Modular graph functions

Eric D'Hoker
Mani L. Bhaumik Institute for Theoretical Physics
Department of Physics and Astronomy, UCLA

Arithmetic, geometry, and modular forms
A conference in honor of Bill Duke
Institute for Mathematical Research, ETH, Zürich
June 2019

Happy Birthday

Bibliography

- ED, M.B. Green and P. Vanhove, JHEP **1508** (2015) 041, arXiv:1502.06698,
  On the modular structure of the genus-one Type II superstring low energy expansion
- ED, M.B. Green, O. Gurdogan and P. Vanhove, CNTP **11** (2017) 165, arXiv:1512.06779,
  Modular Graph Functions
- ED and M.B. Green, Journal of Number Theory, 189 (2018) 25, arXiv:1603.00839,
  Identities between Modular Graph Forms
- ED and J. Kaidi, JHEP **1611** (2016) 051, arXiv:1608.04393,
  Hierarchy of Modular Graph Identities
- ED and W. Duke, JNT **192** (2018) 1-36, arXiv:1708.07998,
  Fourier series of modular graph functions
- ED, M.B. Green, B. Pioline and R. Russo, JHEP 1501 (2015) 031, arXiv:1405.6226,
  Matching the $D^6\mathcal{R}^4$ interaction at two-loops
- ED, M.B. Green and B. Pioline, CMP **366** (2019) 927, arXiv:1712.06135,
  Higher genus modular graph functions, string invariants, and their exact asymptotics

$SL(2,\mathbb{Z})$ in physics

- **Supersymmetric Yang-Mills theory**
  - ⋆ maximal supersymmetry: Montonen-Olive duality symmetry (1976)
  - ⋆ half-maximal supersymmetry: key role in Seiberg-Witten theory (1994)
- **Type IIB superstring theory**
  - ⋆ has $SL(2,\mathbb{Z})$ duality symmetry (Hull & Townsend 1995; Green & Schwarz 1995)
- **Quantum probabilities in string theory** (this talk)
  - ⋆ given by sums over randomly fluctuating Riemann surfaces
  - ⋆ surfaces of genus-one contribution (Shapiro 1972)

Focus of this talk

- **String theory naturally generalizes real-analytic Eisenstein series**

- **Modular graph function for a Riemann surface of genus one**
  - ⋆ maps a graph to an $SL(2,\mathbb{Z})$-invariant function on half-plane $\mathcal{H}_1$

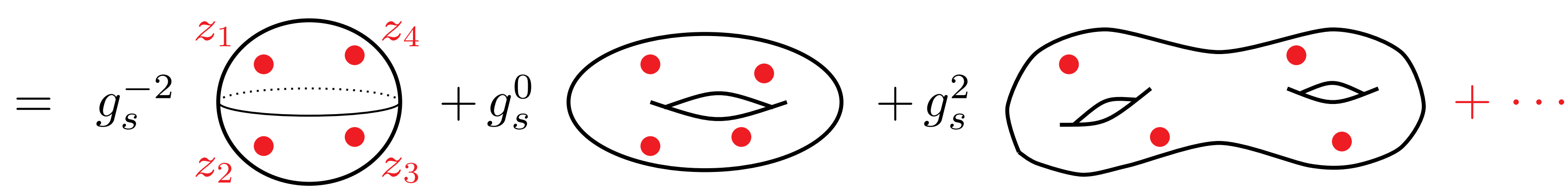
- **Generalization to Riemann surfaces of higher genus**
  - ⋆ maps a graph to a function on the moduli space of Riemann surfaces
  - ⋆ generalizes invariants of Kawazumi and Zhang (2008)

Sums over Riemann surfaces

- **Quantum mechanics predicts probabilities**

$$\text{probability} = \big|\ \text{probability amplitude}\ \big|^2$$

- **Probability amplitude in string theory**



⋆ topological expansion in the string coupling $g_s \in \mathbb{R}$ parameter

⋆ each marked point z_i represents an incoming or outgoing string
⋆ for a give physical process, the number of marked points N is fixed

⋆ for each genus h
  – integrate over N marked points
  – integrate over the moduli space $\mathcal{M}_h$ of Riemann surfaces

Genus zero

- **Genus-zero graviton amplitudes are integrals of the type**

$$\prod_{i=1}^{N-3} \int_{\mathbb{C}} d^2 z_i \, |z_i|^{-2-2s_{i,N-1}} |1-z_i|^{-2s_{iN}} \prod_{j \neq i}^{N-3} |z_i - z_j|^{-2s_{ij}}$$

  - ⋆ three points chosen at $0, 1, \infty$ by conformal invariance
  - ⋆ parameters s_{ij} are kinematic variables
    = the square of energy/momentum of incoming and outgoing strings
  - ⋆ satisfy $s_{ji} = s_{ij}$ and $\sum_{i=1}^{N} s_{ij} = 0$ with $s_{ii} = 0$ for all $i, j = 1, \cdots, N$;
  - ⋆ Meromorphic in s_{ij} with simple poles at non-negative integers

- **Genus-zero four-graviton amplitude**

$$\mathcal{A}_4^{(0)}(s_{ij}) = \frac{1}{s_{12}\, s_{13}\, s_{14}} \frac{\Gamma(1-s_{12})\, \Gamma(1-s_{13})\, \Gamma(1-s_{14})}{\Gamma(1+s_{12})\, \Gamma(1+s_{13})\, \Gamma(1+s_{14})}$$

Genus-one

- **Integral over N marked points $z_i \in \Sigma$**
  - ⋆ torus $\Sigma = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ with modulus $\tau = \tau_1 + i\tau_2 \in \mathcal{H}_1$, $\tau_1, \tau_2 \in \mathbb{R}$

$$\mathcal{B}_N^{(1)}(s_{ij}|\tau) = \prod_{k=1}^{N} \int_\Sigma \frac{d^2 z_k}{\tau_2} \exp\Big\{ \sum_{1 \leq i<j \leq N} s_{ij}\, g(z_i - z_j|\tau) \Big\}$$

  - ⋆ Scalar Green function in terms of $z = u + v\tau$, $u, v \in \mathbb{R}$

$$g(z|\tau) = \sum_{m,n\in\mathbb{Z}}{}^{'} \frac{\tau_2}{\pi |m\tau+n|^2} e^{2\pi i(mu-nv)}$$

  - ⋆ $\mathcal{B}_N^{(1)}(s_{ij}|\tau)$ is invariant under the modular group $SL(2,\mathbb{Z})$
  - ⋆ has simple and double poles at $s_{ij} \in \mathbb{N}$; holomorphic in s_{ij} for $|s_{ij}| < 1$

- **String amplitude for $N=4$ is given by** (Green, Schwarz, 1982)

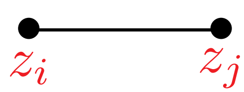
$$\mathcal{A}_4^{(1)}(s_{ij}) = \int_{\mathcal{M}_1} \frac{d^2\tau}{\tau_2^2} \mathcal{B}_4^{(1)}(s_{ij}|\tau) \qquad \mathcal{M}_1 = PSL(2,\mathbb{Z})\backslash\mathcal{H}_1$$

  - ⋆ This amplitude plays a central role in string theory
  - ⋆ Analytic continuation in s_{ij} was proven to exist (ED & Phong 1994)

Graphical Representation of Taylor series of $\mathcal{B}_N^{(1)}(s_{ij}|\tau)$

- **Absolute convergence of the integral $\mathcal{B}_N^{(1)}(s_{ij}|\tau)$ for $|s_{ij}| < 1$ and fixed τ**
  - ⋆ allows for Taylor expansion in the variables s_{ij}
    - $=$ physically corresponds to the "low energy expansion"

- **Represented by Feynman graphs** (Green, Russo, Vanhove 2008)
  - ⋆ Each integration point z_i on Σ is represented by a vertex •
  - ⋆ Each Green function by an edge between vertices z_i and z_j



$$= \quad g(z_i - z_j|\tau)$$

  - ⋆ Each vertex is integrated over Σ
  - ⋆ To a graph with w edges we assign *weight* w

- **Reducibility** : A graph which becomes disconnected
  - ⋆ upon cutting one edge vanishes by $\int_\Sigma g = 0$
  - ⋆ upon removing one vertex factorizes into the product of its components

Modular graph functions

- **To each graph is associated a real-analytic modular function**

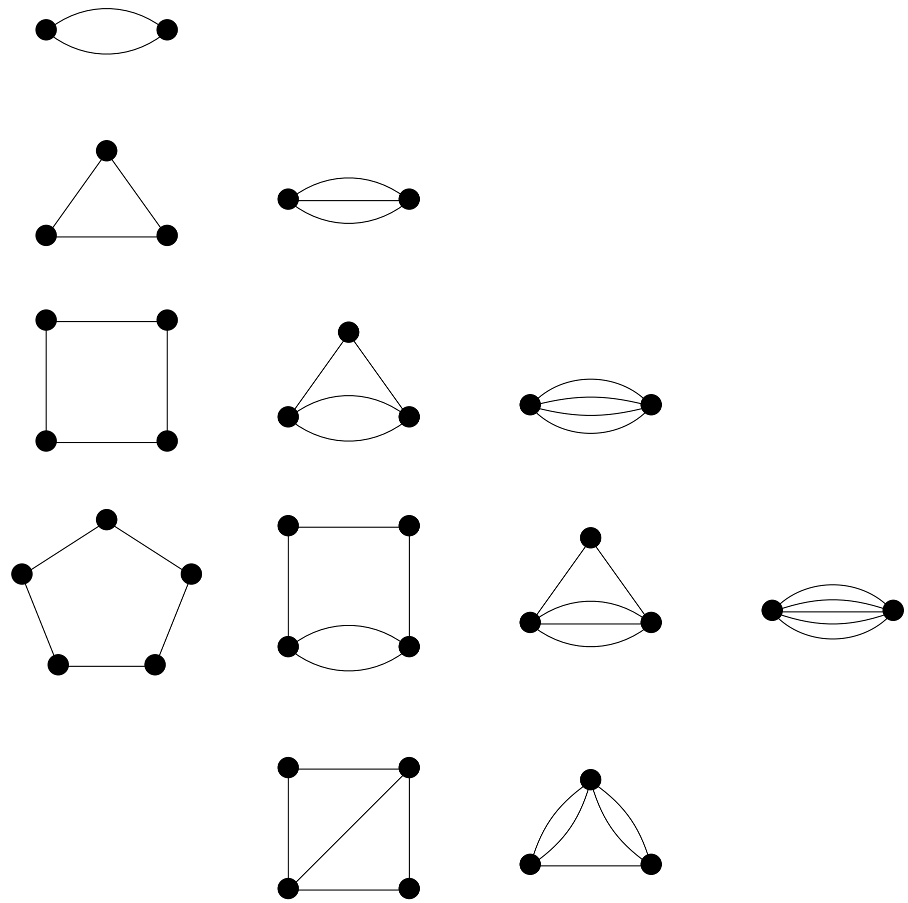
  since $\mathcal{B}_N^{(1)}(s_{ij}|\tau)$ is a modular function, so are its Taylor coefficients

$w = 2$

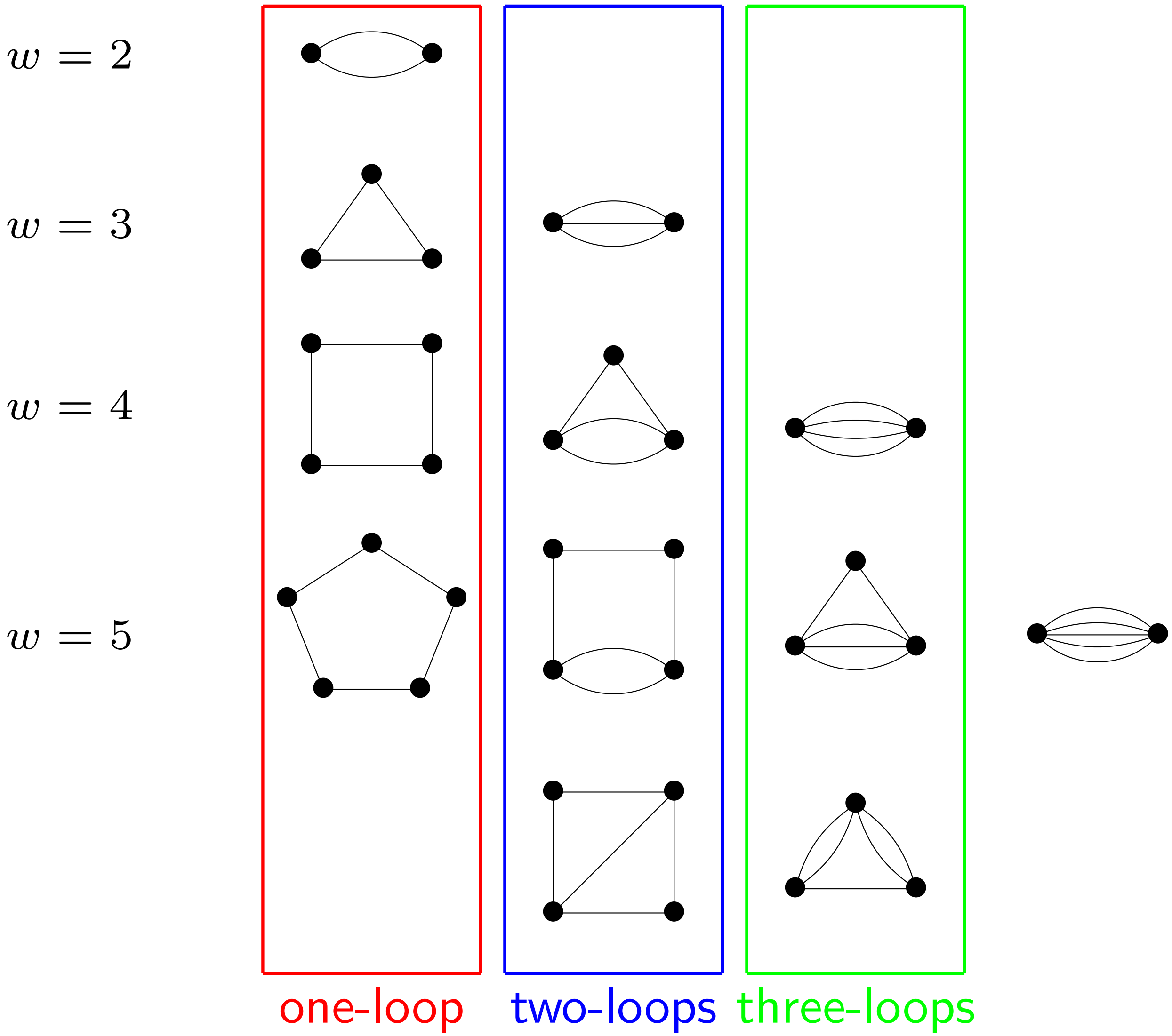
$w = 3$

$w = 4$

$w = 5$



Modular graph functions organized by loop order



One-loop modular graph functions: Eisenstein series

- **One-loop weight w graph has w edges and w vertices**

$$\prod_{i=1}^{w} \int_\Sigma \frac{d^2 z_i}{\tau_2} g(z_i - z_{i+1}|\tau) = \sum_{p \in \Lambda'} \frac{\tau_2^w}{\pi^w |p|^{2w}} = E_w(\tau)$$

⋆ with $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ and $\Lambda' = \Lambda \setminus \{0\}$ or $p = m + \tau n,\ (m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}$
⋆ invariant under the modular group $SL(2,\mathbb{Z})$

- **Real-analytic Eisenstein series $E_s(\tau)$**

⋆ Expansion near the cusp $\tau \to i\infty$

$$E_s = \frac{2\zeta(2s)}{\pi^s}\tau_2^s + \frac{2\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)\,\pi^{s-\frac{1}{2}}}\,\tau_2^{1-s} + \mathcal{O}(e^{-2\pi\tau_2})$$

⋆ Eigenfunction of the Laplace-Beltrami operator $\Delta = 4\tau_2^2 \partial_\tau \partial_{\bar\tau}$ on $\mathcal{H}_1$

$$\Delta E_s = s(s-1)E_s$$

Two-loop modular graph functions

- **Two-loop graphs evaluate to a multiple Kronecker-Eisenstein series**

$$C_{a_1,a_2,a_3}(\tau) = \sum_{p_1,p_2,p_3 \in \Lambda'} \delta\Big(\sum_{r=1}^3 p_r\Big) \prod_{r=1}^3 \left(\frac{\tau_2}{\pi |p_r|^2}\right)^{a_r}$$

  ⋆ Absolutely convergent for $a_r \in \mathbb{N}$, of weight $w = a_1 + a_2 + a_3$
  ⋆ invariant under $SL(2,\mathbb{Z})$; invariant under permutations of a_1, a_2, a_3

- **Theorem 1** (ED & Bill Duke 2017)
  Expansion near cusp $\tau \to i\infty$: Laurent polynomial in τ_2 of degree $(w, 1-w)$

$$C_{a_1,a_2,a_3}(\tau) = c_w (-4\pi\tau_2)^w + \frac{c_{2-w}}{(4\pi\tau_2)^{w-2}} + \sum_{k=1}^{w-1} \frac{c_{w-2k-1}\, \zeta(2k+1)}{(4\pi\tau_2)^{2k+1-w}} + \mathcal{O}(e^{-2\pi\tau_2})$$

  where $c_w, c_{w-2k-1} \in \mathbb{Q}$ are explicitly known with

$$c_{2-w} = \sum_{k=1}^{w-2} \gamma_k\, \zeta(2k+1)\zeta(2w-2k-3) \qquad\qquad \gamma_k \in \mathbb{Z}$$

- **Poincaré series for $\Gamma_\infty \backslash SL(2,\mathbb{Z})$, and Fourier series** (ED & Kaidi 2019)

System of differential identities

- **Theorem 2** (ED, Green, Vanhove 2015)
  Two-loop modular graph functions $C_{a,b,c}$ *of weight* $w = a + b + c$, $a, b, c \in \mathbb{N}$ *obey a system of differential equations of uniform weight* w

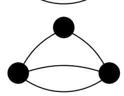
$$2\Delta C_{a,b,c} = 2ab\, C_{a+1,b-1,c} + ab\, C_{a+1,b+1,c-2} - 4ab\, C_{a+1,b,c-1} + a(a-1)\, C_{a,b,c} + \text{ 5 permutations of } (a,b,c)$$

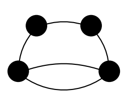
where $\Delta = 4\tau_2^2 \partial_\tau \partial_{\bar\tau}$, *and*

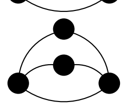
$$C_{a,b,0} = E_a E_b - E_{a+b} \qquad\qquad C_{a,b,-1} = E_{a-1} E_b + E_a E_{b-1}$$

- **Examples suggest "eigenvalues"** $s(s-1)$ **and** $s \in \mathbb{N}$

$C_{1,1,1} =$  $\qquad (\Delta - 0) C_{1,1,1} = 6 E_3$

$C_{2,1,1} =$  $\qquad (\Delta - 2) C_{2,1,1} = 9 E_4 - E_2^2$

$C_{3,1,1} =$  $\qquad (\Delta - 6) C_{3,1,1} = 3 C_{2,2,1} + 16 E_5 - 4 E_2 E_3$

$C_{2,2,1} =$  $\qquad (\Delta - 0) C_{2,2,1} = 8 E_5$

System of differential identities (cont'd)

- **Theorem 3** (ED, Green, Vanhove 2015)

For every weight $w = a + b + c \geq 3$ the two-loop modular graph functions $C_{a,b,c}(\tau)$ are linear combinations of modular functions $\mathfrak{C}_{w;s;\mathfrak{p}}(\tau)$ satisfying

$$\big(\Delta - s(s-1)\big)\mathfrak{C}_{w;s;\mathfrak{p}} = \mathfrak{F}_{w;s;\mathfrak{p}}$$

$$s = w - 2\mathfrak{m} \qquad \mathfrak{m} = 1, \cdots, \left[\frac{w-1}{2}\right] \qquad \mathfrak{p} = 0, \cdots, \left[\frac{s-1}{3}\right]$$

and $\mathfrak{F}_{w;s;\mathfrak{p}}$ is a polynomial of degree 2 and weight w in $E_{s'}$ with $2 \leq s' \leq w$.

(Constructive proof by generating function methods and $SO(2,1;\mathbb{R})$ representation theory)

w	s	$s(s-1)^{(\text{degeneracy})}$
3	1	$0^{(1)}$
4	2	$2^{(1)}$
5	$1, 3$	$0^{(1)} \oplus 6^{(1)}$
6	$2, 4$	$2^{(1)} \oplus 12^{(2)}$
7	$1, 3, 5$	$0^{(1)} \oplus 6^{(1)} \oplus 20^{(2)}$
8	$2, 4, 6$	$2^{(1)} \oplus 12^{(2)} \oplus 30^{(2)}$
9	$1, 3, 5, 7$	$0^{(1)} \oplus 6^{(1)} \oplus 20^{(2)} \oplus 42^{(3)}$

System of algebraic identities

- **Differential identities for odd weight w and zero eigenvalue**
  - ⋆ the inhomogeneous term is linear in E_w for all odd w, e.g.

$$\Delta C_{1,1,1} = 6E_3 \qquad \Delta(C_{3,3,1} + C_{3,2,2}) = 18E_7$$

$$\Delta C_{2,2,1} = 8E_5 \qquad \Delta(9C_{4,4,1} + 18C_{4,3,2} + 4C_{3,3,3}) = 288E_9$$

  - ⋆ Integrate using $\Delta E_s = s(s-1)E_s$
  - ⋆ fix integration constant using asymptotics at cusp

- **Algebraic identities at weight w mixing loop orders**, e.g.

$$C_{1,1,1} = E_3 + \zeta(3) \qquad C_{3,3,1} + C_{3,2,2} = \frac{3}{7}E_7 + \frac{\zeta(7)}{252}$$

$$C_{2,2,1} = \frac{2}{5}E_5 + \frac{\zeta(5)}{30} \qquad 9C_{4,4,1} + 18C_{4,3,2} + 4C_{3,3,3} = 4E_9 + \frac{\zeta(9)}{240}$$

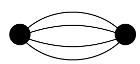
  - ⋆ $C_{1,1,1} = E_3 + \zeta(3)$ proven by direct summation (Zagier, unpublished)
  - ⋆ One algebraic identity for each odd weight. (ED, Green, Vanhove 2015)

Modular graph functions at higher loop order

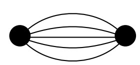
- **Expansion near the cusp** $\tau \to i\infty$
  - ⋆ Laurent polynomial in τ_2 of degree $(w, 1-w)$ $+\mathcal{O}(e^{-2\pi\tau_2})$
  - ⋆ coefficients are "single-valued" multiple zeta-values (ED, Green, Vanhove 2015)
  - ⋆ coefficients include "irreducible" multiple zeta-values (Zerbini 2017)

- **Modular graph functions satisfy algebraic identities of uniform weight**

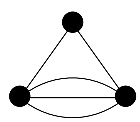
e.g.



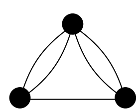
$$= 24C_{2,1,1} + 3E_2^2 - 18E_4$$



$$= 60C_{3,1,1} + 10E_2C_{1,1,1} - 48E_5 + 16\zeta(5)$$



$$= \frac{15}{2}C_{3,1,1} + 3E_2E_3 - \frac{69}{10}E_5 + \frac{7}{40}\zeta(5)$$



$$= 2C_{3,1,1} - \frac{2}{5}E_5 + \frac{3}{10}\zeta(5)$$

- **Laplace-Beltrami operator for 3 loops and higher**
  - ⋆ generically no longer maps the space of modular graph forms into itself

Arbitrary number of loops and exponents

- **Modular graph forms** (ED & Green 2016)

A decorated graph (Γ, A, B) *with* V *vertices and* R *edges consists of*

⋆ *connectivity matrix with components* $\Gamma_{v\,r}$, $v = 1, \cdots, V$, $r = 1, \cdots, R$

⋆ *decoration of the edges by "exponents"* $a_r, b_r \in \mathbb{N}$ *for* $r = 1, \cdots, R$

$A = [a_1, \cdots, a_R]$ and $B = [b_1, \cdots, b_R]$

To the decorated graph (Γ, A, B) *we associate a function on* $\mathcal{H}_1$

$$\mathcal{C}_\Gamma \begin{bmatrix} A \\ B \end{bmatrix} (\tau) = \sum_{p_1, \cdots, p_R \in \Lambda'} \left(\prod_{r=1}^{R} \frac{(\tau_2/\pi)^{a_r}}{(p_r)^{a_r} \, (\bar{p}_r)^{b_r}} \right) \prod_{v=1}^{V} \delta \left(\sum_{r=1}^{R} \Gamma_{v\,r} p_r \right)$$

- **Transformation under** $SL(2, \mathbb{Z})$

$$\mathcal{C}_\Gamma \begin{bmatrix} A \\ B \end{bmatrix} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta} \right) = (\gamma\bar{\tau} + \delta)^{\mu} \, \mathcal{C}_\Gamma \begin{bmatrix} A \\ B \end{bmatrix} (\tau) \qquad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \mathbb{Z})$$

⋆ modular weight $(0, \mu)$ with $\mu = \sum_r (b_r - a_r)$: "modular graph form"

⋆ when $\mu \neq 0$ there is no canonical normalization for powers of τ_2

⋆ $A = B \Rightarrow \mu = 0$ recover modular graph functions for Green function g

Algebraic and differential identities on modular graph forms

- **Momentum conservation at each vertex v implies an algebraic identity**

$$\sum_{r=1}^{R} \Gamma_{v\,r}\, \mathcal{C} \begin{bmatrix} A - S_r \\ B \end{bmatrix} = \sum_{r=1}^{R} \Gamma_{v\,r}\, \mathcal{C} \begin{bmatrix} A \\ B - S_r \end{bmatrix} = 0$$

where $A = [a_1 \ \cdots \ a_R]$, $B = [b_1 \ \cdots \ b_R]$ and $S_r = [0_{r-1}\ 1\ 0_{R-r}]$

- **The Maass operator $\nabla = 2i\tau_2^2 \partial_\tau$ maps between modular graph forms**

$$\nabla\, \mathcal{C}_\Gamma \begin{bmatrix} A \\ B \end{bmatrix} = \sum_{r=1}^{R} a_r\, \mathcal{C}_\Gamma \begin{bmatrix} A + S_r \\ B - S_r \end{bmatrix}$$

of modular weight $(0, \mu) \to (0, \mu - 2)$, and similarly for its complex conjugate

Returning to identities on modular graph functions

- **Algorithm to search for and prove identities on modular graph functions**
  - ⋆ to search for and prove an identity $F = 0$
  - ⋆ solve the weaker condition $\nabla^w F = 0$ for general F of weight w
  - ⋆ using relations between *holomorphic modular forms*, and

- **Lemma 1** *Let F be a modular function with polynomial growth.*
  If F satisfies $\nabla^n F = 0$ for an integer $n \geq 1$, then F is independent of τ

$\Longrightarrow$ **All algebraic identities between modular functions of weight** $w \leq 6$

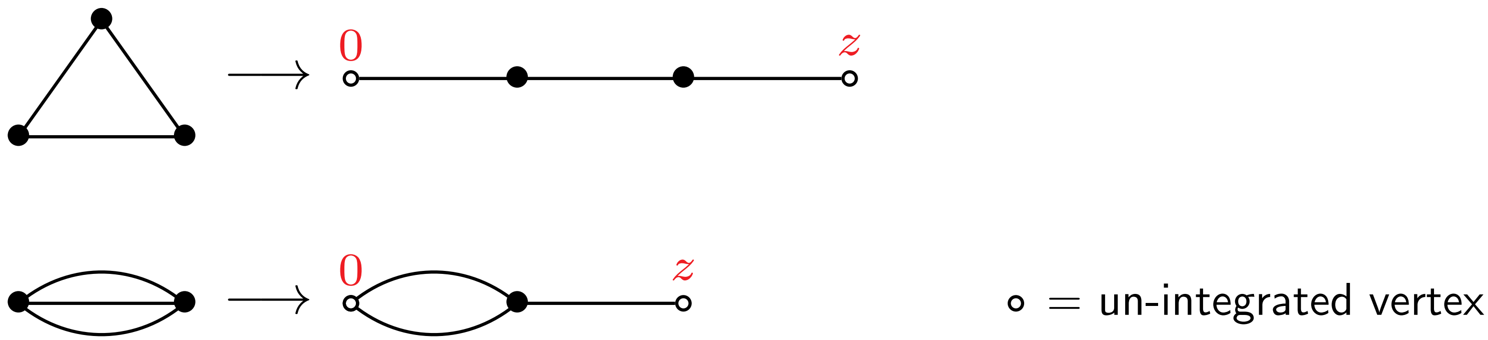
⋆ and many identities at weight 7

(ED & Green 2016; ED & Kaidi 2016)

see also Basu 2016; Kleinschmidt & Verschinnin 2017

Single-valued elliptic multiple polylogarithms

- **Severing a vertex of a modular graph function** (ED, Green, Gurdogan, Vanhove)
  - ⋆ produces a real-valued (single-valued) elliptic function on Σ



- **Linear chain graphs are single-valued elliptic polylogarithms** (Zagier 1990)
  - ⋆ generalizing the Bloch-Wigner dilogarithm
- **Higher loop graphs produce single-valued elliptic multiple polylogarithms**
  - ⋆ Modular graph functions may be viewed as special value at $z = 0$
  - ⋆ and therefore provide generalizations of "single-valued multiple zeta-values"
- **Relation with equivariant iterated integrals**

  (Enriquez 2013; Brown 2017; Broedel, Mafra, Schlotterer 2018)

Higher genus

- **How to generalize the genus-one formula to higher genus ?**
  - ⋆ recall the genus-one generating function

$$\mathcal{B}_N^{(1)}(s_{ij}|\tau) = \prod_{i=1}^{N} \int_{\Sigma_1} \frac{d^2 z_i}{\tau_2} \, \exp\left\{ \sum_{1 \leq i<j \leq N} s_{ij} \, g(z_i - z_j) \right\}$$

- **Compact Riemann surface Σ of genus $h \geq 2$ without boundary**
  - ⋆ we need a scalar Green function $G(z_i, z_j)$
  - ⋆ and a measure $d\mu_N$ on Σ^N

$$\mathcal{B}_N^{(h)}(s_{ij}|\Sigma) = \int_{\Sigma^N} d\mu_N \, \exp\left\{ \sum_{1 \leq i<j \leq N} s_{ij} \, G(z_i, z_j) \right\}$$

Compact Riemann surfaces Σ of genus h

- **Homology and cohomology**
  - ⋆ One-cycles $H_1(\Sigma, \mathbb{Z}) \approx \mathbb{Z}^{2h}$ with intersection pairing $\mathfrak{J}(\cdot,\cdot) \to \mathbb{Z}$
  - ⋆ Canonical basis $\mathfrak{J}(\mathfrak{A}_I, \mathfrak{A}_J) = \mathfrak{J}(\mathfrak{B}_I, \mathfrak{B}_J) = 0$, $\mathfrak{J}(\mathfrak{A}_I, \mathfrak{B}_J) = \delta_{IJ}$ for $1 \leq I, J \leq h$
  - ⋆ Canonical dual basis of holomorphic one-forms ω_I in $H^{(1,0)}(\Sigma)$

$$\oint_{\mathfrak{A}_I} \omega_J = \delta_{IJ} \qquad\qquad \oint_{\mathfrak{B}_I} \omega_J = \Omega_{IJ}$$

  - ⋆ Period matrix Ω obeys Riemann relations $\Omega^t = \Omega, \ \mathrm{Im}\,(\Omega) > 0$

- **Modular group** $Sp(2h, \mathbb{Z}) : H_1(\Sigma, \mathbb{Z}) \to H_1(\Sigma, \mathbb{Z})$ **leaves** $\mathfrak{J}(\cdot,\cdot)$ **invariant**

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad M^t \mathfrak{J} M = \mathfrak{J} \qquad \begin{pmatrix} \mathfrak{B} \\ \mathfrak{A} \end{pmatrix} \to M \begin{pmatrix} \mathfrak{B} \\ \mathfrak{A} \end{pmatrix}$$

  – action on 1-forms ω and period matrix Ω given by

$$\begin{aligned} \omega &\to \omega\,(C\Omega + D)^{-1} \\ \Omega &\to (A\Omega + B)\,(C\Omega + D)^{-1} \end{aligned}$$

Modular graph functions for arbitrary genus

- **Canonical metric on Σ = pull-back of flat metric on Jacobian $J(\Sigma)$**
  - ⋆ modular invariant and smooth canonical volume form on Σ

$$\kappa = \frac{i}{2h} \sum_{I,J} Y^{-1}_{IJ} \, \omega_I \wedge \overline{\omega_J} \qquad \int_\Sigma \kappa = 1 \qquad Y = \operatorname{Im} \Omega$$

- **The Arakelov Green function $G(z,w)$ is defined by**

$$\partial_{\bar w} \partial_w G(w,z) = -\pi \delta(w,z) + \pi \kappa(w) \qquad \int_\Sigma \kappa \, G = 0$$

- **"Natural" generating function for higher genus modular graph functions**

$$\mathcal{C}^{(h)}_N (s_{ij}|\Sigma) = \int_{\Sigma^N} \prod_{i=1}^N \kappa(z_i) \ \exp \left \{ \sum_{1 \leq i < j \leq N} s_{ij} \, G(z_i,z_j) \right \}$$

  - ⋆ Integrals absolutely convergent for $|s_{ij}| < 1$; analytic near $s_{ij}=0$
  - ⋆ Depend only on Σ, not on specific Ω chosen to represent Σ
  - ⋆ Taylor coeffs in s_{ij} give *modular graph functions* (ED, Green, Pioline 2017)

Genus-two string amplitude

- **Actually ... genus 2 string amplitude does NOT correspond to** $\mathcal{C}_4^{(2)}(s_{ij}|\Sigma)$
  - ⋆ volume form κ is unique on Σ
  - ⋆ but κ^N is not unique on Σ^N for $N \geq 2$

- **Genus-two four-graviton string amplitude corresponds to** (ED & Phong 2005)

$$\mathcal{B}_4^{(2)}(s_{ij}|\Sigma) = \int_{\Sigma^4} \frac{\mathcal{Y} \wedge \bar{\mathcal{Y}}}{(\det Y)^2} \exp\Big\{ \sum_{1 \leq i < j \leq 4} s_{ij}\, G(z_i, z_j) \Big\}$$

- **Measure given by a holomorphic** $(1,0)^{\otimes 4}$ **form** $\mathcal{Y}$ **on** Σ^4

$$\mathcal{Y} = (s_{14} - s_{13})\Delta(z_1, z_2) \wedge \Delta(z_3, z_4) + 2 \text{ cycl perms of } (2,3,4)$$

⋆ where Δ is a holomorphic $(1,0)^{\otimes 2}$ form on Σ^2

$$\Delta(z_i, z_j) = \omega_1(z_i) \wedge \omega_2(z_j) - \omega_2(z_i) \wedge \omega_1(z_j)$$

⋆ The volume form $\mathcal{Y} \wedge \bar{\mathcal{Y}}/(\det Y)^2$ is $Sp(4,\mathbb{Z})$-invariant,

⋆ produces a well-defined $\mathcal{B}_4^{(2)}(s_{ij}|\Sigma)$.

Taylor expansion of genus two amplitude

- **Low energy expansion of the genus-two four graviton $\mathcal{B}_4^{(2)}$**

$$\mathcal{B}_4^{(2)}(s_{ij}|\Sigma) = 32\sigma_2 + 64\sigma_3\varphi(\Sigma) + 32\sigma_4\psi(\Sigma) + \mathcal{O}(s_{ij}^5)$$

  ⋆ where $\sigma_k = s_{12}^k + s_{13}^k + s_{14}^k$.

- **Theorem 4** (ED & Green 2013)
  The coefficient $\varphi(\Sigma)$ is the Kawazumi-Zhang invariant for genus two

$$\varphi(\Sigma) = -\frac{1}{4}\sum_{I,J,K,L} Y_{IL}^{-1}Y_{JK}^{-1}\int_{\Sigma^2} G(x,y)\,\omega_I(x)\overline{\omega_J(x)}\omega_K(y)\overline{\omega_L(y)}$$

  ⋆ introduced as a spectral invariant (Kawazumi 2008; Zhang 2008)
  ⋆ related to the genus-two Faltings invariant (De Jong 2010)

- **New higher invariants at every order in s_{ij} , e.g.**

$$\psi(\Sigma) = \int_{\Sigma^4} \frac{|\Delta(1,2)\Delta(3,4)|^2}{(\det Y)^2}\Big(G(1,4) + G(2,3) - G(1,3) - G(2,4)\Big)^2$$

Modular geometry and differential equation

- **Siegel half space** $\mathcal{H}_h = \{\Omega \in \mathbb{C}^{h^2}, \, \Omega^t = \Omega, \, Y = \mathrm{Im}\,(\Omega) > 0\} = \frac{Sp(2h,\mathbb{R})}{U(h)}$
  - ⋆ with $Sp(2h, \mathbb{R})$ invariant Kähler metric

$$ds^2 = \sum_{I,J,K,L=1,\cdots,h} Y^{-1}_{IJ} \, d\bar\Omega_{JK} \, Y^{-1}_{KL} \, d\Omega_{LI}$$

  - ⋆ and Laplace-Beltrami operator on $\mathcal{H}_h$

$$\Delta = \sum_{I,J,K,L} 4 \, Y_{IK} \, Y_{JL} \, \bar\partial^{IJ} \, \partial^{KL} \qquad \qquad \partial^{IJ} = \frac{1}{2}(1+\delta^{IJ}) \frac{\partial}{\partial \Omega_{IJ}}$$

- **Moduli space of genus-two surfaces** $\mathcal{M}_2 = \mathcal{H}_2 / Sp(4, \mathbb{Z})$ (minus diagonal Ω)

- **Theorem 5** (ED, Green, Pioline, Russo 2014; see also Kawazumi 2008)
  $\varphi(\Sigma)$ *satisfies the following inhomogeneous eigenvalue equation on* $\overline{\mathcal{M}_2}$

$$(\Delta - 5)\varphi = -2\pi \delta_{SN}$$

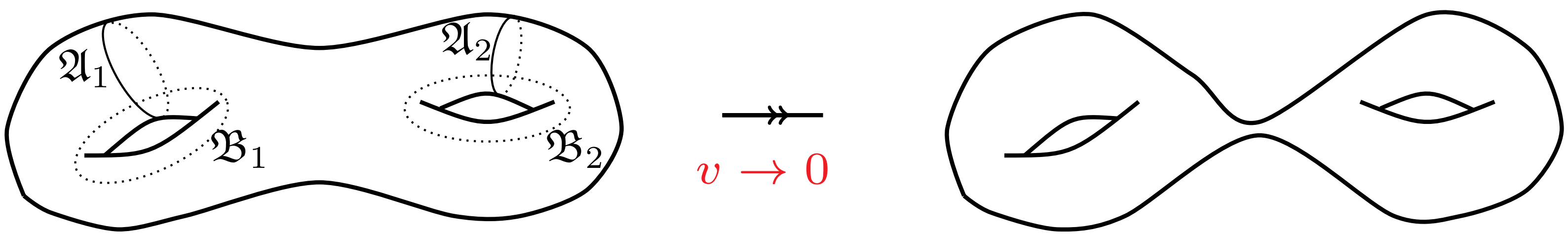
  - ⋆ δ_{SN} has support on separating node (into two genus-one surfaces)
  - ⋆ proven by complex structure deformations theory and methods

Degenerations of genus-two Riemann surfaces

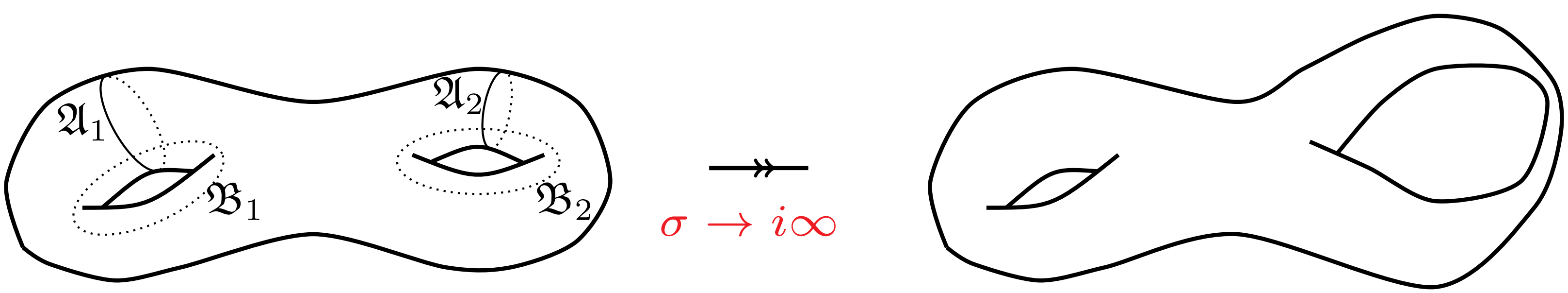
- **Locally parametrize $\mathcal{M}_2 = \mathcal{H}_2/Sp(4,\mathbb{Z})$ by the period matrix**

$$\Omega = \begin{pmatrix} \tau & v \\ v & \sigma \end{pmatrix} \qquad \tau, \sigma, v \in \mathbb{C} \qquad \det(\mathrm{Im}\,\Omega) > 0$$

$\star$ *Separating degeneration*



$\star$ *Non-separating degeneration*



Non-separating degeneration

- Σ **degenerates to torus** Σ_1 **of modulus** τ **with punctures** p_a, p_b
  - $\star$ keep the cycles $\mathfrak{A}_1, \mathfrak{B}_1, \mathfrak{A}_2$ fixed, and let $\mathfrak{B}_2 \to \infty$ as $\mathrm{Im}\,(\sigma) \to \infty$

- **Modular group** $Sp(4,\mathbb{Z})$ **reduces to** $SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^3$ **Fourier-Jacobi group**
  - $=$ the subgroup that leaves $\mathfrak{B}_2$ invariant

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) : \qquad \begin{cases} \tau & \to \quad (a\tau+b)/(c\tau+d) \\ v & \to \quad v/(c\tau+d) \\ \sigma & \to \quad \sigma - cv^2/(c\tau+d) \end{cases}$$

  - $\star$ The degeneration parameter σ is not invariant under $SL(2,\mathbb{Z})$
  - $\star$ e.g. Siegel modular forms decompose into Jacobi forms (Eichler & Zagier 1985)

- **There exists a real-valued** $SL(2,\mathbb{Z})$**-invariant parameter** $t > 0$

$$t \equiv \frac{\det\,(\mathrm{Im}\ \Omega)}{\mathrm{Im}\ \tau} = \mathrm{Im}\,\sigma - \frac{(\mathrm{Im}\,v)^2}{\mathrm{Im}\,\tau} \qquad\qquad \Omega = \begin{pmatrix} \tau & v \\ v & \sigma \end{pmatrix}$$

  - $\star$ the non-separating node is characterized by $t \to \infty$

Degeneration of genus-two string invariants

- **Recall the generating function of genus-two string invariants**

$$\mathcal{B}(s_{ij}|\Sigma) = \int_{\Sigma^4} \frac{\mathcal{Y} \wedge \overline{\mathcal{Y}}}{(\det Y)^2} \exp\left\{ \sum_{i<j} s_{ij} G(z_i, z_j) \right\} = \sum_{w=0}^{\infty} \frac{1}{w!} \mathcal{B}_w(s_{ij}|\Sigma)$$

⋆ Taylor series produces modular graph functions of weight w

$$\mathcal{B}_w(s_{ij}|\Sigma) = \int_{\Sigma^4} \frac{\mathcal{Y} \wedge \overline{\mathcal{Y}}}{(\det Y)^2} \left(\sum_{i<j} s_{ij} G(z_i, z_j) \right)^w$$

- **Theorem 6** (ED, Green, Pioline 2017)

The expansion of $\mathcal{B}_w(s_{ij}|\Sigma)$ near the non-separating node is given by a Laurent polynomial of **finite** *degree $(w, -w)$ in t*

$$\mathcal{B}_w(s_{ij}|\Omega) = \sum_{k=-w}^{w} \mathcal{B}_w^{(k)}(s_{ij}|v, \tau)\, t^k + \mathcal{O}(e^{-2\pi t})$$

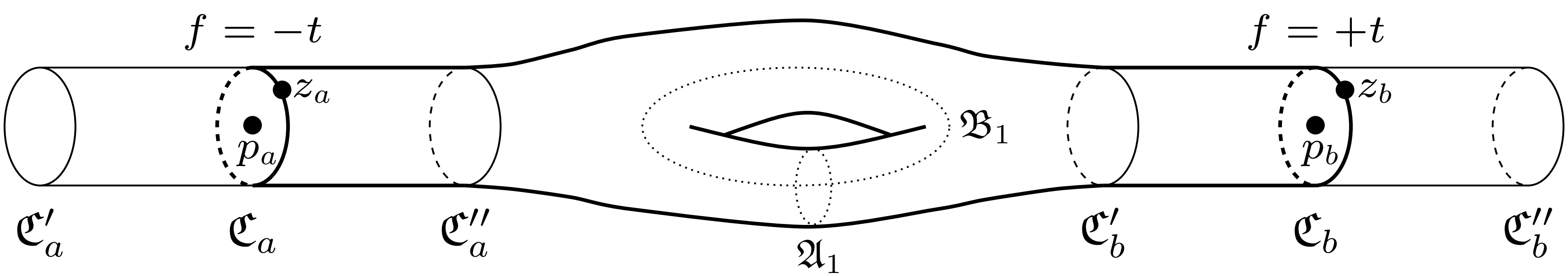
where the coefficients are invariant under $SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^2 \subset Sp(4,\mathbb{Z})$

$$\mathcal{B}_w^{(k)}\left(s_{ij} \middle| \frac{v}{c\tau+d}, \frac{a\tau+b}{c\tau+d} \right) = \mathcal{B}_w^{(k)}(s_{ij}|v, \rho)$$

and are single-valued elliptic polylogarithms

Ingredients in the proof of Theorem 6

- **Funnel construction of the non-separating degeneration** (see e.g. Fay 1973)
  - ⋆ start from genus-one surface with two punctures p_a, p_b
  - ⋆ local coordinates z_a, z_b vanishing at p_a, p_b and identify points $z_a z_b = \mathfrak{t}$
  - ⋆ identify cycles $\mathfrak{C}_a \approx \mathfrak{C}_b$, $\mathfrak{C}'_a \approx \mathfrak{C}'_b$ and $\mathfrak{C}''_a \approx \mathfrak{C}''_b$ all homologous to $\mathfrak{A}_2$
  - ⋆ the cycle $\mathfrak{B}_2$ is a curve connecting z_a to z_b.



- **Key is the existence of a real single-valued harmonic function** $f(z)$ **on** Σ

$$\lim_{z \to p_a} f(z) = -\infty \qquad \lim_{z \to p_b} f(z) = +\infty$$

⋆ large enough t: cycles prescribed by $f(\mathfrak{C}_a) = -2\pi t$ and $f(\mathfrak{C}_b) = +2\pi t$

$$f(z) = -2\pi t + 4\pi \, \mathrm{Im} \int_{z_a}^{z} \Big(\omega_2 - \mathrm{Im}\,(v)\, \omega_1 / \mathrm{Im}\,(\tau) \Big)$$

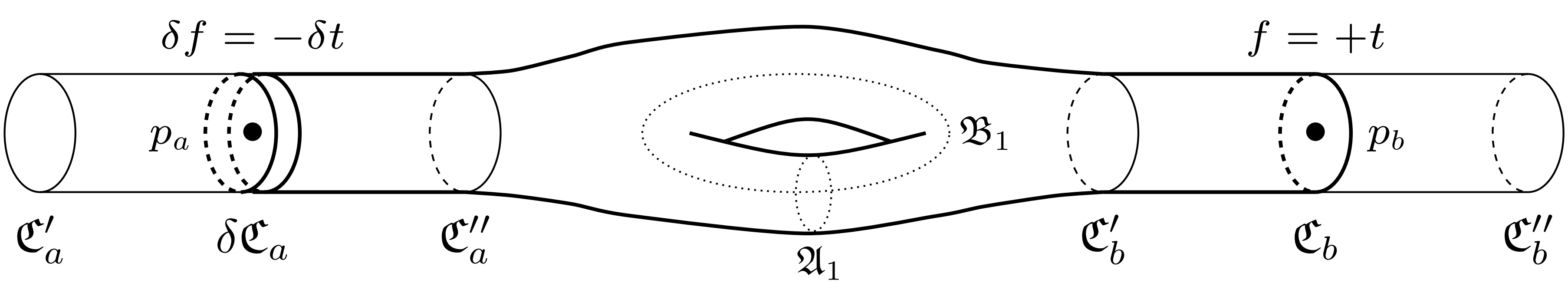
Ingredients in the proof of Theorem 6 (cont'd)

- **Lemma** *Asymptotics of the genus-two Arakelov Green function*

$$G(x,y) = g(x-y) + \frac{1}{8\pi t}(f(x)-f(y))^2 + \eta(x) + \eta(y) + \mathcal{O}(e^{-2\pi t})$$

  – where $g(z)$ is the genus-one Green function
  – and $\eta(x) = \frac{\pi t}{24} - \frac{1}{4}g(z-p_a) - \frac{1}{4}g(z-p_b) + \frac{1}{8}g(p_a-p_b) +$ terms in $\frac{1}{t}$

- **Origin of the finite degree of the Laurent polynomial**
  ⋆ The cycles $\mathfrak{C}_a, \mathfrak{C}_b$ have exponentially vanishing coordinate radius so that $z \in \mathfrak{C}_a$ satisfies $|z-p_a|^2 \approx e^{-2\pi t}$
  ⋆ Power dependence in t arises from singularities in f at p_a, p_b
  ⋆ Extract t-derivative (cfr renormalization group methods in quantum field theory)



Summary and outlook

- **Low energy expansion of string theory reveals a rich structure of**
  - ⋆ Modular graph functions for genus-one Riemann surfaces;
  - ⋆ String invariants include Kawazumi-Zhang for genus-two surfaces.
- **Modular graph functions for genus $h \geq 3$ are well-defined mathematically**
  - ⋆ string amplitude not yet known for genus $h \geq 3$
  - ⋆ Systematic degeneration structure
- **Integration over moduli space** (cfr. ED & Green 2019)
- **Open questions**
  - ⋆ Differential identities for arbitrary higher genus modular graph functions ?
  - ⋆ Theta-lifts for genus 2 and beyond ? (KZ at genus 2, see Pioline 2015)
  - ⋆ Arithmetic significance ?