geometric point of view, the homogeneity condition (4.8) is equivalent to the global existence of a non-vanishing section

$$\mathrm{d} z \mathrm{d} \theta \otimes D_\theta \quad (4.10)$$

of a line bundle $\hat{\kappa}$ which will be called the canonical line bundle of the super Riemann surface. See [29] [35].

### Super Riemann surfaces from supergravity

The definition of a super Riemann surface we just gave is perhaps the most economical, and potentially most suitable for the future developent of super algebraic geometry. To make contact with string theory however, it is crucial to make use of the definition of super Riemann surfaces arising from two-dimensional supergravity. From this point of view a super Riemann surface is characterized by a metric and gravitino multiplet $(g_{mn}, \chi_{\bar{z}}^{+})$, exactly as the fields appearing in the supersymmetric sigma model (3.2). To unify them in a superfield formalism, we consider a (2|2) real supermanifold of coordinates $(\xi^1, \theta^1, \xi^2, \theta^2)$ on which is defined a superzweibein (or superframe) $E^A = \mathrm{d} z^M E_M{}^A$ and a connection $\Omega = \mathrm{d} z^M \Omega_M$. We recall the usual convention of denoting Lorentz indices by early indices (such as $a, A, \alpha$, etc.), Einstein indices by middle indices (such as $m, M, \mu$), commuting coordinates by Latin indices, anti-commuting coordinates by Greek indices. If we define covariant derivatives on rank $n$ Lorentz tensors by

$$\mathcal{D} = \partial_M + in\Omega_M \quad (4.11)$$

or in terms of Lorentz indices by

(4.12) $$\mathcal{D}_A = E_M{}^A \mathcal{D}_M$$

and will satisfy the commutation relations

(4.13) $$\{\mathcal{D}_A, \mathcal{D}_B\} = T_{AB}{}^C D_C + inR_{AB}$$

where $T^C_{AB}$ and $R_{AB}$ are respectively the torsion and curvature tensors. The brackets $\{,\}$ are regular commutator brackets unless both $A$ and $B$ are spinor indices, in which case it becomes the anti-commutator. In supergeometry the torsion is not zero. Rather it is defined by the Wess-Zumino torsion constraints

(4.14) $$T^c_{ab} = T^\gamma_{\alpha\beta} = 0$$
$$T^c_{\alpha\beta} = 2(\gamma^c)_{\alpha\beta} \,.$$

Here $\gamma^a$ are two-dimensional Dirac gamma matrices

$$\{\gamma^a, \gamma^b\} = -\delta_{ab} \,.$$

One way of motivating these constraints is the following. Choose Dirac matrices of the form

$$(\gamma^z)_{++} = (\gamma^{\bar z})_{--} = -(\gamma^z)_+{}^- = (\gamma^{\bar z})_-{}^+ = 1$$
$$(\gamma^a)_{\alpha\beta} = 0 \,, \alpha \neq \beta \,.$$

On scalars, there is no curvature term in (4.13), and the (anti) commutation relation between $\mathcal{D}_+$ and itself just says that the square of $\mathcal{D}_+$ is the operator $\mathcal{D}_z$. Another way of motivating the Wess-Zumino torsion constraints is through the existence of a complex structure [30]. The almost-complex structure

(4.15) $$J_M{}^C = E_M{}^a \epsilon_a{}^b E_b{}^N + E_M{}^\alpha C(\gamma_5)_\alpha{}^\beta E_\beta{}^N$$

is integrable (so that complex coordinates can be defined in patches) provided the torsion constraints

(4.16) $$T_{++}{}^- = T_{++}{}^{\bar z} = T_{+z}{}^- = T_{+z}{}^{\bar z} = 0$$

as well as their complex conjugates are obeyed. Conversely if we have a super surface equipped with a tensor $J_M{}^N$ satisfying the integrability constraints, then there is always a choice of supergeometry $(E_M{}^A, \Omega_M)$ that satisfies the full torsion constraint equations. This can be see as follows: when $J$ is expressed in terms or real coordinates on the supersurface, $J$ is a real tensor, with $J^2 = -1$. Hence it has eigenvalues $i$ and $-i$ each with multiplicity 2. Thus $J_M{}^N$ determines the eigenvectors $E_M{}^+, E_M{}^+$ with eigenvalue $i$ and $E_M{}^-, E_M{}^{\bar{z}}$ with eigenvalue $-i$, up to transformations within the eigenspaces of $i$ and $-i$ respectively: $Gl(1|1) \otimes Gl(1|1)$. For each choice however, (4.15) reproduces the original $J$, and furthermore the torsion constraints (4.16) will automatically be obeyed because the complex structure is integrable. Now we are interested is constructing local $U(1)$ and super-Weyl invariant supergeometries, so those rescalings and rotations in $Gl(1|1) \otimes Gl(1|1)$ should actually correspond to symmetries, and cannot be fixed. Hence $J_M{}^N$ determines $E_M{}^A$ up to

$$\frac{Gl(1|1) \otimes Gl(1|1)}{U(1) \otimes Gl(1)}$$

which has dimension $2|4$, precisely allowing us to pick the remaining torsion constraints in (4.14). Actually, the torsion costraint $T_{+z}{}^{\bar{z}} = 0$ of (4.116) does not figure in (4.14). Rather from (4.14) and the Bianchi identities, one obtains further constraints $T_{\alpha b}{}^c = 0$, which $T_{+z}{}^{\bar{z}} = 0$ is part of. In short, to every supercomplex structure $J$ can always be associated a supergeometry satisfying the torsion costraints (4.13). A more extended study of these issues is in [31]. It should be pointed out that the Wess-Zumino torsion constraints are hyghly non-trivial, in the sense that given any superzweibein, it is not always possible to solve for a connection satisfying these constraints. In fact it suffices to count the number of degrees of freedom: 16 for $E_M{}^A$, 4 for $\Omega_M$, and 14 for the number of constraints in (4.14). This means that the 8 components of $E_+{}^M, E_-{}^M$ cannot be chosen arbitrarily. In particular supergeometries form a complicated subvariety of the space of superzweibeins and connections.

It is instructive to look at flat superspace $C^{1|1}$ from this point of view. There the superzweibein is given by

$$\begin{aligned} E_m{}^a &= \delta_m^a \\ E_m{}^\alpha &= 0 \\ E_\mu{}^a &= (\gamma^a)_\mu^\beta \theta_\beta \\ E_\mu{}^\alpha &= \delta_\mu^\alpha \end{aligned} \tag{4.16}$$

and the covariant superderivatives $\mathcal{D}_+, \mathcal{D}_-$ take the simple form (4.7).

The above is a natural extension of Riemannian geometry to the superspace setting. We still need to make contact with the earlier component formalism ($g_{mn}$, $\chi_{\bar{z}}^+$). This is achieved by going to Wess-Zumino gauge [32]. The idea behind this is that the superzweibein formalism is invariant under a very large group of symmetries, and that many fields in that theory are actually auxiliary fields. To eliminate them, we begin by making a combination of superreparametrizations, super Weyl and super Lorentz transformations so that the Taylor expansions of some of the superzweibeins components coincide with the flat case to first order

$$E_\mu{}^\alpha = \delta_\mu^\alpha + \theta^\nu e^{*\alpha}_{\nu\mu} + \dots \qquad E_\mu{}^a = \theta^\nu e^{**a}_{\nu\mu} + \dots . \tag{4.17}$$

We choose the $e^*, e^{**}$ to have the symmetries as the Dirac gamma matrices. In this gauge we expand the other components of the superweibeins. It can be shown that the expansion for $E_m^a$ must be of the form

$$E_m^a = e_m^a + \theta^\alpha (\gamma^a)_\alpha^\beta \chi_{m\beta} - i\theta\bar{\theta} e A/2 \tag{4.18}$$

where $A$ is a scalar auxiliary field. All other terms in the components $E_m^\alpha, E_\mu^a, E_\mu^\alpha$ can be expressed in function of $e_m^a, \chi_{\bar{z}}^+$ as well. Thus ignoring the auxiliarity field $A$ which has no dynamics, we have a correspondence between supergeometry in the superspace formalism which is described by $E_M{}^A, \Omega_M$, and supergeometryu in the component (Wess-Zumino gauge) formalism, which is described by the multiplet $(g_{mn}, \chi_{\bar{z}}^+)$. The symmetries of the theory in the component formalism can be identified as those of the superspace formalism which perserve Wess-Zumino gauge. Now the infinitesimal generator of superreparametrizations is a supervector field $V^M = (V^m(z,\bar{z},\theta,\bar{\theta}), V^\mu(z,\bar{z},\theta,\bar{\theta}))$. Expanding in $\theta, \bar{\theta}$ gives

$$V^m = v^m + \dots \qquad V^\mu = \zeta^\mu + \dots \tag{4.19}$$

where the dots stand for additional terms involving the $\theta, \bar{\theta}$ variables. In general these terms are of course independent of the leading terms $v^m$ and $\zeta^\mu$. However the requirement that the superreparametrization (4.19) fix Wess-Zumino gauge actually constrains all the additional terms, so that in the component formalism the infinitesimal generators of superreparametrization reduce to just a vector field $v^m$ and a spinor $\zeta^\mu$. The corresponding symmetries acting on the multiplet $(g_{mn}, \chi_{\bar{z}}^+)$ are precisely reparametrization invariance, and $N = 1$ supersymmetry listed in (3.3).

It is now not difficult to see that every supergeometry is super-conformally flat locally, in the sense that after local superreparametrizations, superLorentz, and superWeyl transformations, it can be brought to the flat form of (4.16). Starting from Wess-Zumino gauge, it suffices to find a supersymmetry transformation (3.3) so that $\chi_{\bar{z}^+}$ is 0 . After this a standard Weyl scaling will do the job. In view of (3.3), this means that we need to find a spinor $\chi$ so that

$$\chi_{\bar{z}}{}^{+} = -2\,\partial_{\bar{z}}\zeta^{+} \; . \tag{4.20}$$

This $\bar{\partial}$ equation can always be solved locally, which establishes our claim. In the case of Riemann surfaces, any metric is equivalent by local reparametrization and Weyl scalings to flat Euclidan space, but not so globally because of topological obstructions. Similarly the equation (4.20) cannot always be solved globally. We can determine the codimension of the $\bar{\partial}$ operator on spinors using the index theorem and the dimension of conformal Killing spinors, exactly as we determined the dimension of moduli space earlier (2.12). The outcome is that in addition to the usual (even) moduli parameters for metrics, we also need odd parameters for the gravitino field $\chi_{\bar{z}}{}^{+}$ . The precise count is

$$\dim s\mathcal{M}_h = (3h-3\,|\,2h-2) \tag{4.21}$$

where we recall that supermoduli space is defined to be the inequivalent classes of supergeometries as in (3.4). So far we have of course determined only the dimension of supermoduli space. We shall return to a more precise construction of $s\mathcal{M}_h$ as a fibration later.

**Supermanifolds from sheaves of graded algebras**

We conclude this section with a brief comparison of the approach to super Riemann surfaces adopted here to the more algebraic approach of many other authors [33]. There are two main features which are different: first, in accord with the algebraic point of view a supermanifold is identified by its graded algebra of functions rather that as a space of points; second, the $x$ variables are actually real numbers and the anti-commuting variables are constructed explicitly along, instead of taking values in an auxiliary infinite-dimensional Grassmann algebra as in the DeWitt definition.

In the algebraic point of view, a supermanifold $M^{n|m}$ is given by a standard manifold $M^n$ of dimension $n$, and by a (pre)sheaf of $\mathbf{Z}_2$ – graded algebras $\mathcal{A}$ with $m$ odd generators. We do not distinguish between sheaves for which there is an isomorphism of algebras preserving the grading which commutes with the restriction maps of the persheaf. An example of supermanifolds is given by the sheaf of sections of the full Grassmann algebra of a vector bundle over a manifold $M$ . In this case we even have a more refined $\mathbf{Z}_+$ grading, which gives the $\mathbf{Z}_2$ grading by decompostion into even and odd terms. In Berezin's terminology [33], supermanifolds constructed in this way from vector bundles are the «simple» supermanifolds. From the more general supermanifold $M^{(n|m)}$, we can also construct a vector bundle $N'(M)$. This is accomplished by first considering quotients $M_k = \mathcal{A}|\mathcal{I}^{k+1}$, where $\mathcal{I}^k$ is the $k-$ th power of the ideal generated by the odd generators. The vector bundle $N'(M)$ we were referring to is given by $M_1$ fibering over $M_0$, i.e., by the sheaf $\mathcal{I}/\mathcal{I}^2$. Evidently we have inclusion maps $M_0 \to M_1 \to M_2 \ldots$ Obstructions to existence of projections the other way around are cohomology classes in $H^1(M_0, T \otimes \wedge^{2k} N'(M))$. Thus a supermanifold is simple if the obstructions vanish. When they exist the various projections are parametrized by $H^0(M_0, T \otimes \wedge^{2k} N'(M))$. Another more practical way of determining whether a supermanifold is simple is to see whether one can construct a $\mathbf{Z}_+$ grading of locally free shaves. It is not difficult to give examples of supermanifolds which are not simple. One the other hand in the $C^\infty$ category, supermanifolds are all simple because of existence of partitions of unity.

This approach is especially precise and elegant. However it is not yet appropriate for the supergravity formalism we have relied on so far, because all functions of $x$ are still commuting unless they explicitly involve the odd generators (compare with, say, the expansion (6.1) below for scalar superfields, where the coefficients $\psi_\pm^\mu(x)$ are anti-commuting after having expanded in the odd generators $\theta, \bar\theta$). It is necessary then to introduce an auxiliary. Grassmann algebra with an inductively infinite number of generators, as well as consider families of supermanifolds in the algebraic sense, in order to arrive at the concepts we need here. We shall leave this task to the future, and henceforth restrict ourselves to supermanifolds in the DeWitt formulation.

## 5. HOLOMORPHIC STRUCTURE OF SUPERMODULI SPACE

In this section we shall show that supermoduli space has a complex structure, with respect to which the operators $\mathcal{D}_-$ vary holomorphically. Thus the same question arises for the superstring that had been an wered for the bosonic string: whether there are superholomorphic anomalies and whether they cancel. We shall evaluate the superholomorphic anomalies explicitly and show that they cancel in the critical dimension $d = 10$ for the superstring and heterotic string with rank 16 gauge groups [30] [3].

We begin by introduce the complex structure. Infinitesimal deformations of supergeometries can be parametrized by $\delta E_M^A$. Since it is more convenient to work with Lorentz indices, we introduce instead

$$H_A{}^B = E_A{}^M \delta E_M{}^B . \tag{5.1}$$

Because of the constraints, not all components of $H_A{}^B$ are independent. In fact they can all be solved in terms of $H_-{}^-, H_-{}^{\bar z}, H_-{}^z$ and their conjugates. Similarly the variation $\delta\Omega_A$ of the connection can also be solved for. We can set to 0 any component that can be removed by Weil, super Weyl, and super Lorentz gauge transformations. This is in particular the case for the components $H_-, H_-{}^{\bar z}$ themselves. Thus deformations of super Riemann surfaces structures can be parametrized by $H_-{}^z, H_+{}^{\bar z}$ alone. In Wess-Zumino gauge, $H_-{}^z$ can be expanded as

$$H_-{}^z = \bar\theta(e_{\bar z}{}^m \delta e_m{}^z - \theta\delta\chi_{\bar z}{}^+) \tag{5.2}$$

which shows clearly that a variation of supergeometry does correspond to a variation of conformal structure $g_{mn}$ and of gravitino field $\chi_{\bar z}{}^+$. If we denote $H_-{}^z$ by $H$, and its conjugate $H_+{}^{\bar z}$ by $\bar H$, the complete set of components $H_A^B$ is given by

$$\begin{aligned} & H_-{}^- = H_-{}^{\bar z} = H_{\bar z}{}^{\bar z} = 0 \\ & H_{\bar z}{}^z = \mathcal{D}_- H, H_-{}^+ = -\mathcal{D}_+ H/2 \\ & H_{\bar z}{}^+ = -\frac{1}{2}\mathcal{D}_-\mathcal{D}_+ H - \frac{i}{2} R_{+-} H \\ & H_{\bar z}{}^- = -\frac{1}{2}\mathcal{D}_-^2 H \\ & \delta\Omega_- = i\mathcal{D}_z H + \frac{1}{2}\mathcal{D}_+ H\Omega_+ - H\Omega_z \end{aligned} \tag{5.3}$$

with conjugate formulas for $H_+{}^+, H_+{}^z, H_z{}^z, H_z{}^{\bar z}, H_z{}^-, H_z{}^+, \delta\Omega_+$. The key feature of the above expressions is that they involve only $H$ and not $\bar H$. This means that the variations of the supergeometry $\delta E_+{}^M$ and of the superderivative

$$\delta\mathcal{D}_- = \delta E_-{}^M \mathcal{D}_M + \mathrm{in}\,\delta\Omega_- \tag{5.4}$$

depends also only on the parameter $H$. The tangent space to supermoduli space is then seen to split into two conjugate complements. To first order $H$ can serve as holomorphic coordinates for supermoduli space near each supergeometry. This is the holomorphic structure that we were looking for.

To determine whether we can quantize and preserve the holomorphic dependence on supermoduli parameters, we have to examine whether the logarithm of the determinants of the corresponding Laplacians are pluriharmonic. For this we need variations of supergeometries and super derivatives to second order. Unlike the case for moduli space where these can be chosen essentially arbitrarily (such choices are just choices of conformal gauges), these have to be solved for due to the more restrictive nature of the constraints of supergeometries. In the remainder of this section we shall denote by $E_A{}^M$ the reference supergeometry, and by $\hat{E}_A{}^M$ the new supergeometry obtained from the deformation $H_-^z$. The result is (ignoring terms of the form $\bar{H}^2$ and $H^2$ which do not affect the pluriharmonicity)

$$\begin{aligned}
\hat{E}_-{}^M &= E_-{}^M + \frac{1}{2}(\mathcal{D}_+ H + \mathcal{D}_- M) E_+{}^M - H E_z{}^M \\
\hat{E}_{\bar{z}}{}^M &= E_{\bar{z}}{}^M - \mathcal{D}_- H E_z{}^M + \frac{1}{2}(\mathcal{D}_-\mathcal{D}_+ H + \mathcal{D}_{\bar{z}} M) E_+{}^M \\
&\quad + \frac{1}{2}(\mathcal{D}_z H + \mathcal{D}_- K) E_-{}^M \\
\delta\Omega_- &= i\mathcal{D}_z H + i\mathcal{D}_- K + \frac{1}{2}\mathcal{D}_+ H \Omega_+ - H\Omega_z \\
&\quad - \frac{1}{2} R_{-+} M + \frac{1}{2}\mathcal{D}_- M \Omega_+
\end{aligned} \tag{5.5}$$

where the new terms $K$ and $M$ mix both $H$ and $\bar{H}$, and are given by

$$\begin{aligned}
M &= \bar{H}\mathcal{D}_- H - \frac{1}{2}\mathcal{D}_-\bar{H} H \\
K &= \frac{1}{2}\mathcal{D}_+\mathcal{D}_-\bar{H} H + \bar{H}\mathcal{D}_+\mathcal{D}_- H - \frac{3i}{2} R_{+-}\bar{H} H\,.
\end{aligned} \tag{5.6}$$

This dependence on both $H$ and $\bar{H}$ at the second order requires explanation. For this we need a seemingly more abstract definition of the complex structure on supermoduli space. Recall that associated to each supergeometry $E_M{}^A$ is a super Weyl invariant tensor $J_M{}^N$ given by (4.15). Thus supermoduli space $s\mathcal{M}_h$ can also be described as

$$s\mathcal{M}_h = \frac{\{J_M{}^N, J_M{}^P J_P{}^N = -\delta_M{}^N\}}{s\mathrm{Diff}(M)}\,. \tag{5.7}$$

This implies that the tangent space to $s\mathcal{M}_h$ at each $J_M{}^N$ can be viewed as the space of tensors $\delta J_M{}^N$ satisfying

$$T_{(J_M{}^N)}(s\mathcal{M}_h) = \{\delta J_M{}^P J_P{}^N + J_M{}^P \delta J_P{}^N = 0\}\,. \tag{5.8}$$

On this tangent space we can define an operator $\mathcal{I}$ whose square is $-1$

$$\begin{aligned}
&\mathcal{I} : T(s\mathcal{M}) \to T(\mathcal{M}) \\
&\mathcal{I}(\delta J_M{}^N) = J_M{}^P \delta J_P{}^N\,.
\end{aligned} \tag{5.9}$$

This almost-complex structure on supermoduli space is integrable, since the one-forms

$$\Gamma_M{}^N = dJ_M{}^N - i\mathcal{I}(dJ_M{}^N)$$

satisfy

$$(5.10)\qquad \begin{aligned} d\Gamma_M{}^N &= \frac{i}{4}(\Gamma_M{}^P \wedge \bar{\Gamma}_P{}^N + \bar{\Gamma}_M{}^P \wedge \Gamma_P{}^N) \\ &= 0 \text{ modulo } \Gamma\ . \end{aligned}$$

Superholomorphic functions can be defined as functions satisfying

$$(5.11)\qquad \mathcal{I}\partial f = i\partial f\,.$$

It follows that the components $J_-{}^z, J_-{}^+, J_{\bar{z}}{}^z, J_{\bar{z}}{}^+$ viewed as functions on supermoduli space are holomorphic to second order. If we evaluate them in terms of $H$ and $\bar{H}$, we find

$$(5.12)\qquad \begin{aligned} J_-{}^z &= 2iH \\ J_-{}^+ &= -i(\mathcal{D}_+H + \mathcal{D}_-M) \\ J_{\bar{z}}{}^z &= \mathcal{D}_-J_-{}^z \\ J_{\bar{z}}{}^+ &= \mathcal{D}_-J_-{}^+\,. \end{aligned}$$

Thus the equations (6.3) (6.4) really say that

$$(5.13)\qquad \mathcal{E}_-{}^M = E_-{}^M + \frac{1}{2}iJ_-{}^+E_+{}^M + \frac{1}{2}iJ_-{}^zE_z{}^M$$

and the variations (5.5) (5.6) of supergeometries are indeed holomorphic to second order. It remains to discuss the connection and superderivative. Rearranging the expression for $\delta\Omega$ in (5.5) gives

$$(5.14)\qquad \delta\Omega_- = -\mathcal{D}_+J_-{}^+ + \frac{1}{2}iJ_-{}^+\Omega_+ + \frac{1}{2}iJ_-{}^z\Omega_z + i\mathcal{D}_-(K + \mathcal{D}_+M)$$

which means that the new superderivative $\bar{\mathcal{D}}_-$ can be written

$$(5.15)\qquad \begin{aligned} \bar{\mathcal{D}}_- &= \mathcal{D}_- + \frac{1}{2}iJ_-{}^+\mathcal{D}_+ + \frac{1}{2}J_-{}^z\mathcal{D}_z \\ &\quad - \mathrm{i}n\mathcal{D}_+J_-{}^+ - n\mathcal{D}_-(K + \mathcal{D}_+M)\,. \end{aligned}$$

By Weyl and $U(1)$ local invariance, we may conjugate with $\exp n(K + \mathcal{D}_-M)$ and obtain a holomorphic dependence. This last observation has its analogue in the bosonic case, where it is the operators $\bar{\partial}_n$ which are holomorphic, and not the covariant derivatives $\nabla_{\bar{z}}$ and $D_{\bar{z}}$ on Einstein and Lorentz tensors. Thus (5.5) (5.6) (5.13) (5.15) exhibit variations of supergeometries and superderivatives which are all holomorphic to second order. They can then be used to evaluate the super analogue of the holomorphic anomaly.

By a super heat kernel regularization, we obtain the following formula

$$\delta_H \delta_{\bar{H}} \log \frac{s\det' \mathcal{D}_+^{(n-1/2)} \mathcal{D}_-^{(n)}}{s\det < \Phi_I | \Phi_J > s\det < \Psi_I | \Psi_J >}$$

$$(5.16) \qquad = \frac{1-4n}{2\pi}(-1)^{2n} \int \mathrm{d}\mathbf{z}\, s\det E \left[ -\frac{1}{2} R^2 H \bar{H} + \mathcal{D}_z H \mathcal{D}_{\bar{z}} \bar{H} + \frac{1}{2} i R (4 \mathcal{D}_+ \bar{H} \mathcal{D}_- H + H \mathcal{D}_+ \mathcal{D}_- \bar{H} + \bar{H} \mathcal{D}_- \mathcal{D}_+ H \right]$$

where $\Phi_I$ and $\Psi_J$ are the zero modes of $\mathcal{D}_-^{(n)}$ and $\mathcal{D}_+^{(n-1/2)}$ respectively.

## 6. FERMIONIC STRINGS IN SUPERFIELD FORMALISM

We have already seen how the component version $(g_{mn}, \chi_{\bar{z}}^+)$ of a supergeometry can be given a succint explanation in superfield language. To complete our translation of component into superfield formalisms and vice versa, we still need a superfield version of the matter fields $x^\mu, \psi^\mu$ and of the supersymmetric sigma model action (3.2). For this introduce scalar superfields $X^\mu(z, \theta, \bar{\theta})$. Expanding them in $\theta, \bar{\theta}$ gives

$$(6.1) \qquad X^\mu = x^\mu + \theta \psi_+^\mu + \bar{\theta} \psi_-^\mu + \theta \bar{\theta} F^\mu .$$

Except for the auxiliary fields $F^\mu$ which will have no dynamics, the coefficients of the superfield once again reproduce the fields of the component formalism. The action we need is just the naive generalization of the sigma model action (2.3)

$$(6.2) \qquad I(E_M{}^A, X^\mu) = \frac{1}{8\pi} \int \mathrm{d}^2\mathbf{z}\, (s\det E_M{}^A) \mathcal{D}_- X^\mu \mathcal{D}_+ X^\mu .$$

This actually coincides with the component version of the action (3.2). In Wess-Zumino gauge, the superdeterminant $s\det E_M{}^A$ can be evaluated to be

$$(6.3) \qquad s\det E_M{}^A = e \left[ 1 + \frac{1}{8} \theta \bar{\theta} \epsilon^{mn} \chi_m \gamma_5 \chi_n \right]$$

where we have denoted by $e$ the determinant of the zweibein $e_m{}^a$, ignored all auxiliary fields, and taken the gravitino $\chi_m{}^\alpha$ to be traceless. The covariant derivative are given by

$$\mathcal{D}\_X = \chi_z^- + \bar\theta\left(\partial_{\bar z}x + \frac{1}{2}\chi_{\bar z}^+\psi_+^\mu\right) + \theta\bar\theta\left(-\frac{1}{4}\chi_{\bar z}^+\chi_z^-\psi_-^\mu - \frac{1}{2}\chi_{\bar z}^+\partial_z x - \partial_{\bar z}\psi_+^\mu\right) \tag{6.4}$$

where we have again ignored auxiliary fields. Integrating out the odd variables $\theta, \bar\theta$ according to fermionic integration rules (4.5) gives at once the desired answer.

In the superspace formalism, the gauge-fixing procedures for the supersymmetric sigma model action (3.2) are essentially the same as those developed for the bosonic string. We shall write down the versions which are the analogues of the general gauge-fixed formula (2.32), and of the formulation with ghosts.

### General gauge fixed-formula

Let $\hat S$ be a slice of $(3h-3|2h-2)$ dimensions in the space of supergeometries, transversal to the orbits of the guage groups. Let $(m_J) = (\tau_j, \zeta_a), J = (j,a), j = 1,\ldots,3h-3, a = 1,\ldots,2h-2$ be coordinates for the slice, with $\tau_j$ the even, and $\zeta_a$ the odd coordinates. As before we need the super analogues of Beltrami differentials to parametrize the tangents along the slice $\hat S$. Recall that infinitesimal variation of supergeometries are of the form $H_A^B$, and infinitesimal deformations of super Riemann surfaces structure can be parametrized by the sole components $H = H_-{}^z, \bar H = H_+{}^{\bar z}$ (c.f. (5.3)). This suggests introducing the «super Beltrami differentials»

$$\mu_j = (H_-{}^z)_J = E_-{}^M\partial_{m_J}E_M{}^z \tag{6.5}$$

which can be viewed as tangents to the superslice $\hat S$ at the geometry $E_A{}^M$. We shall also require for each supergeometry $E_A{}^M$ a basis $\Phi_K, K = (k,b), k = 1,\ldots,3h-3, b = 1,\ldots,2h-2$ of zero modes for the adjoint $\mathcal{D}_-^{3/2}$ of the gauge-fixing operator. They will be called super quadratic differentials (although their $U(1)$ rank is rather $3/2$ !). Their number is the same as the dimension of supermoduli space. The super quadratic differentials corresponding to $K = k$ will be odd, while those corresponding to $K = b$ will be even. We can now write down the expression for the gauge-fixed path integral corresponding to the action (3.2), which is the same as the action (4.23):

$$
(6.6)\qquad \begin{aligned} Z_\delta = \int_{\hat{S}} \mathrm{d}^{5h-5} m_J \mathrm{d}^{5h-5} \bar{m}_J \frac{|s\det < \mu_J|\Phi_K > |^2}{s\det < \Phi_J|\Phi_K >} \\ \left[ s\det{}' \mathcal{D}_-^\dagger \mathcal{D}_-^{3/2} \right]^{1/2} \left[ \frac{8\pi^2 s\det \mathcal{D}_-^\dagger \mathcal{D}_-^0}{\int \mathrm{d}\, \mathbf{z} s \det E} \right]^{-5} . \end{aligned}
$$

As in the bosonic string, the terms involving the determinants of $\mathcal{D}_-^{(0)}$ result from the free Gaussian integrals over the scalar superfields $X^\mu$. The remaining terms arise from the factoring out of the volumes of the gauge groups. The super Weyl symmetry is anomalous, and the symmetry in only restored in dimension $d = 10$, when the anomalies between scalar superfields and determinants $\mathcal{D}_-^{3/2}$ cancel. We have restricted ourselves to this case. We have also written explicitly an index $\delta$ to emphasize that so far the path integrals have been taken with respect to the chirally symmetric action (3.2), and that all left and right spinors are taken with respect to the same spin structure $\delta$. In particular the GSO projection is not yet enforced.

To do so requires cancellation of holomorphic anomalies. We can now make use of the explicit formula (5.14) for holomorphic anomalies. Since the $U(1)$ ranks of the ghost determinants and matter fields are $-1$ and $0$, we see that the holomorphic anomalies of matter and ghosts cancel in the critical dimension $d = 10$. Now the (conjugates of) the zero modes of $\mathcal{D}_-^{1/2}$ do not appear in the gauge-fixed formula (6.6), although they are required for a full cancellation of holomorphic anomalies. A similar situation occurred in the gauge-fixed bosonic partition function (2.16), which was responsible for the term $\det(\mathrm{Im}\,\Omega)^{13}$ in the final formula (2.30). More systematically, let $\hat{\omega}_I$ be a basis of these zero modes, henceforth referred to as super abelian differentials. For even spin structures $\delta$, there are generically $h$ of them. We temporarily delay the discussion of the odd spin structures case, since they present new difficulties, and their contributions to the partition function $Z_\delta$ vanish due to the presence of zero modes. On super Riemann surfaces, there is an instrinsic notion of integral of $1/2$ – forms

$$
(6.7)\qquad \int \mathrm{d}\, \mathbf{z} \hat{\omega}
$$

along cycles on the surface. After a choice of homology basis as in (2.23), we can fix a basis of super abelian differentials by

$$
(6.8)\qquad \int_{A_I} \mathrm{d}\, \mathbf{z} \hat{\omega}_J = \delta_{IJ}
$$

and define a «super period matrix» $\hat{\Omega}_{IJ}$ by

(6.9) $$\int_{B_I} \mathrm{d}\, z\hat{\omega}_J = \hat{\Omega}_{IJ} \,.$$

We shall describe these notions more explicitly in terms of $g_{mn}$ and $\chi_{\bar{z}}{}^{+}$ in Section 9. An adaptation to the super case of the arguments for the Riemann identities gives

6.10) $$< \hat{\omega}_I | \hat{\omega}_J >= \mathrm{Im}\, \hat{\Omega}_{IJ}$$

and the partition function (6.6) can be put in the form

(6.11) $$Z_\delta = \int \mathrm{d}^{5h-5} m_J \mathrm{d}^{5h-5} \bar{m}_J |\mathcal{F}_\delta|^2 (\det \mathrm{Im}\, \hat{\Omega})^5$$

where we have absorbed the term $s\det < \mu_J | \Phi_K >$ in the factor $\mathcal{F}_\delta$ which is a holomorphic function on supermoduli space.

We can now carry out the GSO projection. Since the super period matrix depends on the spin structure, it is necessary to separate also left and right movers in the term $\mathrm{Im}\, \hat{\Omega}$. This is achieved by rewriting this term as a Gaussian integral over parameters $p_I^\mu, \mu = 1, \ldots, 10$

(6.12) $$Z_\delta = \int \mathrm{d}^{5h-5} m_J \mathrm{d}^{5h-5} \bar{m}_J \int \mathrm{d}\, p_I^\mu |e^{\pi i p_I^\mu \hat{\Omega} p_J^\mu}|^2 |\mathcal{F}_\delta|^2 \,.$$

Finally the superstring partition function can be constructed as

(6.13) $$\begin{aligned} Z_{\mathrm{TypeII}} &= \int \mathrm{d}^{5h-5} m_J \mathrm{d}^{5h-5} \bar{m}_J \int \mathrm{d}\, p_I^\mu \\ &\left( \sum_\delta (-1)^{\sigma(\delta)} e^{\pi i p_I^\mu \hat{\Omega}_{IJ} p_J^\mu} \mathcal{F}_\delta \right) \left( \sum_{\bar{\delta}} e^{\pi i p_I^\mu \hat{\bar{\Omega}}_{IJ} p_J^\mu} (-1)^{\bar{\sigma}(\bar{\delta})} \bar{\mathcal{F}}_{\bar{\delta}} \right) \end{aligned}$$

where the relative signs $(-1)^{\sigma(\delta)}$ between even and odd spin structures is the same for left and right factors in the case of the TypeIIA string, and opposite in the case of the TypeIIB string. For later reference we write down also the ghost version of the gauge-fixed partition function.

## Superghost Formulation

The gauge-fixed $Z_\delta$ of (5.5) can be conveniently rewritten in terms of free integrals by introducing superghost fields $B, C$ of $U(1)$ ranks $3/2$ and $-1$ respectively. The action for the superghosts is taken to be

(6.14) $$I_{sgh}(B, \bar{B}; C, \bar{C}) = \frac{1}{2\pi} \int \mathrm{d}^2 z E (B \mathcal{D}_- C + \bar{B} \mathcal{D}_+ \bar{C})$$

which admits the same symmetries as the original matter action (4.23). To convert the jacobians in (6.6) to an integral over superghosts, we note that

$$\int \mathcal{D}(B\bar{B}C\bar{C})e^{-I_{sgh}'}\prod_K |\delta(<\mu_K|B>|^2 = \frac{s\det <\mu_K|\Phi_J>|^2}{s\det <\Phi_K|\Phi_J>}(s\det \mathcal{D}_-^\dagger \mathcal{D}_-^{3/2}) \tag{6.15}$$

and thus the full expression (5.5) can be recast as

$$Z_\delta = \int \mathrm{D}(B\bar{B}C\bar{C}X^\mu)e^{(-I(E_M^A,X^\mu)+I_{sgh}(B,C;\bar{B},\bar{C}))} \prod_{a=1}^{2h-2}|\delta(<\mu_a|B>)|^2 \prod_{j=1}^{3h-3} <\mu_j|B>|^2 . \tag{6.16}$$

Here we have made use of the fact that for integrals over anti-commuting variables $\theta$, the Dirac delta function $\delta(\theta)$ is the same as $\theta$.

## 7. HOLOMORPHIC STRUCTURE OF SUPERSTRINGS IN COMPONENT FORMALISM

Although we have succeded in constructing a holomorphic integrand for the superstring partition function in the previous section, it is certainly by no means evident how to accomplish this in presence of vertex operators corresponding to emission and absorption of string states. Also the case of odd spin structures would have to be trated then, since they do contribute to the general scattering amplitudes. For this we need the component formalism. As a by product, we shall obtain a rather complete translation into concrete geometry of the rather abstract objects of analytic supergeometry encountered in Section 6. Such a translation is indispensable if we are to make contact with the better understood geometry of moduli space and modular forms.

### Gauge-fixing in component formalism

We need to rewrite the gauge-fixed formula (6.6) for the partition function in component language. We shall begin with the superghosts, since they do not pose any problem from the point of view of holomorphic splitting. Expanded in components, the superghost fields $B$ and $C$ become

$$\begin{aligned} B &= \beta + \theta b \\ C &= c + \theta\gamma \end{aligned} \tag{7.1}$$

where we have neglected auxiliary fields. The superghost action (5.6) can be worked out to be in components

$$
I_{sgh} = I_{sgh}^{\text{free}} + < \chi | S_{gh} > + < \bar{\chi} | \bar{S}_{gh} >
$$
$$
= \frac{1}{2\pi} \int \mathrm{d}^2 z (b\partial_{\bar{z}} c + \beta \partial_{\bar{z}} \gamma) - \frac{1}{2\pi} \int \mathrm{d}^2 z \chi_{\bar{z}}{}^{+} S_{gh} + \text{c.c.} \tag{7.2}
$$

where $S_{gh} = \frac{1}{2} b\gamma - \frac{3}{2}\beta\partial_z c - (\partial_z \beta) c$ is the ghost part of the full supercurrent

$$
S = S_{gh} - \frac{1}{2}\psi_{+}^{\mu}\partial_z x^{\mu} \, . \tag{7.3}
$$

Substituting in (6.6) gives the gauge-fixed partition function in component language

$$
Z_\delta = \int \mathrm{D}(b\bar{b}c\bar{c}\beta\bar{\beta}\gamma\bar{\gamma}x^{\mu}\psi_{+}^{\mu}\psi_{-}^{\mu}) \prod_{i=1}^{3h-3} | < \mu_i | B > |^2 \prod_{a=1}^{2h-2}
$$
$$
|\delta(< \mu_a | B >)|^2 e^{(<\chi_{\bar{z}}^{+}|S> + <\chi_{z}^{-}|\bar{S}>)} e^{-\frac{1}{8\pi}\int \mathrm{d}^2 z \chi_{z}^{-}\chi_{\bar{z}}^{+}\psi_{+}^{\mu}\psi_{-}^{\mu}} e^{-I_{\text{total}}^{\text{free}}} \, . \tag{7.4}
$$

Here the total kinetic energy for both matter and ghost fields is

$$
I_{\text{total}}^{\text{free}} = \frac{1}{4\pi}\int \mathrm{d}^2 z (\partial_z x^{\mu}\partial_{\bar{z}} x^{\mu} - \psi_{+}^{\mu}\partial_{\bar{z}}\psi_{+}^{\mu} - \psi_{-}^{\mu}\partial_z \psi_{-}^{\mu})
$$
$$
+ \frac{1}{2\pi}\int \mathrm{d}^2 z (b\partial_{\bar{z}} c + \beta\partial_{\bar{z}}\gamma + \text{c.c.}) \, . \tag{7.5}
$$

From this it is evident that the ghost contributions are holomorphically split. Thus we need concentrate only on the matter field contributions. We shall actually do it not just for the partition function, but also for any insertion of massless particles vertices.

## Quantum scalar superfields

We have seen the vertex operator for emission of massless particles for the bosonic string in (2.8). One of the advantages of the superfield formalism is that it allows us to guess easily the vertex for massless particles in the superstring. The answer is

$$
V(\mathbf{z}, k^{\mu}) = \epsilon_{\mu\bar{\mu}} \mathcal{D}_{-} X^{\mu} \mathcal{D}_{+} X^{\bar{\mu}} e^{ik^{\rho}X_{\rho}} \, . \tag{7.6}
$$

We have left out the usual overall integration in the super insertion point $\mathbf{z}$. As in the case of the partition function, the superstring amplitudes for scattering of massless bosons are only obtained after separating out holomorphic from antiholomorphic factors in the functional integrals, and summing afterwards over spin structures. To carry this out, it is convenient to work with the generating function for the vertices (7.6) rather than with the vertices themselves:

$$V(z,\bar{z},\zeta,\bar{\zeta};k) = \exp[ikX + \zeta^\mu \sqrt{E}\mathcal{D}_+ X^\mu + \bar{\zeta}^{\bar{\mu}} \sqrt{E}\mathcal{D}_- X^{\bar{\mu}}](z) \,. \tag{7.7}$$

Thus we need to consider amplitudes of the form

$$\left\langle \prod_{i=1}^{N} V(z_i,\bar{z}_i,\zeta_i,\bar{\zeta}_i;k_i) \right\rangle_X = \int \mathrm{D}X^\mu e^{-I} \prod_{i=1}^{N} V(z_i,\bar{z}_i,\zeta_i,\bar{\zeta}_i;k_i) \tag{7.8}$$

with of course the case of no insertion reducing to the partition function discussed at length in Section 6 in the superfield formalism.

We have already seen that even in the case of the partition function alone, separation of left and right movers requires the introduction of internal loop momenta. It is easier to trace the origin of the difficulties in component language.

- The functional integrals in $x^\mu$ and $\psi^\mu$ in (7.8) will lead to a product of determinants and combinations of Green's function on scalars and spinors. Although the determinants themselves will combine with their ghost counterparts to produce holomorphically split terms according to the Belavin-Knizhnik theorem, no such theorem holds for the Green's functions. For example, the scalar Green's function $G(z,\omega)$ defined by

$$G(z,\omega) = < x(z)x(\omega) > \tag{7.9}$$

is a real-valued function which satisfies

$$\begin{aligned} \partial_{\bar{z}}\partial_z G(z,\omega) &= -2\pi\delta(z,\omega) + \frac{2\pi g_{z\bar{z}}}{\int d^2z\sqrt{g}} \\ \partial_z\partial_{\bar{\omega}} G(z,\omega) &= 2\pi\delta(z,\omega) - \pi \sum_{I,J=1}^{h} \omega_{I(z)} (\mathrm{Im}\,\Omega)^{-1}_{IJ} \bar{\omega}_J(\omega) \,. \end{aligned} \tag{7.10}$$

The additional terms on the right hand side of (7.10) are due to the presence of zero modes for the operator $\partial_{\bar{z}}$ on scalars and its adjoint. They are clearly obstructions to holomorphicity in both variables $z,\omega$, and the implicit moduli parameter. Similarly the Green's function for spinors could have zero modes. Generically the number of zero modes is 0 for even spin structures, and 1 for odd spin structures. This is one of the main reasons why the odd spin structure case is more difficult than the even one.

• The holomorphic structure of supergeometries and supermoduli space force the inclusion of complicated terms coupling both chiralities. We have seen an example of this when discussing variations to second order of supergeometries in Section 6. Even more exmlicitly, the very action of the supersymmetric sigma model contains the chirally mixing term $\chi_{\bar z}^{+}\chi_{z}^{-}\psi_{+}^{\mu}\psi_{-}^{\mu}$ , and the same is true for superderivatives such as those appearing in (6.4).

It is remarkable that nevertheless the correlation functions of (7.8) do split holomorphically after introduction of internal loop momenta. To show this, we shall provide explicit rules for evaluating them that will only involve notions that are manifestly holomorphic in supermoduli parameters, insertion points $\mathbf{z}_i$ , and polarization tensors $\zeta_i$ .

We need some ingredients of function theory on Riemann surfaces. The basic notion is that of the prime form $E(z,\omega)$ , which will serve as a version of $z-\omega$ which nevertheless encodes the complex structure of the underlying Riemann surface. Recall that given a Riemann surface with a choice of homology basis (2.28), the complex structure is characterized by the period matrix $\Omega$ (2.29). The theta function with characteristics $\vartheta[\delta]$ is defined by [39]

$$\vartheta[\delta](z_I,\Omega) = \sum_{n\in\mathbf{Z}^h} \exp[\pi i(n_I+\delta_I')\Omega_{IJ}(n_J+\delta_J') + 2\pi i(n_I+\delta_I')(z_I+\delta_I'')] \ . \tag{7.11}$$

As a function of $z_I$ , it has the same parity as $4\delta'\delta''$ , that is, as the spin structure $\delta$ . If $\delta$ is odd, there is at least one holomorphic spinor $h_\delta$ (generically exactly one), which can be written down explicitly as

$$h_\delta(z) = \left(\sum_{I+1}^{h}\partial_I\vartheta[\delta](0,\Omega)\omega_I(z)\right)^{1/2} \ . \tag{7.12}$$

The prime form $E(z,\omega)$ can now be defined as [40]

$$E(z,\omega) = \frac{\vartheta[\delta]\left(\int_\omega^z\omega_I,\Omega\right)}{h_\delta(z)\,h_\delta(\omega)} \ . \tag{7.13}$$

The prime form is actually independent of the odd spin structure $\delta$ . It has the property of vanishing only when $z=\omega$ , of being holomorphic in all variables $z,\omega,\Omega_{IJ}$ . Strictly speaking it is defined only on the universal covering of the Rieamann surface, where it is a $(-1/2,0)$ form in each of the variables $z$ and $\omega$ . Thus it is multiplevalued on the Riemann surface. However the multiplevaluedness will cancel out in all physical string amplitudes.

It is convenient to summarize the results in the following

**Chiral splitting theorem**

(A) Up to a local superholomorphic anomaly, the amplitudes (7.8) can be expressed as

$$< \prod_{i=1}^{N} V(z_i, \bar{z}_i, \zeta_i, \bar{\zeta}_i; k_i) >_X = \tag{7.14}$$
$$(2\pi)^{10} \delta\left(\sum_{i=1}^{N} k_i\right) \int_{\mathcal{I}} \mathrm{d} p_I^\mu |\mathcal{F}_\delta(z_i, \zeta_i, \Omega, \chi_{\bar{z}}^+; k_i, p_I^\mu)|^2$$

where the conformal blocks $\mathcal{F}_\delta^m$ satisfy

- $\mathcal{F}_\delta^m$ is meromorphic in $z_i$, holomorphic in $\zeta_i, \Omega_{IJ}$, and $\chi_{\bar{z}}^+$;
- $\mathcal{F}_\delta^m$ is a $(1/2, 0)$ form in each of the insertion points $z_i$;
- $\mathcal{F}_\delta^m$ has non-trivial monodromy

$$\mathcal{F}_\delta^m(z_i + \delta_{ij} A_K, \zeta_i^\mu, \Omega, \chi_{\bar{z}}^+; k_i^\mu, p_I^\mu) = \mathcal{F}_\delta^m(z_i, \zeta_i^\mu, \Omega, \chi_{\bar{z}}^+; k_i^\mu, p_I^\mu)$$

$$\mathcal{F}_\delta(z_i + \delta_{ij} B_K, \zeta_i^\mu, \Omega, \chi_{\bar{z}}^+; k_i, p_I^\mu) = \tag{7.15}$$
$$F_\delta(z_i, \zeta_i^\mu, \Omega, \chi_{\bar{z}}^+; k_i, p_I^\mu + \delta_{IK} k_j^\mu)$$

- $\mathcal{F}_\delta^m$ is unique up to a constant phase.

(B) The conformal bloks $\mathcal{F}_\delta^m$ are given by the following explicit formulas in components

$$\mathcal{F}_\delta^m = Z_0^{-10} Z_{1/2,\delta}^{10} < \exp \mathcal{L} > \quad \text{evev } \delta \tag{7.16}$$
$$\mathcal{F}_\delta^m = Z_0^{-10} Z_{1/2,\delta}^{10} \int \mathrm{d}\lambda < \exp[\mathcal{L} + \lambda(h_\delta | j^\mu)] > \quad \text{odd } \delta$$

where

$$\mathcal{L} = \frac{1}{4\pi} \int \mathrm{d}^2 z \chi_{\bar{z}}^+ \psi_+^\mu \partial_z x_+^\mu + i \sum_{I=1}^{h} p_I^\mu \oint_{B_I} \mathrm{d} z \partial_z x_+^\mu \tag{7.17}$$
$$+ \sum_{i=1}^{N} [ i k_i^\mu X_+^\mu(z_i) + \zeta_i^\mu \partial_+ X_+^\mu(z_i)]$$

and

$$j^\mu = -\frac{1}{4\pi} \chi_{\bar{z}}^+ \partial_z x_+^\mu + \sum_{i=1}^{N} (i k_i \theta_i + \zeta_i^\mu) \delta(z - z_i) \; .$$

The effective chiral superfield $X_+^\mu$ and derivatives $\partial_+$ are defined by

$$
\begin{aligned}
X_+^\mu &= x_+^\mu + \theta\psi_+^\mu \\
\partial_+ &= \partial_\theta + \theta\partial_z
\end{aligned}
\tag{7.18}
$$

and the expectation values $<>$ are understood in the sense of Wick contractions of $x_+^\mu$ and $\psi_+^\mu$ according to the following rules

$$
\begin{aligned}
&< x_+(z)x_+(\omega) > = -\log E(z,\omega) \\
&-< \psi_+(z)\psi_+(\omega) > = S_\delta(z,\omega) = \frac{1}{E(z,\omega)}\frac{\vartheta[\delta](\int_\omega^z \omega_I,\Omega)}{\vartheta[\delta](0,\Omega)} \quad \text{even } \delta \\
&-< \psi_+(z)\psi_+(\omega) > = \frac{1}{E(z,\omega)}\frac{\sum_{I=1}^h \partial_{z_I}\vartheta[\delta](\int_\omega^z \omega_I,\Omega)W_I}{\sum_{I+1}^h \partial_{z_I}\vartheta[\delta](0,\Omega)W_I} \quad \text{odd } \delta
\end{aligned}
\tag{7.19}
$$

with $W_I$ an arbitrary generic $h-$ vector. Finally $Z_0(\Omega)$ and $Z_{1/2,\delta}(\Omega)$ can be viewed as the chiral scalar and Dirac determinants respectively, up to a local anomaly which will cancel out in superstring amplitudes

$$
\begin{aligned}
\frac{8\pi^2 \det{}'\Delta g}{\det(\operatorname{Im}\Omega)\int d^2z\sqrt{g}} &= |Z_0(\Omega)|^4 \\
\frac{\det{}'\bar\partial^\dagger\bar\partial_{1/2}}{< h_\delta|h_\delta >} &= |Z_{1/2,\delta}(\Omega)|^2 .
\end{aligned}
$$

The point of these rules is that contractions with effective chiral fields will always give manifestly holomorphic expression with respect to all variables, unlike contractions with the original fields $x^\mu$ and $\psi_+^\mu$.

(C) Alternatively the conformal blocks $\mathcal{F}_\delta^m$ are given by explicit formulas in a manifestly supersymmetric way (say for even spin structures)

$$
\begin{aligned}
F_\delta^m = & Z_{\mathcal{D},\delta}^{-10}(\hat\Omega,\chi_{\bar z}^+) < \exp[ip_I^\mu \oint_{B_I} dz\partial_+ X_+^\mu + \\
& \sum_{i=1}^N (ik_i^\mu X_+^\mu(\mathbf{z}_i) + \zeta_i^\mu\partial_+ X_+^\mu(\mathbf{z}_i))] >
\end{aligned}
$$

where

$$
Z_{\mathcal{D},\delta}^{-1} = Z_0^{-1} Z_{1/2} < \exp(-\frac{1}{4\pi}\int d^2z\chi_{\bar z}^+\psi_+^\mu x_+^\mu) >_{x_+,\psi_+} . \tag{7.20}
$$

Wick contractions are perfomed with

$$
< X_+(\mathbf{z})X_+(\omega) >= -\log \mathcal{E}_\delta(\mathbf{z},\omega)
$$

and $\hat{\Omega}$, $\mathcal{E}_\delta$ are supersymmetric versions of the period matrix and the prime form to be discussed in Section 8. The odd case will also be given there.

(D) Finally the contributions of the superghosts are manifestly split into $|F_\delta^{gh}|^2$ (up to a local anomaly) without recourse to internal loop momenta $p_I^\mu$. Thus the amplitudes for the Types II A,B superstrings at loop order $h$ are given by

$$(7.21)\qquad \begin{aligned} &< \prod_{i=1}^{N} V(z_i, \bar{z}_i, \zeta_i, \bar{\zeta}_i, k_i) >_{\text{Type II}} = \int \mathrm{d}\, p_I^\mu \int_{s\mathcal{M}_h} \mathrm{d}\, m_J \mathrm{d}\, \bar{m}_J \times \\ &\Big(\sum_\delta (-1)^{\sigma(\delta)} \mathcal{F}_\delta^{gh} \mathcal{F}_\delta^m\Big)\Big(\sum_{\bar{\delta}} (-1)^{\bar{\sigma}(\bar{\delta})} \bar{\mathcal{F}}_{\bar{\delta}}^{gh} \bar{\mathcal{F}}_{\bar{\delta}}^m\Big) . \end{aligned}$$

The choice of phases of $\mathcal{F}_\delta$ is such that they are un affected by modular transformations. As in Section 6 the term $\sigma(\delta)$ is $0$ or $1$ on even or odd spin structures. For Type II A(B) $\sigma$ and $\bar{\sigma}$ are the same (opposite).

We note that these effective rules essentially amount to replacing all fields by their formal chiral components, drop all the chirality mixing terms (in particular this reduces all the complicated super derivatives of supergeometry by their flat counterparts, so that we formally get back to the familiar setting of conformal field theory in conformal gauge), on the following key conditions. First the propagators should be the ones built out of the prime form; second, we have to introduce internal loop momenta flowing through the $B_I$ cycles. These are the prescriptions that encode the non-trivial topology and the complex structure of the underlying surface.

The full proofs for the chiral splitting theorem are in [38].

## 8. THE SUPER PERIOD MATRIX

We would now like to analyze the ambiguities in superstring scattering amplitudes which had been alluded to in the introduction. First we explain the nature of the ambiguities found, and then we indicate how they can be resolved.

### A. Ambiguities

Some of the difficulties uncovered recently in perturbation theory for superstrings are the following:

- In [10] it is pointed out that zero modes for the scalar laplacian seem to lead to a positive additional term in the partition function for the Type II superstring, when compared with the partition function for the heterotic string. Since the partition function can be interpreted as the space-time cosmological constant, this certainly is not consistent with the expectation that both should vanish in flat space-time due to space-time supersymmetry;

• In [12] [41] the value of the cosmological constant for the heterotic string at two loops is found to be zero by taking the $\mu_a$ of (5.8) to be delta functions supported at branch points. A generic choice of points on the surface however seems to lead to a non-vanishing answer. Why are branch points singled out, otherwise than by mathematical expediency? Since choices of points are jus choices of gauge slices, this would signify a breakdown of gauge invariance unless we can explain the privileged role of branch points;

• Finally in the BRST formalism [5], different choices of insertions for the picture-changing operator lead to total derivatives on moduli space for the integrand in the partition function [42] [11] [13]. These total derivatives are however defined only locally on moduli space, so the cosmological constant appears to be ill-defined, a situation resembling the Wu-Yang analysis of the action for an electric charge in the field of a magnetic monopole.

In view of our discussion of holomorphic splitting, we can readily see that the prescription of matching left and right movers at the same internal loop momenta does reproduce the heterotic string from a chiral halr of the bosonic string and the other chiral half of the Type II superstring. In fact, heterotic supergeometry corresponds to $\chi_z^- = 0$ . The scalar superfields contractions in a heterotic supergeometry background can be evaluated as before, giving the same answer as (7.10) but with $\chi_z^-$ set to $0$

$$< \prod_{i=1}^{N} V(z_i,\bar{z}_i,\zeta_i,\bar{\zeta}_i;k_i) >_{X,HET} = < (2\pi)^{10}\delta\left(\sum_{i=1}^{N} k_i\right) \int_{\mathcal{I}} d\, p_I^\mu \mathcal{F}_\delta(z_i,\zeta_i,\Omega,\chi_{\bar{z}}^+;k_i,p_I^\mu)\bar{\mathcal{F}}_\delta(\bar{z}_i,\bar{\zeta}_i,\bar{\Omega},0;k_i,p_I^\mu) \ . \tag{8.1}$$

On the other hand the chiral halves of the bosonic string can similarly be obtained from the Type II superstring with both $\chi_{\bar{z}}^+$ and $\chi_z^-$ set to $0$ . We can recognize then (8.1) as being built from $\mathcal{F}_\delta(z_i,\zeta_i,\Omega,\chi_{\bar{z}}^+;k_i,p_i^\mu)$ , which is a chiral half of the Type II, and $\bar{\mathcal{F}}_\delta(\bar{z}_i,\bar{\zeta}_i,\bar{\Omega},0;k_i,p_I^\mu)$ , which is the half of the bosonic string of the opposite chirality. Thus even after summing over spin structures and incorporating ghost contributions, the heterotic string and the Type II string partition functions share the key common factor which will cause them to vanish simultaneously. In retrospect, the first apparent contradictions noted above are due to the absence of internal loop momenta.

To get an idea of what could be the probelm with the next two objections, we discuss briefly how the picture-changing formalism emerges from the guage-fixed superstring (7.4). Since prescription of internal loop momenta can reproduce the superstring from the heterotic string, we shall limit our discussion to this latter case and drop the $\chi_z^-$ terms from the action (7.4). In (7.4) no specification has been made as yet about the choice of gauge super slice $\hat{S}$. The goal is to reduce the integral over supermoduli space in (6.5) (7.4) by a suitable choice of superslice, and integrate out the odd variables. The choice is the following: select a $3h-3$ slice $S$ of metrics $g_{mn}$ for moduli space parametrized by $\tau_i, i = 1, dots, 3h-3$, and $2h-2$ sections $\chi_a$. Then the superslice $\hat{S}$ is taken to be the $(3h-3|2h-2)$ slice given by $g_{m,n}, \chi = \sum_{a=1}^{2h-2} \zeta_a \chi_a)$. In particular $g_{mn}$ is independent of $\zeta_a$. With this choice the super Beltrami differentials $\mu_i$ of (5.8) are the standard Beltrami differentials associated to the slice $S$, while the super Beltrami differentials $\mu_a$ are just the $\chi_a$. The gauge-fixed partition function becomes

$$(8.1) \qquad \begin{aligned} Z_\delta = \int & d^{3h-3}\tau_j d^{3h-3}\bar{\tau}_j \\ & \prod_{a=1}^{2h-2} d\zeta_a d\bar{\zeta}_a | < \mu_j | b > |^2 |\delta(< \chi_a|\beta >)|^2 e^{\sum_{a=1}^{2h-2} \zeta_a < \chi_a | S >} e^{-I_{\rm total}^{\rm free}} \end{aligned}$$

where $S$ is the supercurrent (7.3) and by $I_{\rm total}^{\rm free}$ we have denoted the kinetic energy terms of both matter and ghost fields (7.5). The fermionic integral with respect to $\zeta_a$ can now be evaluated immediately, and we obtain

$$(8.2) \qquad \begin{aligned} Z_\delta = \int & d^{3h-3}\tau_j d^{3h-3}\bar{\tau}_j \\ & \prod_{a=1}^{2h-2} d\zeta_a d\bar{\zeta}_a | < \mu_j | b > |^2 |\delta(< \chi_a|\beta >) < \chi_a | S > |^2 e^{-I_{\rm total}^{\rm free}} . \end{aligned}$$

The operator $\delta(< \chi_a|\beta >) < \chi_a|S >$ is of course rather complicated. We only obtain a local operator by taking the $\chi_a$ to be Dirac delta functions supported at points $z_a$. As shown by Verlinde and Verlinde [42] from operator product expansions, the operator $Y(z_a)$ obtained this way

$$(8.3) \qquad Y(z_a) = \delta(\beta(z_a))S(z_a)$$

is exactly the well-known BRST invariant picture-changing operator of Friedan, Martinec, and Shenker [5] which had already played an important role in superstring field theory [43]. In this way from the functional integrals one arrives at the prescription suggested by BRST invariance

$$(8.4) \qquad Z_\delta = \int d^{3h-3}\tau_i d^{3h-3}\bar{\tau}_i \prod_{i=1}^{3h-3} | < \mu_i | b > |^2 | \prod_{a=1}^{2h-2} Y(z_a)|^2 e^{-I_{\rm total}^{\rm free}} .$$

In this derivation the insertion points $z_a$ are arbitrary generic points, and the value of $Z_\delta$ should not depend on their choice. However, a BRST argument shows that a change of $z_a$ is the same as a BRST transformation, and will not leave the integrand unchanged but rather result in a total derivative in moduli parameters. This is actually a very serious difficulty. In fact the points $z_a$ should be properly thought of as sections of the universal curve above moduli space. This curve admits no global continuous section. This means that we are forced to cover moduli space by patches, and choose insertion points on each patch. The total derivatives caused by changes of insertions on the overlaps prevent the sum of the integrals over parches from having any intrinsic meaning.

What could be the reason for these ambiguities? A close look at the above derivation of the picture-changing formalism shows that a key step is the use of the apparently natural projection

$$\begin{array}{c}(g_{mn},\chi_m^\alpha)\\ \downarrow\\ g_{mn}\end{array} \tag{8.5}$$

to reduce integrals over supermoduli to integrals over moduli space. The problem is that this projection is not invariant under sypersymmetry! That is, it will in general be the case that under a supersymmetry transformations

$$\begin{array}{ccc}(g_{mn},\chi_m^\alpha) & \rightarrow & (g_{mn}+\delta g_{mn},\chi_m^\alpha+\delta\chi_m^\alpha)\\ g_{mn} & \rightarrow & g_{mn}+\delta g_{mn}\ .\end{array}$$

We are then led to the question of finding a supersymmetric, conformally invariant, modification of the metric $g_{mn}$. There is no obvious candidate for a modification of $g_{mn}$ as a tensor. However if we view the conformal class of $g_{mn}$ as characterized rather by its period matrix, then the natural candidate is the super period matrix $\hat\Omega_{IJ}$, which has appeared earlier in the superfield formalism (6.19). We shall now show that the holomorphically split amplitudes of (7.14)-(7.20) also lead naturally to the same matrix, so that holomorphic splitting and the search for the correct superstring measure reinforce one another in suggesting the super period matrix as the fundamental starting notion.

## B. The super period matrix as covariance of holomorphic amplitudes

Returning to the holomorphically split amplitudes (7.14)-(7.20), it suffices to consider the case of no insertions, so that we are dealing with the measure alone. It is natural from this point of view to introduce the covariance of these amplitudes, viewed as Gaussians in the internal loop momenta $p_I^\mu$. It is evidently a modification of the usual period matrix $\Omega$. Since the local anomalies have cancelled out, this matrix is supersymmetric. We can expect it to be the super period matrix, which will turn out to be indeed the case

$$\hat\Omega_{IJ}=\frac{1}{2\pi i}\partial^2_{p_I^\mu p_J^\mu}\log\mathcal{F}_\delta^m(\Omega,\chi_{\bar z}^+;p_I^\mu)\ . \tag{8.6}$$

### C. The super period matrix as a supersymmetric correction to the period matrix

To obtain a supersymmetric correction, we begin by introducing an effective Dirac propagator $\hat{S}_\delta(z,w)$ . It is defined by the equation

$$(8.7)\qquad \partial_{\bar z}\hat{S}_\delta(z,w) + \frac{1}{8\pi}\chi_{\bar z}^{+}\int d^2u\,\chi_{\bar u}^{+}\partial_z\partial_u \log E(z,u)\hat{S}_\delta(u,w) = 2\pi\delta(z,w)$$

which can be solved by a perturbative expansion. This expansion will terminate, since in practice the gravitino $\chi_{\bar z}{}^{+}$ will depend on only $2h-2$ Grassmann valued parameters. The super period matrix is then given by

$$(8.8)\qquad \hat{\Omega}_{IJ} = \Omega_{IJ} - \frac{i}{8\pi}\int d^2z d^2w\,\omega_I(z)\chi_{\bar z}{}^{+}\hat{S}_\delta(z,w)\chi_{\bar w}{}^{+}\omega_J(w)\ .$$

From the transformation laws (3.3), it is not difficult to check that $\delta\hat{\Omega} = 0$ under supersymmetry.

### D. The super period matrix as periods of super abelian differentials

This was how the super period matrix appeared in the first place. On a super Riemann surface with even spin structure there will be $h$ superholomorphic $1/2$ – forms, which generalize the usual abelian differentials of the first kind. Given a homology basis $A_I, B_I$ , we can choose a basis $\hat{\omega}_I$ dual to the cycles $A_I$ . The super period matrix is then the matrix of periods of the $\hat{\omega}_I$ around the cycles $B_J$ , with a suitable notion of line integrals (c.f. equations (6.8) and (6.9)). The component version as well as the odd spin structure case is presented in Section 9. It can then be checked that this agrees with the previous definitions.

Thus both considerations of supersymmetry and holomorphic splitting strongly indicate that we should view parametrize supermoduli space in terms of $(\hat{\Omega}_{IJ},\chi_{\bar z}^{+})$ rather than $(\Omega_{IJ},\chi_{\bar z}^{+})$ . The projection $(\hat{\Omega}_{IJ},\chi_{\bar z}^{+}) \to \hat{\Omega}_{IJ}$ is well-defined under supersymmetry, and should be the one used for integrating out the odd variables in supermoduli space. We shall discuss this projection as well as a more precise construction of supermoduli space as a fiber bundle over the space of super period matrices in Section 10. Presently we would like to point out that this could give an explanation for why insertion at branch points seem to be the only ones leading to the expected value for the two-loop cosmological constant, which is one of the challenging puzzles mentioned in the above list.

At $h = 2$ the number of odd supermoduli is $2h - 2 = 2$ , and for arbitrary insertions $z_1, z_2$ the gravitino $\chi_{\bar z}^{+}$ is of the form $\chi_{\bar z}^{+} = \zeta_1\delta(z,z_1) + \zeta_2\delta(z,z_2)$ .

Substituting $\chi_{\bar{z}}^{+}$ in (8.8) gives

$$\hat{\Omega}_{IJ} = \Omega_{IJ} - \frac{i}{4\pi}\zeta_1\zeta_2\omega_I(z_1)S_\delta(z_1,z_2)\omega_J(z_2) \tag{8.9}$$

where $S_\delta(z,\omega)$ is the Szego kernel of (7.9). However explicit formulas for the Szego kernel at genus 2 are available in terms of the surface as a double cover of the plane with cuts [40]. These formulas show that the Szego kernel vanishes at branch points, so that with this choice of insertions, the super period matrix and the period matrix coincide. This is the reason why the superstring measure obtained earlier using the standard period matrix gave the correct answer only at branch points.

## 9. SUPERANALYTIC FUNCTION THEORY

We have obtained in Section 7 explicit prescriptions for holomorphic splitting of all amplitudes needed for scattering of massless bosons in component language. We have quoted the answer for a manifestly supersymmetric formulation in terms of the super prime form. We shall now provide a more complete discussion of analytic function theory on super Riemann surfaces.

### Super Abelian Differentials

We begin by a more explicit formula for super Abelian differentials. In components, a super Abelian differential is a $U(1)$ rank $1/2$ tensor satysfying $\mathcal{D}_{-}^{(1/2)}\hat{\omega} = 0$. Expanding $\hat{\omega}$ gives

$$\hat{\omega} = \hat{\omega}_{+} + \theta\hat{\omega}_{z} \tag{9.1}$$

and the equation for holomorphicity in presence of a non-vanishing gravitino field becomes

$$\begin{aligned} &\partial_{\bar{z}}\hat{\omega}_{+} + \frac{1}{2}\chi_{\bar{z}}^{+}\hat{\omega}_{z} = 0 \\ &\partial_{\bar{z}}\hat{\omega} + \frac{1}{2}\partial_{z}(\chi_{\bar{z}}^{+}\hat{\omega}_{+}) = 0 \,. \end{aligned} \tag{9.2}$$

For such forms we can define the line integrals

$$\int_{z}^{w}\hat{\omega} = \int_{z}^{w}(d z\hat{\omega}_{z} - \frac{1}{2}d\bar{z}\chi_{\bar{z}}^{+}\hat{\omega}_{+}) + \theta_{w}\hat{\omega}_{+}(w) - \theta_{z}\hat{\omega}_{+}(z) \,. \tag{9.3}$$

which are invariant under continuous deformations from $z$ to $w$. The line integrals satisfy

$$\mathcal{D}_{+}\int^{z}\hat{\omega} = \hat{\omega} \quad \mathcal{D}_{-}\int^{z}\hat{\omega} = 0$$

For even spin structures there are precisely $h$ solutions to the equation (9.2). With canonical normalization (6.8), they can be written explicitly as

$$(9.4)\qquad \begin{aligned} (\hat{\omega}_z) &= \omega_I(z) - \frac{1}{16\pi^2}\int d^2w d^2y \partial_z \partial_w \log E(z,w) S_\delta(w,y) \chi_{\bar{y}}^{+}(\hat{\omega}_y)_I \\ (\hat{\omega}_+)_I &= -\frac{1}{4\pi}\int d^2w S_\delta(z,w) \chi_{\bar{w}}^{+}(\hat{\omega}_w)_I(w) \, . \end{aligned}$$

The super period matrix $\hat{\Omega}$ is defined as usual by (6.9). We get yet another formula for it

$$(9.5)\qquad \hat{\Omega}_{IJ} = \Omega_{IJ} + \frac{i}{2}\int d^2w \omega_J(w) \chi_{\bar{w}}^{+}(\hat{\omega}_+)_I(w) \, .$$

The case of odd spin structures is more difficult, due to the presence on an additional super abelian differential $\hat{\omega}_0$ generated perturbatively from the holomorphic spinor $h_delta$ of (7.12)

$$(9.8)\qquad \begin{aligned} \hat{\omega}_0 &= (\hat{\omega}_+)_0 + \theta(\hat{\omega}_z)_0 \\ (\hat{\omega}_+)_0(z) &= h_\delta + \frac{1}{4\pi}\int d^2y \hat{S}_\delta(z,y) \chi_{\bar{y}}^{+}(\hat{\omega}_z)_0(y) \\ (\hat{\omega}_z)_0(z) &= \frac{1}{4\pi}\int d^2w \partial_z \partial_w \log E(z,w) \chi_{\bar{w}}^{+}(\hat{\omega}_+)_0(w) \, . \end{aligned}$$

Its periods are given by

$$(9.9)\qquad \oint_{A_I} \hat{\omega}_0 = 0 \quad \oint_{B_I} \hat{\omega}_0 = \hat{\Omega}_{I0} \, .$$

For generic super Riemann surfaces the equation (9.2) has no odd Grassmann valued solutions. The reason is that the zero mode $h_\delta$ imposes a constraint

$$\int d^2z h_\delta \chi_{\bar{z}}^{+} \hat{\omega}_z = 0$$

which due to the Grassmann nature of $\chi_{\bar{z}}^{+}$ forces $\hat{\omega}_z$ to vanish. However one may consider more general solutions of (9.2) which may be meromorphic, or holomorphic with non-trivial monodromy. One such set of solutions $\hat{\omega}_I$ is given be the analogues of (9.4), but with the propagator $S_\delta(z,w)$ replaced by the multiple valued propagator of (7.19). Their monodromy can be evaluated easily

$$(9.10)\qquad \begin{aligned} \hat{\omega}_K(w + A_L) &= \hat{\omega}_K(w) \\ \hat{\omega}_K(w + B_L) &= \hat{\omega}_K(w) + \hat{\Omega}_{K0} \hat{W}_L \hat{\omega}_0(w) \end{aligned}$$

where

$$\hat{W}_K = W_K \left[ \sum_{i=1}^{h} W_I \oint_{A_I} \hat{\omega}_0(\mathbf{w}) \int^{\mathbf{w}} \hat{\omega}_0 \right]^{-1} . \tag{9.11}$$

The super period matrix is no longer well-defined. However we shall still define it by the formula (9.5), keeping in mind that it is a formal object. This will also be the case with the super prime form below in the odd spin structure case. We can view their use at this point as a shorter way of rewriting the amplitudes $\mathcal{F}_\delta^m$ of (7.14)-(7.20). It is however quite likely that a proper setting for such objects can be found, upon which their role may become central to our understanding of superholomorphic function theory.

Next the super prime form $\mathcal{E}_\delta(\mathbf{z}, \mathbf{w})$ generalizes the ordinary prime form, and its main properties are

- superholomorphic dependence on $z, w$ and supermoduli parameters;
- it transforms as $(-1/2, 0) \times (-1/2, 0)$ form in $\mathbf{z}$ and $\mathbf{w}$;
- $\mathcal{E}_\delta(\mathbf{z}, \mathbf{w}) = 0$ and $\mathcal{D}_+ \mathcal{E}_\delta(\mathbf{z}, \mathbf{w}) = 0$ if and only if $\mathbf{z} = \mathbf{w}$;
- it has non-trivial monodromy

$$\begin{aligned} &\mathcal{E}_\delta(\mathbf{z} + A_K, \mathbf{w}) = \mathcal{E}_\delta(\mathbf{z}, \mathbf{w}) \\ &\mathcal{E}_\delta(\mathbf{z} + B_K, \mathbf{w}) = \mathcal{E}_\delta(\mathbf{z}, \mathbf{w}) \exp\left[ -i\pi \hat{\Omega}_{KK} - 2\pi i \int_{\mathbf{z}}^{\mathbf{w}} \hat{\omega}_K \right] \Phi_K \end{aligned} \tag{9.12}$$

with

$$\begin{aligned} &\Phi_K = 1 \quad \delta \text{ even} \\ &\Phi_K = \exp\left[ \int^{\mathbf{z}} \hat{\omega}_0 \hat{W}_K \int^{\mathbf{w}} \hat{\omega}_0 \right] \quad \delta \text{ odd} . \end{aligned} \tag{9.13}$$

In components the full expression for the super prime form is

$$\begin{aligned} \log \mathcal{E}_\delta(\mathbf{z}, \mathbf{w}) &= \log E(z, w) - \theta_z \theta_w \hat{S}_\delta(z, w) \\ &- \frac{1}{4\pi} \theta_z \int d^2 y \chi_{\bar{y}}^+ \partial_y \log \frac{E(y, w)}{E(y, z)} \hat{S}_\delta(z, y) \\ &- \frac{1}{4\pi} \theta_w \int d^2 y \partial_y \log \frac{E(y, z)}{E(y, w)} \hat{S}_\delta(w, y) \\ &\frac{1}{32\pi^2} \int d^2 x d^2 y \chi_{\bar{x}}^+ \partial_x \log \frac{E(x, z)}{E(x, w)} \hat{S}_\delta(x, y) \chi_{\bar{y}}^+ \partial_y \log \frac{E(y, w)}{E(y, z)} \end{aligned} \tag{9.14}$$

where for odd spinstructure it is understood that we use multiple-valued fermionic propagators. This completes our description of super Riemann sufrace function theory.

## 10. SUPERMODULI SPACE AND THE SPACE OF SUPER PERIOD MATRICES

We return now to the geometry of supermoduli space, from the viewpoint of super period matrices. Let $\hat{\mathcal{P}}_h$ be the space of matrices $\hat{\Omega}_{IJ}$ which arise as super period matrices of some two-dimensional supergeometry. The space of supergeometries admits the natural projection

$$\begin{array}{c}(g_{m,n}, \chi_m^\alpha)\\ \downarrow\\ \hat{\Omega}\end{array} \tag{10.1}$$

onto $\hat{\mathcal{P}}$. A point in $s\mathcal{M}_h$ represented by a supergeometry $(g_{m,n}, \chi_m^\alpha)$ can be represented also by $(\hat{\Omega}, \chi_m^\alpha)$. Supersymmetry transformations acting on the space of supergeometries leave the corresponding super period matrices invariant. The modular group $Sp(2h, \mathbf{Z})$ acts on the on the space of supergeometries and the space of super period matrices $\hat{\mathcal{P}}_h$, since it corresponds to a change of homology bases, and one definition of $\hat{\Omega}$ exhibits it as the periods of dual bases of super abelian differentials (6.8) (6.9)

$$\begin{aligned} &M : (\hat{\Omega}, \chi_m^\alpha) \to (M\hat{\Omega}, M\chi_m^\alpha)\\ &M\hat{\Omega} = (A\hat{\Omega} + B)(C\hat{\Omega} + D)^{-1} . \end{aligned} \tag{10.2}$$

Thus we can take quotients of the above projection by both supersymmetry and modular group actions, and obtain the projection between coset spaces

$$\begin{array}{ccc}(g_{mn}, \chi_m^\alpha) & \in & \frac{\text{Super geometries}}{\{SUSY \times \text{Diff} \times \text{Weyl} \times s\text{Weyl}\}} = s\mathcal{M}_h\\ \downarrow & & \downarrow\\ \hat{\Omega} & \in & \hat{\mathcal{P}}_h .\end{array} \tag{10.3}$$

We now return to the issue of the superstring measure. We have seen that due to non-invariance of the period matrix $\Omega_{IJ}$ under supersymmetry transformations, we cannot parametrize supermoduli space by patches of the form $(g_{mn} \in B_\alpha, \sum_{a=1}^{2h-2} \zeta_a \chi_a)$ with $B_a$ a covering parch for moduli space. Besides, in the DeWitt notion of supermanifolds which we rely on for supergravity, even the metric $g_{mn}$ and hence the period matrix $\Omega_{IJ}$ has inherent soul coordinates. Thus even a statement linking directly $\Omega_{IJ}$ and moduli space at this point is inappropriate. Rather we should start form a covering $\hat{B}_\alpha$ of the space $\hat{\mathcal{P}}_h$ of super period matrices. If the covering is fine enough, it is reasonable to expect a generic choice of $\chi_a$ on each patch will give a parametrization for supermoduli space. We can now use these slices for gauge-fixing, using the gauge-fixing methods and holomorphic splittin of Sections 6 and 7. Invariance of the super period matrix under supersymmetry allows us to integrate out the odd variables $\zeta_a$ without any ambiguities. The answers will agree exactly on overlapping parches, and not just up to a total derivative. In this way we arrive at an integral on the space $\hat{\mathcal{P}}_h$. A key observation now is that integrals on supermanifolds should be viewed as a multi-dimensional versions of line integrals over cycles within the supermanifld, which can be deformed to the body. Thus the integrals over the space $\hat{\mathcal{P}}_h$ can now be deformed to integrals over the regular moduli space of Riemann surfaces. In practice, what this means is that after integrating out the $\zeta_a$ we can just identify $\hat{\Omega}_{IJ}$ with a bona fide period matrix. In this way we can reduce superstring scattering amplitudes to integrals of modular forms over moduli space.

Implementation of this program requires a number of tools. Although we now know how to express any correlation function in terms of $\hat{\Omega}$, we still need good formulations for the determinants of the $\bar{\partial}$ operators themselves (c.f. (7.14)). These can of course be written in terms of $\Omega$ by bosonization, but the passage from $\Omega$ to $\hat{\Omega}$ is often unwieldy. The superghosts $\beta, \gamma$ also demand special care, because they are a commuting first order system. Even choices of branch points as insertion points cause degeneracy problems which need regularization. Nevertheless these are more thecnical rather than conceptual problems, and we can be optimitistic that with the correct rules of holomorphic splitting and tratment of odd variables, we shall soon arrive at a full proof of consistency and finiteness of superstring perturbation theory.

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