Exploring transcendentality in string amplitudes

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Amplitudes in String and Field Theory

Bibliography

Based on

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Builds on earlier work

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Introduction

- **Quantum Field Theory and String amplitudes**
  - exhibit structure beyond the implications from known symmetries.
- **QFT amplitudes are periods** (in the sense of Kontsevich and Zagier)
  - integrals whose integrand and boundary conditions are *algebraic*,
  - include multiple polylogarithms graded by transcendentality weight,
  - planar $\mathcal{N} = 4$ SYM exhibits *maximal transcendentality* (i.e. *grading*),
- **Closed Type II superstring amplitudes**
  - **Can one define transcendentality ? If yes, it a grading ?**
- **This talk: genus-one four-graviton amplitude in low energy expansion**
  - **evaluate analytic and non-analytic contributions**
  - **transcendentality grading exists (plus finer arithmetic properties)**
- **Related goal: KLT beyond tree-level**

Tree-level amplitude

- **Four-graviton amplitude in Type II in string perturbation theory**

$$\mathbf{A}(k_i, \varepsilon_i) = \sum_{h=0}^{\infty} g_s^{2h-2} \mathcal{A}^{(h)}(s_{ij}) \, \mathcal{R}^4 \qquad 1 \leq i, j \leq 4$$

  – External momenta k_i with $k_i^2 = 0$ and $\sum_i k_i = 0$
  – *Dimensionless* Lorentz-invariants $s_{ij} = -\alpha'(k_i + k_j)^2/4$
  – all dependence on polarization tensors ε_i captured by $\mathcal{R}^4$

- **Tree-level amplitude**

$$\mathcal{A}^{(0)}(s_{ij}) = \frac{1}{stu} \frac{\Gamma(1-s)\,\Gamma(1-t)\,\Gamma(1-u)}{\Gamma(1+s)\,\Gamma(1+t)\,\Gamma(1+u)}$$

  – $s = s_{12}, \ t = s_{13}, \ u = s_{14}$ with $s + t + u = 0$
  – simple poles in s, t, u at non-negative integers;
  – Γ-factors holomorphic for $|s|, |t|, |u| < 1$ and admit Taylor series

Low energy expansion of tree-level amplitude

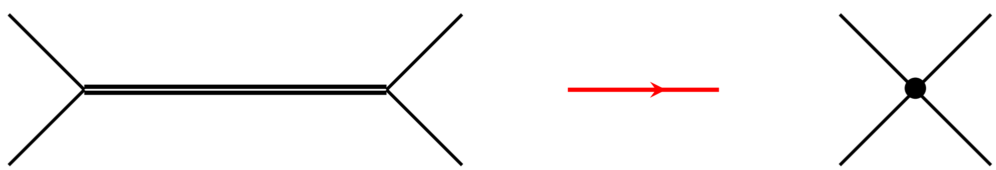
- **Low energy expansion of $\mathcal{A}^{(0)}(s_{ij})$ in powers of s, t, u**

$$\mathcal{A}^{(0)}(s_{ij}) = \frac{1}{stu} \exp \left\{ - \sum_{n=1}^{\infty} \frac{2\zeta(2n+1)}{2n+1} (s^{2n+1} + t^{2n+1} + u^{2n+1}) \right\}$$

- **Generates (non-local) massless exchange plus local effective interactions**

$$\mathcal{A}^{(0)}(s_{ij}) = \underbrace{\frac{1}{stu}}_{\text{massless}} + \underbrace{2\zeta(3)}_{\mathcal{R}^4} + \underbrace{\zeta(5)(s^2+t^2+u^2)}_{D^4\mathcal{R}^4} + \underbrace{2\zeta(3)^2 stu}_{D^6\mathcal{R}^4} + \cdots$$

– Exchange of massive string states produces local effective interactions.



Zeta values, multiple zeta values, and polylogs

- **Appearance of Riemann zeta-values is familiar from QFT amplitudes**
  - low order amplitudes expressible in terms of polylogs (see Julio's talk)

- **Standard multiple polylogs may be defined by**

$$L_{a_1,\cdots,a_\ell}(z_1,\cdots,z_\ell) = \sum_{n_1>n_2>\cdots n_\ell>0} \frac{z_1^{n_1}\cdots z_\ell^{n_\ell}}{n_1^{a_1}\cdots n_\ell^{a_\ell}}$$

  - Multiple zeta values (MZV) $\zeta(a_1,\cdots,a_\ell) = L_{a_1,\cdots,a_\ell}(1,\cdots,1)$
  - The Riemann zeta function corresponds to $\ell = 1$.
  - Polylogs and MZVs satisfy shuffle and stuffle relations; e.g. stuffle

$$\begin{aligned} L_a(y)L_b(z) &= L_{a,b}(y,z) + L_{b,a}(z,y) + L_{a+b}(yz) \\ \zeta(a)\zeta(b) &= \zeta(a,b) + \zeta(b,a) + \zeta(a+b) \qquad \text{(Euler 1730)} \end{aligned}$$

  - For $a,b \in \mathbb{N}, a+b$ odd $\zeta(a,b)$ is reducible to zeta-values
  - For $a+b$ even $\zeta(a,b)$ is not always reducible to zeta-values

Assigning transcendentality weight

- **The transcendentality** *weight* w **for polylogs and MZVs is defined by,**

$$w = \sum_{i=1}^{\ell} a_i \qquad e.g.\ w[\zeta(a)] = a,\ w[\pi] = 1$$

  - **The weight is conserved by the shuffle and stuffle relations and provides a** ***grading*** **on the algebra of polylogs and MZVs**
  - The number of loops ℓ or *depth* is conserved by shuffle, but not by stuffle.

- **Can we assign a definite weight to a string amplitude ?**

- **Yes, for tree-level**, recalling the 4-graviton amplitude,

$$\mathcal{A}^{(0)}(s_{ij}) = \frac{1}{stu} \exp\left\{ - \sum_{n=1}^{\infty} \frac{2\zeta(2n+1)}{2n+1} \left(s^{2n+1} + t^{2n+1} + u^{2n+1}\right) \right\}$$

  - has definite weight iff we assign $w[s] = w[t] = w[u] = -1$.
  - Generalizes to tree-level amplitudes with more states using MZVs.

- **How about genus-one amplitudes ?**
  - no explicit formula available $\Rightarrow$ much more complicated/interesting.

Genus-one four-graviton amplitude in Type IIB

- **The reduced amplitude is given by an integral**
  - of a $SL(2,\mathbb{Z})$-invariant function $\mathcal{B}(s_{ij}|\tau)$ of $\tau \in \mathbb{H} = \{\tau \in \mathbb{C}, \mathrm{Im}(\tau) > 0\}$,

$$\mathcal{A}^{(1)}(s_{ij}) = \int_{\mathcal{M}} \frac{d^2\tau}{\tau_2^2} \, \mathcal{B}(s_{ij}|\tau)$$

  - over the moduli space $\mathcal{M} = SL(2,\mathbb{Z})\backslash\mathbb{H}$ of tori, represented by

$$\mathcal{M} = \{\tau = \tau_1 + i\tau_2, \, \tau_1, \tau_2 \in \mathbb{R}, \, \tau_2 > 0, \, |\tau| \geq 1, \, |\tau_1| \leq 1/2\}$$

- $\mathcal{B}$ **is an integral over 4 copies of** $\Sigma_\tau = \mathbb{C}/\Lambda_\tau$ **with** $\Lambda_\tau = \mathbb{Z}\tau + \mathbb{Z}$

$$\mathcal{B}(s_{ij}|\tau) = \left(\prod_{i=1}^{4} \int_{\Sigma_\tau} \frac{d^2 z_i}{\tau_2} \right) \exp\left\{ \sum_{1 \leq i < j \leq 4} s_{ij} \, G(z_i - z_j|\tau) \right\}$$

  - The scalar Green function $G(z|\tau)$ on Σ_τ is defined by

$$\tau_2 \partial_z \partial_{\bar z} G(z|\tau) = -\pi\delta(z) + \pi \qquad\qquad \int_{\Sigma_\tau} d^2 z \, G(z|\tau) = 0$$

Low energy expansion of $\mathcal{B}(s_{ij}|\tau)$

- **For fixed modulus τ the integrals defining $\mathcal{B}(s_{ij}|\tau)$**
  - are absolutely convergent for $\mathrm{Re}(s_{ij}) < 1$ and analytic near $s_{ij} = 0$;
  - have convergent Taylor expansion for $|s|, |t|, |u| < 1$;
  - in terms of symmetric functions $\sigma_k = s^k + t^k + u^k$ for $k = 2, 3$,

$$\mathcal{B}(s_{ij}|\tau) = \sum_{p,q=0}^{\infty} \mathcal{B}_{(p,q)}(\tau) \, \frac{\sigma_2^p \, \sigma_3^q}{p! \, q!}$$

- **Assigning the same weight to s, t, u as for tree-level, namely -1,**
  - we can assign a definite weight, namely 0, to $\mathcal{B}(s_{ij}|\tau)$ iff

$$w[\mathcal{B}_{(p,q)}(\tau)] = 2p + 3q$$

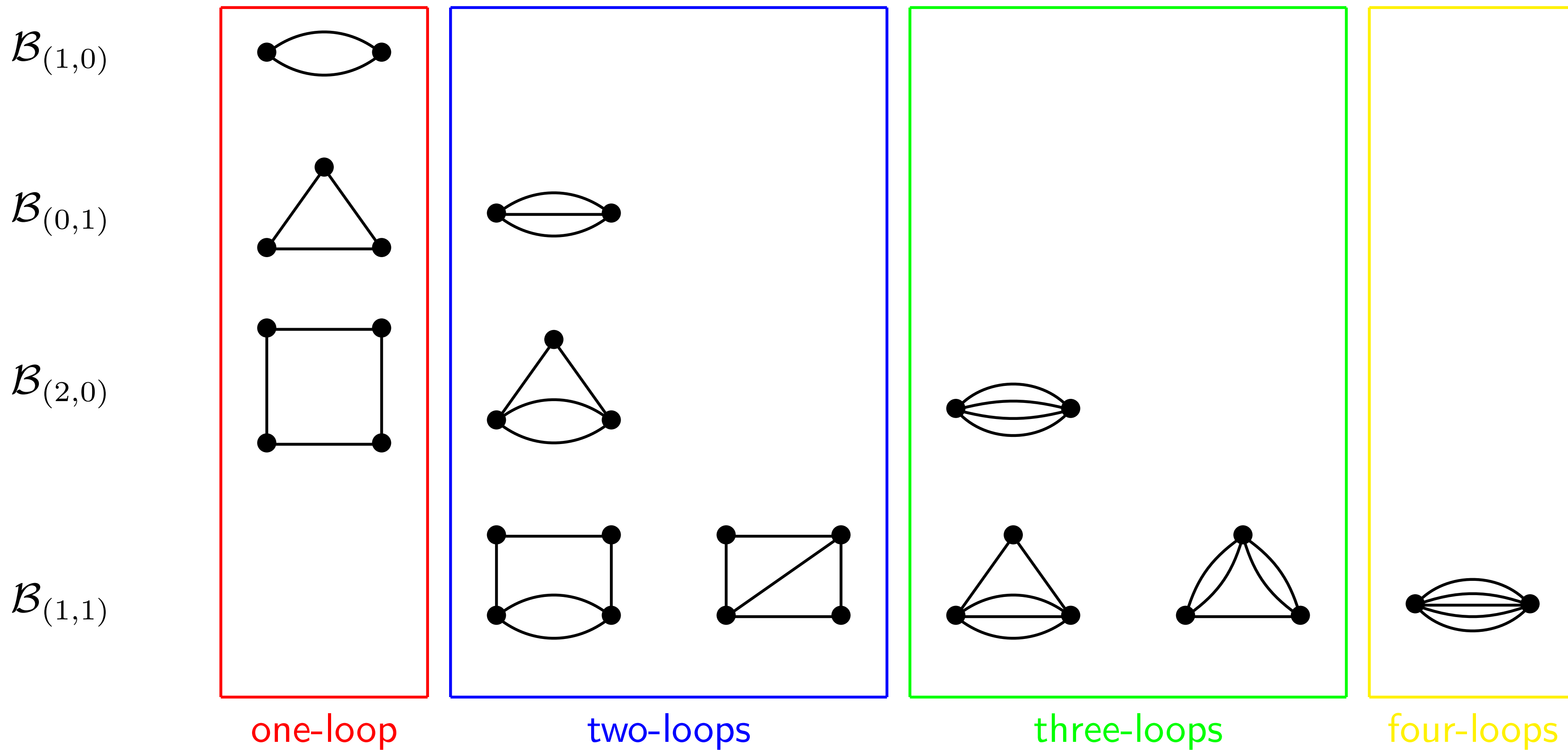
- **The coefficients $\mathcal{B}_{(p,q)}(\tau)$ admit a natural graphical expansion**
  - an integration point z_i is represented by a vertex of the graph,
  - a Green functions $G(z_i - z_j|\tau)$ by an edge joining z_i and z_j;
  - a Green function cannot begin and end on the same vertex.

Modular graph functions

- **To every graph corresponds a modular function**

Green function : 

$= G(z_i - z_j|\tau)$



One-loop modular graph functions

- **Fourier series for the scalar Green function,**

$$G(z|\tau) = \sum_{p \in \Lambda'_\tau} \tau_2 \, \frac{e^{2\pi i(n\beta - m\alpha)}}{\pi \, |p|^2} \qquad \begin{cases} z = \alpha + \beta\tau & \alpha, \beta \in \mathbb{R} \\ p = m\tau + n & m, n \in \mathbb{Z} \end{cases}$$

- **One-loop graphs with $a \geq 2$ vertices $z_1, z_2, \cdots, z_a, z_{a+1} = z_1$**

$$\prod_{i=1}^{a} \int_\Sigma \frac{d^2 z_i}{\tau_2} G(z_i - z_{i+1}|\tau) = \sum_{p \in \Lambda'} \frac{\tau_2^a}{\pi^a |p|^{2a}} = E_a(\tau)$$

  - real analytic Eisenstein series : $\Delta E_a = a(a-1)E_a$ with $\Delta = 4\tau_2^2 \partial_{\bar\tau} \partial_\tau$
  - invariant under $SL(2,\mathbb{Z})$; assign weight $w[E_a(\tau)] = a$
  - Asymptotic expansion in $y = \pi\tau_2$ near the cusp $\tau \to i\infty$,

$$E_a(\tau) = \frac{2\zeta(2a)}{\pi^{2a}} y^a + \frac{\Gamma(2a-1)\zeta(2a-1)}{2^{2a-3}\,\Gamma(a)^2\, y^{a-1}} + \mathcal{O}(e^{-2y})$$

assigning $w[\tau] = 0$ and $w[y] = 1$ gives both terms weight a.

Two-loop modular graph functions

- **Two-loop graphs evaluate to Kronecker-Eisenstein sums**

$$C_{a,b,c}(\tau) = \sum_{p_r \in \Lambda'_\tau} \delta\Big(\sum_r p_r\Big) \frac{\tau_2^w}{\pi^w |p_1|^{2a} |p_2|^{2b} |p_3|^{2c}}$$

– Convergent for $a,b,c \geq 1$; invariant under $SL(2,\mathbb{Z})$; assign weight

$$w[C_{a,b,c}] = a+b+c = w$$

– Satisfy algebraic and differential identities which conserve the weight

$$\begin{aligned} 2\Delta C_{a,b,c} &= a(a-1)\, C_{a,b,c} + 2ab\, C_{a+1,b-1,c} - 4ab\, C_{a+1,b,c-1} \\ &\quad + ab\, C_{a+1,b+1,c-2} + \text{ 5 permutations of } (a,b,c) \end{aligned}$$

– Asymptotic expansion in $y = \pi\tau_2$ near the cusp conserves the weight

$$C_{a,b,c}(\tau) = c_w y^w + c_{2-w} y^{2-w} + \sum_{k=1}^{w-1} c_{w-2k-1} \zeta(2k+1) y^{w-2k-1} + \mathcal{O}(e^{-2y})$$

– $c_w, c_{w-2k-1} \in \mathbb{Q}$, c_{2-w} is bilinear in odd zeta values of weight $2w-2$.
(ED & Bill Duke 2017)

Higher-loop modular graph functions

- **Higher-loop graphs all evaluate to higher Kronecker-Eisenstein sums**
  - RULE of THUMB: summing over momenta, each πp_r has weight zero !

e.g. $$E_a(\tau) = \sum_{p \in \Lambda'} \frac{(\pi \tau_2)^a}{|\pi p|^{2a}}$$

  - All algebraic and differential identities conserve the weight.

- **Evaluation of coefficients $\mathcal{B}_{(p,q)}$ in terms of Kronecker-Eisenstein sums**
  - gives a linear combination of modular graph functions,
  - each one of which has weight $w = 2p + 3q$,
  - Asymptotic expansion near cusp generally involves MZVs (Zerbini 2017)

⇒ **Consistent assignment of transcendentality weight**

$$w[\mathcal{B}_{(p,q)}(\tau)] = 2p + 3q$$

  - String integrand $\mathcal{B}(s_{ij}|\tau)$ consistently has weight 0.

Genus-one Type II string amplitude

- **Obtained by integrating $\mathcal{B}(s_{ij}|\tau)$ over modulus τ**

$$\mathcal{A}^{(1)}(s_{ij}) = \int_{\mathcal{M}} \frac{d^2\tau}{\tau_2^2} \, \mathcal{B}(s_{ij}|\tau)$$

  – Inherits the OPE pole singularities of $\mathcal{B}(s_{ij}|\tau)$ at $s_{ij} \in \mathbb{N}$.
  – Further singularities due to the non-uniform behavior of G in τ;
  – To see this, extract from G the constant Fourier mode in α,

$$G(z|\tau) = 2\pi\tau_2 \left(\beta^2 - |\beta| + \tfrac{1}{6}\right) + g(z|\tau) \qquad\qquad z = \alpha + \tau\beta$$

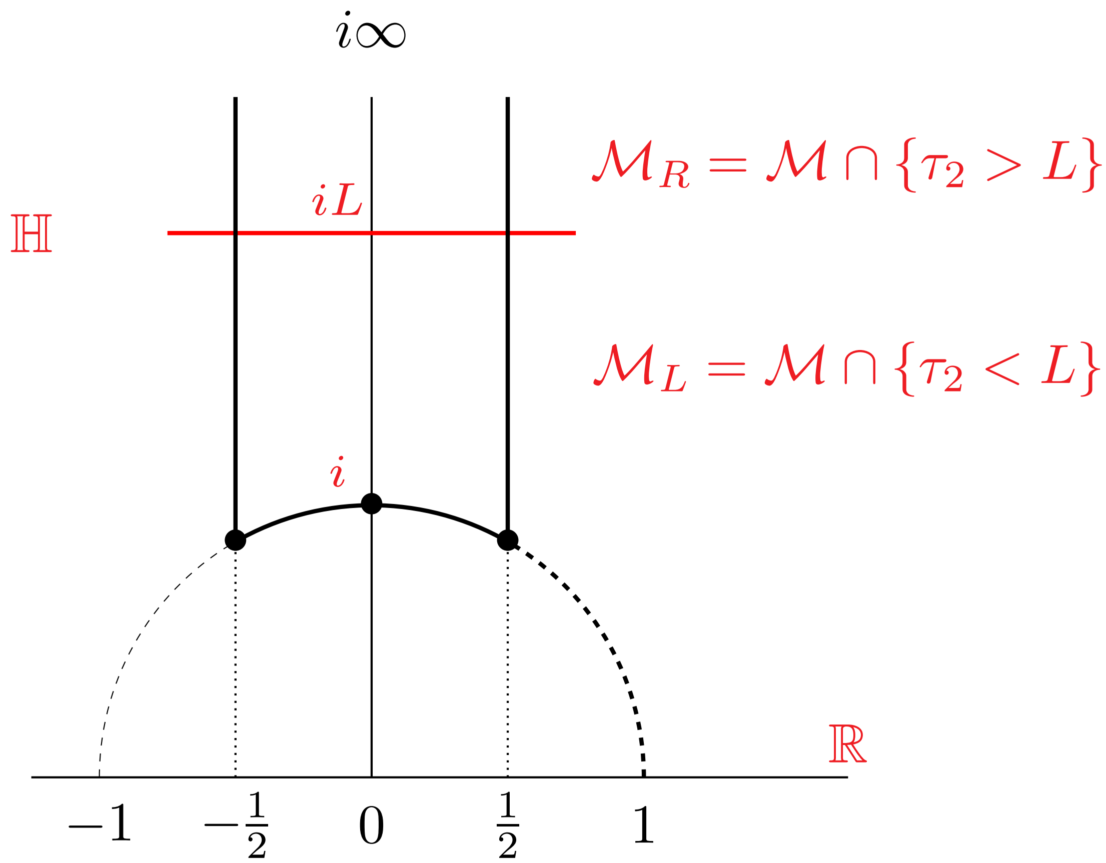
  where the non-constant Fourier modes are given by $g(z|\tau)$,

$$g(z|\tau) = \sum_{m \neq 0} \sum_{k \in \mathbb{Z}} \frac{1}{|m|} \, e^{2\pi i m(\alpha + \tau_1(k+\beta)) - 2\pi\tau_2 |m(k+\beta)|}$$

- **Absolute convergence of τ-integral requires s_{ij} purely imaginary**
  – **analytic continuation in s_{ij} was shown to exist** (ED, Phong 1994)
  – **produces poles and branch cuts in s_{ij} expected from unitarity.**
  – **The amplitude is UV convergent !**

Partitioning the moduli space of the torus

- **Isolate a neighborhood of the cusp**, for $L > 1$,



Partitioning the moduli space of the torus (cont'd)

– **The contribution from $\mathcal{M}_L$ is analytic in s, t, u for $|s|, |t|, |u| < 1$**

$$\mathcal{A}_L(L; s_{ij}) = \sum_{p,q=0}^{\infty} \frac{\mathcal{A}_{(p,q)}(L)}{p!\, q!} \qquad \mathcal{A}_{(p,q)}(L) = \int_{\mathcal{M}_L} \frac{d^2\tau}{\tau_2^2} \mathcal{B}_{(p,q)}(\tau)$$

– **The contribution from $\mathcal{M}_R$**

$$\mathcal{A}_R(L; s_{ij}) = \int_L^{\infty} \frac{d\tau_2}{\tau_2^2} \int_0^1 d\tau_1 \mathcal{B}(s_{ij}|\tau)$$

- **must be analytically continued in s_{ij}**
- **the result will be non-analytic in s_{ij} at $s_{ij} = 0$**

– **By construction, all L-dependence cancels in $\mathcal{A}_L(L; s_{ij}) + \mathcal{A}_R(L; s_{ij})$**

Partitioning the moduli space of the torus (cont'd)

- **Use the freedom in choosing L to organize calculations**

  - Choose $L \gg 1$ and omit negative powers and exponentials of L.
  - E.g. the asymptotics of modular graph functions for $w = 2p + 3q$

$$\mathcal{B}_{(p,q)}(\tau) = \sum_{k=1-w}^{w} \mathfrak{b}_k \tau_2^k + \mathcal{O}(e^{-2\pi\tau_2})$$

  - determines $\mathcal{A}_L$, up to an L-independent constant $\mathcal{A}^{(0)}_{(p,q)}$

$$\mathcal{A}_{(p,q)}(L) = \mathcal{A}^{(0)}_{(p,q)} + \mathfrak{b}_1 \ln L + \sum_{\substack{k=1-w \\ k \neq 1}}^{w} \frac{\mathfrak{b}_k L^{k-1}}{k-1} + \mathcal{O}(e^{-2\pi L})$$

  - Since L-dependence cancels in total amplitude, only $\mathcal{A}^{(0)}_{(p,q)}$ contributes
  - for later use, also keep $\ln L$ terms

Calculating the analytic part

- **Exploit algebraic and differential identities to simplify $\mathcal{B}_{(p,q)}$**

  (ED, Green, Vanhove 2015-16; ED, Kaidi 2016; Broedel, Sclotterer, Zerbini 2018)

  - **Expose well-understood Eisenstein series**
  - **Expose Laplacian of modular functions**
  - **Laplacian of modular function f integrated by Stokes's theorem**

$$\int_{\mathcal{M}_L} \frac{d^2\tau}{\tau_2^2} \Delta f(\tau) = \int_0^1 d\tau_1 \, \partial_{\tau_2} f(\tau) \Big|_{\tau_2=L}$$

  - Up to weight 6 we have $\mathcal{B}_{(0,0)} = 1, \; \mathcal{B}_{(1,0)} = E_2,$

$$\begin{aligned}
3\mathcal{B}_{(0,1)} &= 5E_3 + \zeta(3) \\
\mathcal{B}_{(2,0)} &= \Delta C_{2,1,1} - 10E_4 + 2E_2^2 \\
180\,\mathcal{B}_{(1,1)} &= 70\,\Delta C_{3,1,1} - 1612E_5 + 580E_2E_3 + 60E_2\zeta(3) + 29\zeta(5) \\
6\,\mathcal{B}_{(3,0)} &= \Delta\left(-9C_{2,2,1,1} + 18E_3^2 + 9E_2E_4 + 6C_{4,1,1} + 156C_{3,2,1} + 41C_{2,2,2}\right) \\
&\quad +72E_2C_{2,1,1} - 12E_3^2 - 36E_2E_4 - 2652E_6 \\
27\,\mathcal{B}_{(0,2)} &= \Delta\left(9C_{2,2,1,1} - 18E_3^2 - 9E_2E_4 - 6C_{4,1,1} + 258C_{3,2,1} + 64C_{2,2,2}\right) \\
&\quad -36E_2C_{2,1,1} + 483E_3^2 + 30\zeta(3)E(3) + 18E_2E_4 + 6E_2^3 - 3186E_6 + 3\zeta(3)^2
\end{aligned}$$

Calculating the analytic part (cont'd)

- **Integrals over $\mathcal{M}_L$ of product of Eisenstein series** (see also Zagier)
  - may be evaluated using the eigenvalue equations and Stokes's theorem

$$\int_{\mathcal{M}_L} \frac{d^2\tau}{\tau_2^2}(E_a \Delta E_b - E_b \Delta E_a) = (a-b)(1-a-b)\int_{\mathcal{M}_L} \frac{d^2\tau}{\tau_2^2} E_a E_b$$
$$= \int_0^1 d\tau_1 (E_a \partial_{\tau_2} E_b - E_b \partial_{\tau_2} E_a)\Big|_{\tau_2=L}$$

  - Using the asymptotic Laurent series of E_a, E_b the integral evaluates to,

$$\int_{\mathcal{M}_L} \frac{d^2\tau}{\tau_2^2} E_a E_b = \# L^{a+b-1} + \# L^{a-b} + \# L^{b-a} + \# L^{1-a-b} + \mathcal{O}(e^{-4\pi L})$$

  - For $a \neq b \geq 2$ there are no terms of order L^0 and $\ln L$, thus set to 0.
  - Only contributions of order L^0 and $\ln L$ come from $a = b \geq 2$,

$$\frac{16\Gamma(2a-1)\zeta(2a)\zeta(2a-1)}{(2\pi)^{2a-1}\Gamma(a)^2}\left[\ln(2L) + \frac{\zeta'(2a)}{\zeta(2a)} - \frac{\zeta'(2a-1)}{\zeta(2a-1)} - \Psi(2a-1) + \Psi(a)\right]$$

  - where $\Psi(a) = \Gamma'(a)/\Gamma(a)$.
  - Similarly for integrals of E_2^3 and $E_2 C_{2,1,1}$.

Calculating the analytic part (cont'd)

- **Summary of contributions from analytic part up to weight 6**
  - Omitting all non-zero power and exponential contributions in L
  - Dividing by the volume of moduli space $\mathrm{Vol}(\mathcal{M}) = \frac{\pi}{3}$

$$\mathcal{A}_{(0,0)} = 1 \qquad \mathcal{A}_{(0,1)} = \frac{\zeta(3)}{3} \qquad \mathcal{A}_{(1,0)} = 0 \qquad \mathcal{A}_{(1,1)} = \frac{29\zeta(5)}{180}$$

which all manifestly conserve transcendentality. Next, we have,

$$\mathcal{A}_{(2,0)} = \frac{4\zeta(3)}{15}\left[\ln(2L) + \frac{\zeta'(4)}{\zeta(4)} - \frac{\zeta'(3)}{\zeta(3)} - \frac{1}{4}\right]$$

$$\mathcal{A}_{(3,0)} = \frac{\zeta(5)}{105}\left[33\ln(2L) - 2\frac{\zeta'(6)}{\zeta(6)} - 33\frac{\zeta'(5)}{\zeta(5)} + 70\frac{\zeta'(4)}{\zeta(4)} - 35\frac{\zeta'(2)}{\zeta(2)} + \frac{13}{2}\right]$$

$$\mathcal{A}_{(0,2)} = \frac{\zeta(5)}{135}\left[18\ln(2L) + 23\frac{\zeta'(6)}{\zeta(6)} - 18\frac{\zeta'(5)}{\zeta(5)} - 10\frac{\zeta'(4)}{\zeta(4)} + 5\frac{\zeta'(2)}{\zeta(2)} - \frac{31}{12}\right] + \frac{\zeta(3)^2}{9}$$

  - Transcendentality assignments are clearly more tricky !
  - Postpone making assignments until the non-analytic part has been evaluated.

Calculating the non-analytic part

- **Non-analytic part has branch cuts in s, t, u starting at integers ≥ 0**
  - In low energy expansion, non-analyticity in s, t, u starts at zero.
  - Split Green function into constant and non-constant Fourier modes

$$G(z|\tau) = 2\pi\tau_2 \left(\beta^2 - |\beta| + \tfrac{1}{6}\right) + g(z|\tau) \qquad\qquad z = \alpha + \beta\tau$$

  - Treat constant mode exactly, and expand in powers of $g(z|\tau)$

- **To disentangle absolute values: partition into 6 channels**

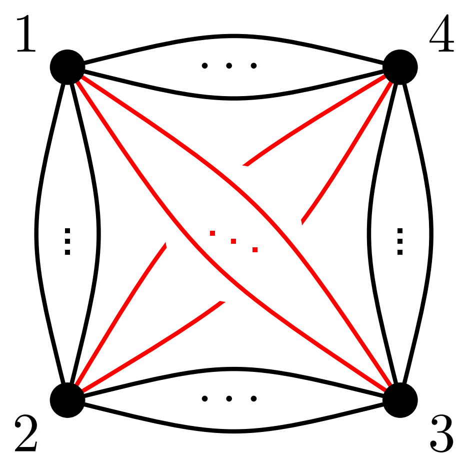
$$\begin{aligned} \mathcal{A}_R(L; s_{ij}) &= \mathcal{A}_*(L; s, t) + \text{ 5 permutations of } s, t, u \\ \mathcal{A}_*(L; s, t) &= \int_L^\infty \frac{d\tau_2}{\tau_2^2} \int [dx] e^{4\pi\tau_2(s x_1 x_3 + t x_2 x_4)} \mathcal{F}(s_{ij}; x_i, \tau_2) \end{aligned}$$

  - changed variables from β_i to differences x_i of β_i
  - measure $[dx] = dx_1 \cdots dx_4 \, \delta(1 - x_1 \cdots - x_4)$ with $x_i \in [0, 1]$

$$\mathcal{F}(s_{ij}; x_i, \tau_2) = \int_0^1 d\tau_1 \int_0^1 d\alpha_1 \cdots \int_0^1 d\alpha_4 \exp\Big\{ \sum_{i<j} s_{ij} g(z_i - z_j|\tau) \Big\}$$

Two Lemmas

- **Graphical expansion in powers of $g_{ij} = g(z_i - z_j|\tau)$ in the s-channel**



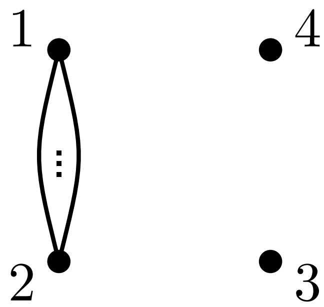
Lemma 1 **Any contribution to $\mathcal{A}_*(L; s, t)$ from the Taylor expansion of $\mathcal{F}$ in powers of s, t, u involving at least one factor of g_{13} or g_{24} is of order $\mathcal{O}(L^{-3})$.**

Lemma 2 **Any contribution to $\mathcal{A}_*(L; s, t)$ from the expansion of $\mathcal{F}$ in powers of s, t, u involving at least one factor of $g_{ij}g_{ik}$ with $k \neq j$ is of order $\mathcal{O}(L^{-3})$.**

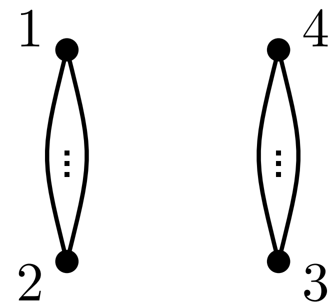
Only two contributions remain in the s-channel

- **Representing the non-constant Fourier mode by full black lines**

$\mathcal{R}_{12} =$

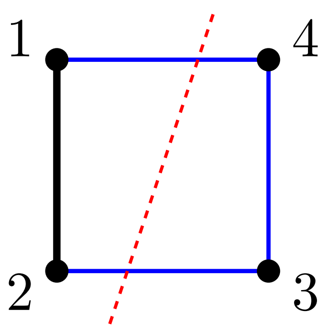


$\mathcal{R}_{12;34} =$

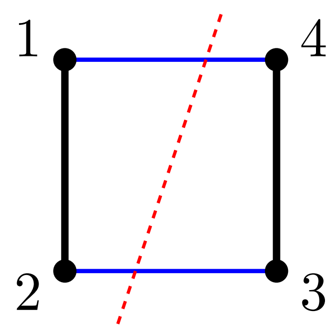


- **Compare with massless cut structure in s-channel**
  - massive exchanges in black lines
  - massless exchanges in blue lines
  - s-channel cut in red lines

$\mathcal{R}_{12} :$



$\mathcal{R}_{12;34} :$



Non-analytic part

Theorem To all orders in s, t, orders $L^0, \ln L$ in L, $\mathcal{A}_*(L; s, t)$ is given by,

$$\mathcal{A}_*(L; s, t) = \sum_{N=2}^{\infty} \frac{s^N}{N!} \mathcal{R}_{12}^{(N)}(L; s, t) + \sum_{M,N=2}^{\infty} \frac{s^{M+N}}{M!\,N!} \mathcal{R}_{12;34}^{(M,N)}(L; s, t) + (s \leftrightarrow t)$$

– The coefficient functions are given by

$$\mathcal{R}_{12}^{(N)} = \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} C_{k,\ell} S(N, k+1) s^{k-\ell+2} t^{\ell} \times \Big(\ln(-4\pi L s) + \Psi(k-\ell+2) - 2\Psi(k+7) \Big)$$

$$\mathcal{R}_{12;34}^{(M,N)} = \sum_{k,m,n=0}^{\infty} D_{k,m,n} S(M, k+m+1) S(N, k+n+1) s^{k+m+n+3} t^k \times \Big(\ln(-4\pi L s) + \Psi(k+m+n+4) - 2\Psi(2k+m+n+8) \Big)$$

where $C_{k,\ell}$ and $D_{k,m,n}$ are rational combinatorial coefficients, and,

$$S(N, k) = \sum_{\substack{m_r \neq 0 \\ r=1,\cdots,N}} \frac{\delta(\sum_r m_r)}{|m_1 \cdots m_N|(|m_1| + \cdots |m_N|)^k}$$

Multi-zeta value decomposition

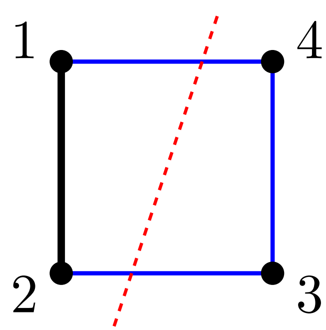
- **The coefficients $S(N,k)$ are linear combinations of MZVs** (Zagier 2008)

$$S(N,k) = \sum_{\substack{a_1,\cdots,a_r \\ a_1+\cdots+a_r=N-2}} N!\, 2^{2r+2-N-k} \zeta(k+2,a_1,\cdots,a_r)$$

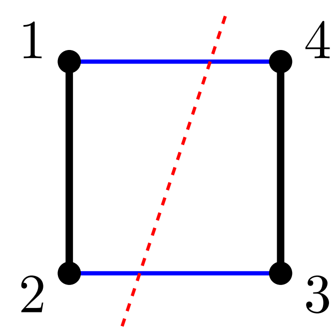
THEOREM The s-channel partial amplitude $\mathcal{A}_*(L;s,t)+\mathcal{A}_*(L;s,u)$

- **is free of irreducible multiple zeta-values**
  i.e. it is in the $\mathbb{Q}$-ring generated by odd zeta-values
- **its discontinuity is the product of tree-level 4-point amplitudes**

$\mathcal{R}_{12}$:



$\mathcal{R}_{12;34}$:



Transcendentality counting for $D^8\mathcal{R}^4$

- $D^8\mathcal{R}^4$ **is the lowest order at which non-analyticities appear**

$$\mathcal{A}_L = \frac{4\zeta(3)\sigma_4}{15}\left[\ln(2L) + \frac{\zeta'(4)}{\zeta(4)} - \frac{\zeta'(3)}{\zeta(3)} - \frac{1}{4}\right]$$

$$\mathcal{A}_R = \frac{4\zeta(3)s^4}{15}\left[-\ln(-4\pi L s) - \gamma + \frac{17}{5}\right] + \text{ 2 perms of } s,t,u$$

– Their sum gives (we have factored out the volume $\mathrm{Vol}(\mathcal{M}) = \frac{\pi}{3}$)

$$\mathcal{A}_L + \mathcal{A}_R = \frac{4\zeta(3)s^4}{15}\left[-\ln(-2\pi s) + \frac{\zeta'(4)}{\zeta(4)} - \frac{\zeta'(3)}{\zeta(3)} - \gamma + \frac{63}{20}\right] + \text{ 2 perms of } s,t,u$$

RULES towards conserving weight

$\surd$ The argument $-2\pi s$ of the log consistently has weight 0;

$\surd$ Assign weight 1 to $\ln(-2\pi s)$

$\surd$ Assign weight 1 to the combination $\frac{\zeta'(4)}{\zeta(4)} - \frac{\zeta'(3)}{\zeta(3)} - \gamma$

?? Assign weight 1 to $\frac{63}{20}$

Transcendentality counting for $D^{10}\mathcal{R}^4$

- $D^{10}\mathcal{R}^4$ **is purely analytic so that** $\mathcal{A}_R = 0$

$$\mathcal{A}_{(1,1)}\sigma_2\sigma_3 = \frac{29\,\zeta(5)\,\sigma_2\sigma_3}{180}$$

  – manifestly conserves transcendentality

Transcendentality counting for $D^{12}\mathcal{R}^4$

- **Lowest order where there are two different kinematic invariants**
  - We introduce the following combinations

$$Z_n \equiv \frac{\zeta'(n)}{\zeta(n)} - \frac{\zeta'(n-1)}{\zeta(n-1)} - \gamma \qquad\qquad n \geq 3$$

  - The full amplitude to this order is given by

$$\begin{aligned}\mathcal{A}_L + \mathcal{A}_R \;\; = \;\; & -\frac{84s^6 + 2\sigma_2 s^4}{420}\zeta(5)\ln(-2\pi s) + \text{ 2 perms of } s,t,u \\ & + \frac{\zeta(5)\sigma_2^3}{18}\left[-\frac{2}{35}Z_6 - Z_5 + Z_4 + Z_3 + \frac{8339}{2450}\right] \\ & + \frac{\zeta(5)\sigma_3^2}{54}\left[\frac{23}{5}Z_6 + Z_5 - Z_4 - Z_3 + \frac{51011}{420}\right] + \frac{\zeta(3)^2\sigma_3^2}{18}\end{aligned}$$

RULES towards conserving weight

√ The argument $-2\pi s$ of the log consistently has weight 0;
√ Assign weight 1 to $\ln(-2\pi s)$;
√ Assign weight 1 to the combination Z_n for all $n \geq 3$
?? Assign weight 1 to other rationals ?

Interpreting the combination $Z_n = \frac{\zeta'(n)}{\zeta(n)} - \frac{\zeta'(n-1)}{\zeta(n-1)} - \gamma$

- **Is Z_n a period ?**

- **Does it make sense to assign weight 1 to Z_n ?**
  - Look at series representation

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \qquad\qquad \zeta'(z) = -\sum_{n=1}^{\infty} \frac{\ln n}{n^z}$$

  - Since $\ln n$ is assigned weight 1 for $n \in \mathbb{Q}$ it "adds" weight 1

- **The following combination can be assigned weight 1**

$$Z_m - Z_n = \frac{d}{dz} \ln \frac{\zeta(m+z)\zeta(n-1+z)}{\zeta(m-1+z)\zeta(n+z)} \Big|_{z=0}$$

  - The argument of the log has weight zero
  - The log itself produces weight 1.

- **What is the role of γ ?**
  - Can one relate this to $\zeta(1+z) = z^{-1} + \gamma + \mathcal{O}(z)$?

The weight of finite harmonic sums

- **The rational numbers all result from finite harmonic sums**
  - **in the analytic part** they received contributions from

$$\int_{\mathcal{M}_L} \frac{d^2\tau}{\tau_2^2} E_a^2 \sim \ln(2L) + \gamma + Z_{2a} - \Psi(2a-1) + \Psi(a)$$

  - **in the non-analytic part** they received contributions e.g. from $\mathcal{R}_{12}$

$$\mathcal{R}_{12}^{(N)} \sim \sum_{k,\ell} \#_{k,\ell} \Big(\ln(-4\pi L s) + \gamma + \Psi(1) + \Psi(k-\ell+2) - 2\Psi(k+7) \Big)$$

  - The differences of Ψ-functions at integers are finite harmonic sums

$$\Psi(m) - \Psi(n) = \sum_{k=n}^{m-1} \frac{1}{k}$$

  - although they evaluate to rational numbers, one can assign weight 1

    (conversations at UCLA with Julio Parra Martinez and at NBI with David Broadhurst)

    (Kotikov, Lipatov 2002; Beccaria, Forini 2009; etc)

Conclusions and outlook

- **Explicit evaluation of non-analytic part of one-loop string amplitude**
  - Local effective interactions require its normalization;
  - General structure of analytic part known

- **Consistent assignment of conserved transcendentality weight**

- **Simple, but striking, observation at order $D^{12}\mathcal{R}^4$**
  - recall two invariants: σ_2^3 and σ_3^2

$$\frac{\zeta(3)^2\sigma_3^2}{18} + 0\cdot\zeta(3)^2\sigma_2^3$$

  - First term can arise from "square of tree-level", but not the second term !
  - Why ? Is there a pattern ?

- **Extension to higher-point functions at genus one and higher genus**
  - Cuts involve tree-level higher-point functions and MZVs
  - Asymptotics of genus-two modular graph functions (ED, Green, Pioline 2018)