Modular structure of Type IIB low energy expansion

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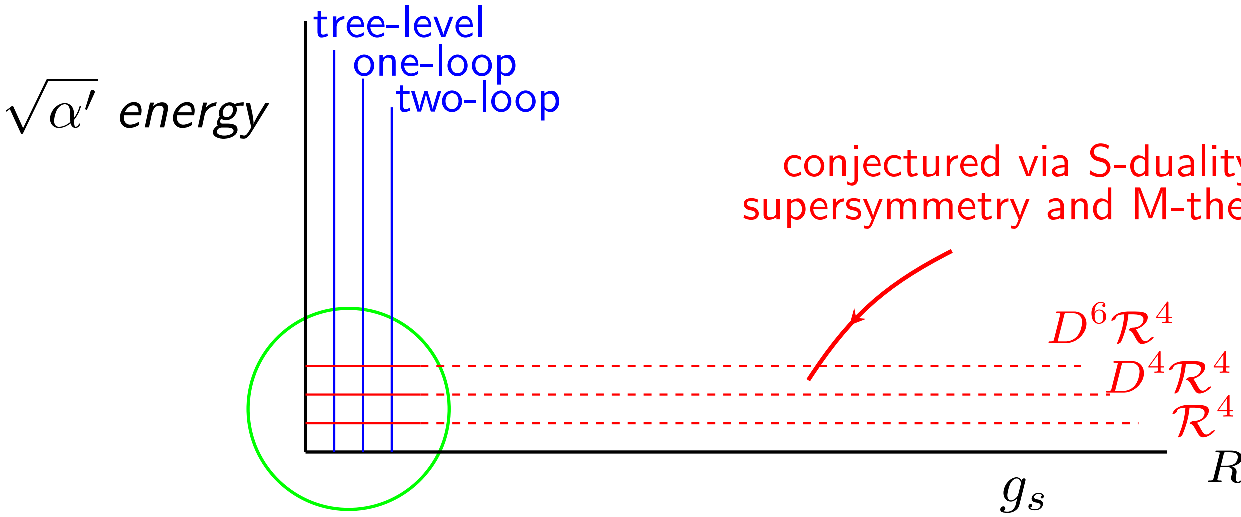
Perimeter Institute – 2015

Bibliography

Based on

- ED, Michael Green, Pierre Vanhove, arXiv:1502.06698,
  On the modular structure of the genus-one Type II superstring low energy expansion
- ED, Michael Green, Boris Pioline, Rudolfo Russo, arXiv:1405.6226; JHEP 1501 (2015) 031,
  Matching the $D^6\mathcal{R}^4$ interaction at two-loops
- ED, Michael Green, arXiv:1308.4597; Journal of Number Theory, Vol 144 (2014) 111-150,
  Zhang-Kawazumi invariants and Superstring Amplitudes

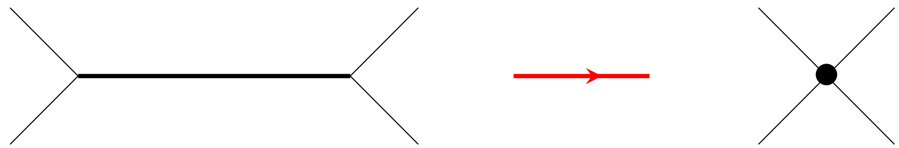
Expansions of Type IIB Superstring Theory



- **Superstring Perturbation Theory in powers of** g_s
  - holds for weak coupling g_s
  - but for all energies
- **Classical supergravity** R
  - leading low energy expansion of string theory
  - holds for all couplings g_s
- **String induced effective interactions** $\mathcal{R}^4, D^4\mathcal{R}^4, D^6\mathcal{R}^4$
  - Evaluated in perturbation theory for $g_s \ll 1$
  - Conjectured for all couplings via S-duality, supersmmetry and M-theory

Effective Interactions

Exchange of massive string states produces local effective interactions.



- **Four-graviton amplitude in Type II at tree-level,**

$$\mathcal{A}^{(0)} = \kappa^2 \, \mathcal{R}^4 \, \frac{1}{stu} \, \frac{\Gamma(1-s)\,\Gamma(1-t)\,\Gamma(1-u)}{\Gamma(1+s)\,\Gamma(1+t)\,\Gamma(1+u)}$$

  - κ^2 = Newton's constant in 10 dimensions;
  - $\mathcal{R}^4$ = unique maximally supersymmetric contraction of 4 Weyl tensors
  - $s_{ij} = -\alpha' k_i \cdot k_j/2$, $s = s_{12}$, $t = s_{13}$, $u = s_{14}$ with $s+t+u=0$

- **Low energy expansion corresponds to** $|s|, \, |t|, \, |u| \ll 1$

$$\frac{1}{stu} \; + \; 2\zeta(3) + \zeta(5)(s^2+t^2+u^2) + 2\zeta(3)^2 stu + \cdots$$

massless $\qquad \mathcal{R}^4 \qquad D^4\mathcal{R}^4 \qquad D^6\mathcal{R}^4$

D-instantons and Eisenstein series

- **Full $\mathcal{R}^4$ effective interaction conjectured from D-instanton** [Green Gutperle 1997]

$$(T_2)^{\frac{1}{2}} E_{\frac{3}{2}}(T) \mathcal{R}^4 \qquad T = T_1 + iT_2 \qquad T_2 = \frac{1}{g_s}$$

- **The "non-holomorphic" Eisenstein series is defined by,**

$$E_s(T) = \sum_{(m,n)\neq(0,0)} \frac{(T_2)^s}{\pi^s |mT+n|^{2s}}$$

– Modular invariant under S-duality group $SL(2,\mathbb{Z})$ of Type IIB;
– satisfies a Laplace-eigenvalue equation,

$$\Delta E_s = s(s-1)E_s \qquad \Delta = 4\, T_2^2\, \partial_T \partial_{\bar T}$$

– and admits the following asymptotics near the cusp $T_2 \to \infty$,

$$E_s(T,\bar T) = \frac{2\zeta(2s)}{\pi^s} T_2^s + \frac{2\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)\pi^{s-\frac{1}{2}}} T_2^{1-s} + \mathcal{O}(e^{-2\pi T_2})$$

– Perturbative contributions to $\mathcal{R}^4$ arise from genus 0 and 1 **only**.

Supersymmetry and S-duality

- **Laplace-eigenvalue eq from space-time supersymmetry** [Green, Sethi, 1998]
  - Eisenstein series = unique modular solution with polynomial growth at cusp

- **Predicts vanishing contributions for high enough loop order,**

$\mathcal{R}^4$	1/2 BPS	$h \geq 2$	$E_{\frac{3}{2}}$
$D^4\mathcal{R}^4$	1/4 BPS	$h \geq 3$	$E_{\frac{5}{2}}$
$D^6\mathcal{R}^4$	1/8 BPS	$h \geq 4$	$(\Delta - 12)\mathcal{E}_{D^6\mathcal{R}^4} = (E_{\frac{3}{2}})^2$

[Green, Gutperle, Vanhove 1997; Green, Vanhove 2005]

- **Predicts relations between non-vanishing contributions** (e.g. with tree-level),

$\mathcal{R}^4$	$h = 1$	[Green, Gutperle 1997]
$D^4\mathcal{R}^4$	$h = 2$	[ED, Gutperle, Phong 2005]
$D^6\mathcal{R}^4$	$h = 2$	[ED, Green, Pioline, Russo 2014]
	$h = 3$	[Gomez, Mafra 2013]

Focus of this talk

- **Effective interaction $D^6\mathcal{R}^4$ at two-loops.**
  - involves a new modular object, the "Zhang-Kawazumi-invariant".

- **Structure of $D^{2w}\mathcal{R}^4$ effective interactions for $w \geq 4$.**
  - no longer governed by BPS;
  - at one loop produces rich structure of non-holomorphic modular forms.
  - admits natural generalization to two-loops (beyond the scope of this talk)

- **In both cases, we will find that the *integrands* on moduli space**
  - ⋆ of compact Riemann surfaces (without punctures),
  - ⋆ having integrated over all vertex operator positions,
    - **obey families of interesting differential and algebraic equations**;
    - specify $D^{2w}\mathcal{R}^4$ for un-compactified or compactified space-times.

The effective interaction $D^6\mathcal{R}^4$ at genus-two

- **Start with Type II four-graviton amplitude at genus 2,** [ED, Phong 2005]

$$\mathcal{A}^{(2)} = \frac{\pi}{64}\kappa^2\mathcal{R}^4 \int_{\mathcal{M}_2} d\mu_2\, \mathcal{B}^{(2)}(s,t,u|\Omega)$$

$$\mathcal{B}^{(2)} = \int_{\Sigma^4} \mathcal{Y} \wedge \bar{\mathcal{Y}} \exp \sum_{i<j} s_{ij} G(i,j)$$

  - $\mathcal{M}_2$ is the moduli space with Siegel volume form $d\mu_2$;
  - $G(i,j)$ is the scalar Green function;
  - $\mathcal{Y} = (s-t)\Delta(1,3) \wedge \Delta(4,2) + 2$ permutations;
  - $\Delta(i,j)$ is a holomorphic $(1,0)_i \otimes (1,0)_j$ form independent of s,t,u.

- Contributions produced to local effective interactions
  - $\mathcal{R}^4$: zero, since $\mathcal{Y}$ vanishes for $s=t=u=0$;
  - $D^4\mathcal{R}^4$: non-zero, $\mathcal{B}^{(2)}$ constant on $\mathcal{M}_2$;
  - $D^6\mathcal{R}^4$: non-zero, one power of G brought down in integral over Σ^4;

$$\mathcal{B}^{(2)} = 32(s^2+t^2+u^2) + 192\, stu\, \varphi(\Omega) + \mathcal{O}(s^4, \cdots, u^4)$$

  - $\varphi(\Omega)$ coincides with the Zhang -Kawazumi invariant [ED, Green 2013].

The Zhang-Kawazumi invariant for genus-two

- **Definition of the ZK-invariant**

  – Let A_I, B_I be canonical homology basis for $H_1(\Sigma, \mathbb{Z})$,
  – ω_I dual holomorphic (1,0) forms normalized via,

$$\oint_{A_I} \omega_J = \delta_{IJ} \qquad\qquad \oint_{B_I} \omega_J = \Omega_{IJ} = X_{IJ} + iY_{IJ}$$

  then the ZK-invariant takes the following form,

$$8\varphi(\Omega) = \sum_{I,J,K,L} \left(Y^{-1}_{IJ} Y^{-1}_{KL} - 2 Y^{-1}_{IL} Y^{-1}_{JK} \right) \int_{\Sigma^2} G(x,y) \omega_I(x) \overline{\omega_J(x)} \omega_K(y) \overline{\omega_L(y)}$$

  – invariant under the modular group $Sp(4, \mathbb{Z})$
  – equivalent to definition via Arakelov geometry [Zhang 2007, Kawazumi 2008]
  – related to the genus-two Faltings invariant [De Jong 2010]

- **Direct evaluation of $\int_{\mathcal{M}_2} d\mu_2 \, \varphi(\Omega)$ appeared out of reach ... until ...**

ZK satisfies a Laplace eigenvalue equation

- **Theorem : genus-two ZK invariant satisfies remarkably simple equation,**

$$(\Delta - 5)\varphi = -2\pi\delta_{SN}$$

  – Δ is the Laplace-Beltrami operator on $\mathcal{M}_2$ with Siegel metric;
  – δ_{SN} has support on separating node (into two genus-one surfaces)

[ED, Green, Pioline, R. Russo 2014]

- **Using Theorem, the integral over $\mathcal{M}_2$ reduces to an integral over $\partial\mathcal{M}_2$,**

$$\int_{\mathcal{M}_2} d\mu_2 \, \varphi = \frac{1}{5}\int_{\mathcal{M}_2} d\mu_2 \big(\Delta\varphi + 2\pi\delta_{SN}\big) = \frac{2\pi^3}{45}$$

  – agrees with prediction from Supersymmetry, S-duality, M-theory on $\mathbb{T}^2$

[Green, Vanhove 2006]

Evidence

- **Initial indications from $D^6\mathcal{R}^4$ interaction for compactification on $\mathbb{T}^d$,**

$$\mathcal{E}^{(2)}_{D^6\mathcal{R}^4} = \pi \int_{\mathcal{M}_2} d\mu_2\, \varphi(\Omega)\, \Gamma_{d,d,2}(\rho_d|\Omega)$$

  – $\Gamma_{d,d,2}$ is the torus partition function
    ⋆ dependent on $\rho_d = G + B \in SO(d,d,\mathbb{R})/SO(d,\mathbb{R}) \times SO(d,\mathbb{R})$
    ⋆ satisfies $\left(2\Delta - \Delta_{SO(d,d)} + 3d - d^2\right)\Gamma_{d,d,2} = 0$;

  – Susy & duality conjectured relation with genus-one $\mathcal{E}^{(1)}_{\mathcal{R}^4}$ (for $d \neq 2$)

$$\left(\Delta_{SO(d,d)} - (d+2)(5-d)\right)\mathcal{E}^{(2)}_{D^6\mathcal{R}^4} = -\left(\mathcal{E}^{(1)}_{\mathcal{R}^4}\right)^2$$

  – Elimination of $\Delta_{SO(d,d)}$ gives,

$$\int_{\mathcal{M}_2} d\mu_2\, \varphi(\Omega)(\Delta - 5)\Gamma_{d,d,2}(\rho_d,\Omega) = -\frac{\pi}{2}\left(\int_{\mathcal{M}_1} d\mu_1 \Gamma_{d,d,1}(\rho,\tau)\right)^2$$

  integration by parts, and notice no d-dependence in eigenvalue !

- **Further evidence from asymptotics of φ** [via De Jong 2012, Wentworth 1991]

Proof via deformations of complex structures

- **Laplacian Δ on genus-two moduli space $\mathcal{M}_2$**
  - $=$ Laplace-Beltrami operator for the Siegel metric on upper half space
  - In terms of the period matrix $\Omega_{IJ} = X_{IJ} + iY_{IJ}$, with $I, J = 1, 2$

$$\Delta = \sum_{I \leq J} \sum_{K \leq L} Y_{IK} Y_{JL} \frac{\partial}{\partial \bar{\Omega}_{IJ}} \frac{\partial}{\partial \Omega_{KL}}$$

- **Variations in Ω_{IJ} result from variation by Beltrami differential μ**

$$\delta_\mu \phi = \frac{1}{2\pi} \int_\Sigma d^2 w \, \mu_{\bar{w}}{}^w \, \delta_{ww} \phi$$

  - $\delta_{ww}\phi$ is obtained by variation of $\bar{\partial}$ or insertion of the stress tensor T_{ww}

$$\begin{aligned} \delta_{ww} \Omega_{IJ} &= 2\pi i \, \omega_I(w) \omega_J(w) \\ \delta_{ww} \omega_I(x) &= \omega_I(w) \partial_x \partial_w \ln E(x, w) \\ \delta_{ww} G(x, y) &= -\partial_w G(w, x) \partial_w G(w, y) + \cdots \end{aligned}$$

- **Careful calculation of mixed derivatives proves $(\Delta - 5)\varphi = 0$ inside $\mathcal{M}_2$**
  - contribution from separating node results from asymptotics of φ

[ED, Green, Pioline, R. Russo 2014]

Generalizations of KZ-invariant

- **The KZ-invariant exists for all genera** $h \geq 2$ [Zhang 2007, Kawazumi 2008]
  - but does not satisfy a simple Laplace-eigenvalue equation for $h \geq 3$;
  - most likely is not the correct object for string theory at $h \geq 3$.
- But the <u>*integrands*</u> on $\mathcal{M}_2$ for the coefficients of $D^8\mathcal{R}^4$, $D^{10}\mathcal{R}^4, \cdots$
  - do naturally emerge from string theory;
  - are modular invariants which generalize ZK;
  - satisfy more complicated Laplace-type equations

    [ED, Green, Vanhove] ... in progress ...
- **The corresponding genus-one problem remains to be explored** $\cdots$

Genus-one effective interactions

- **The four-graviton amplitude is an integral over moduli space $\mathcal{M}_1$,**

$$\mathcal{A}_4^{(1)} = 2\pi\kappa^2 \mathcal{R}^4 \int_{\mathcal{M}_1} d\mu_1 \, \mathcal{B}^{(1)}(s,t,u|\tau)$$

- **$\mathcal{B}^{(1)}$ reduces to an integral over four copies of the torus Σ,**

$$\mathcal{B}^{(1)}(s,t,u|\tau) = \left(\prod_{i=1}^{4} \int_\Sigma \frac{d^2 z_i}{\tau_2} \right) \exp \Big\{ \sum_{1\leq i<j\leq 4} s_{ij} \, G(z_i - z_j|\tau) \Big\}$$

– The scalar Green function on Σ is a Fourier sum of torus momenta $(m,n) \in \mathbb{Z}^2$, where $z = \alpha + \beta\tau$ with $\alpha, \beta \in \mathbb{R}/\mathbb{Z}$,

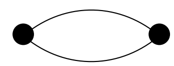
$$G(z|\tau) = \sum_{(m,n)\neq(0,0)} \frac{\tau_2}{\pi|m\tau + n|^2} \, e^{2\pi i (m\alpha - n\beta)}$$

Worldsheet Feynman diagrams

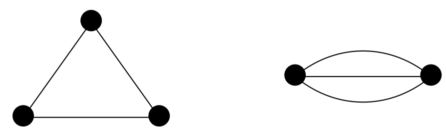
- **Expansion in powers of s_{ij} organized in "worldsheet Feynman diagrams"**
  - Each integration point z_i on Σ is represented by a vertex;
  - Each Green function $G(z_i - z_j|\tau)$ by a line —— between z_i and z_j;
  - Diagrams with a single G ending in a point vanish by $\int_\Sigma d^2z\, G(z|\tau) = 0$
  - A diagram with w lines of G,
    - $\star$ has *weight* w;
    - $\star$ contributes to $D^{2w}\mathcal{R}^4$.

Worldsheet Feynman diagrams (connected)

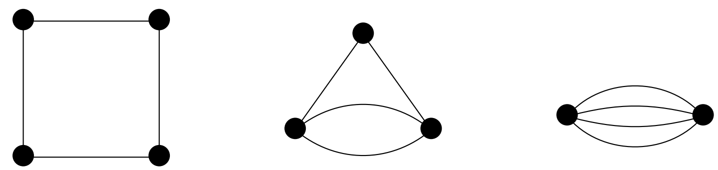
$D^4\mathcal{R}^4$



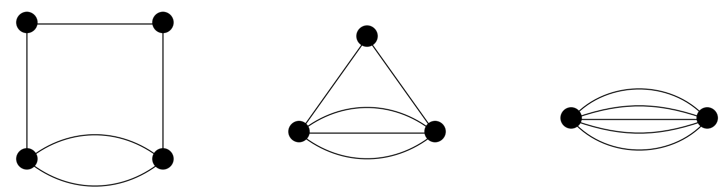
$D^6\mathcal{R}^4$

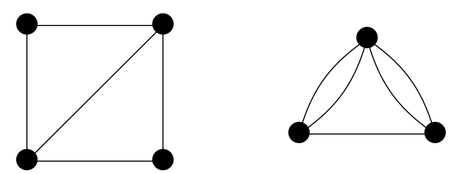


$D^8\mathcal{R}^4$

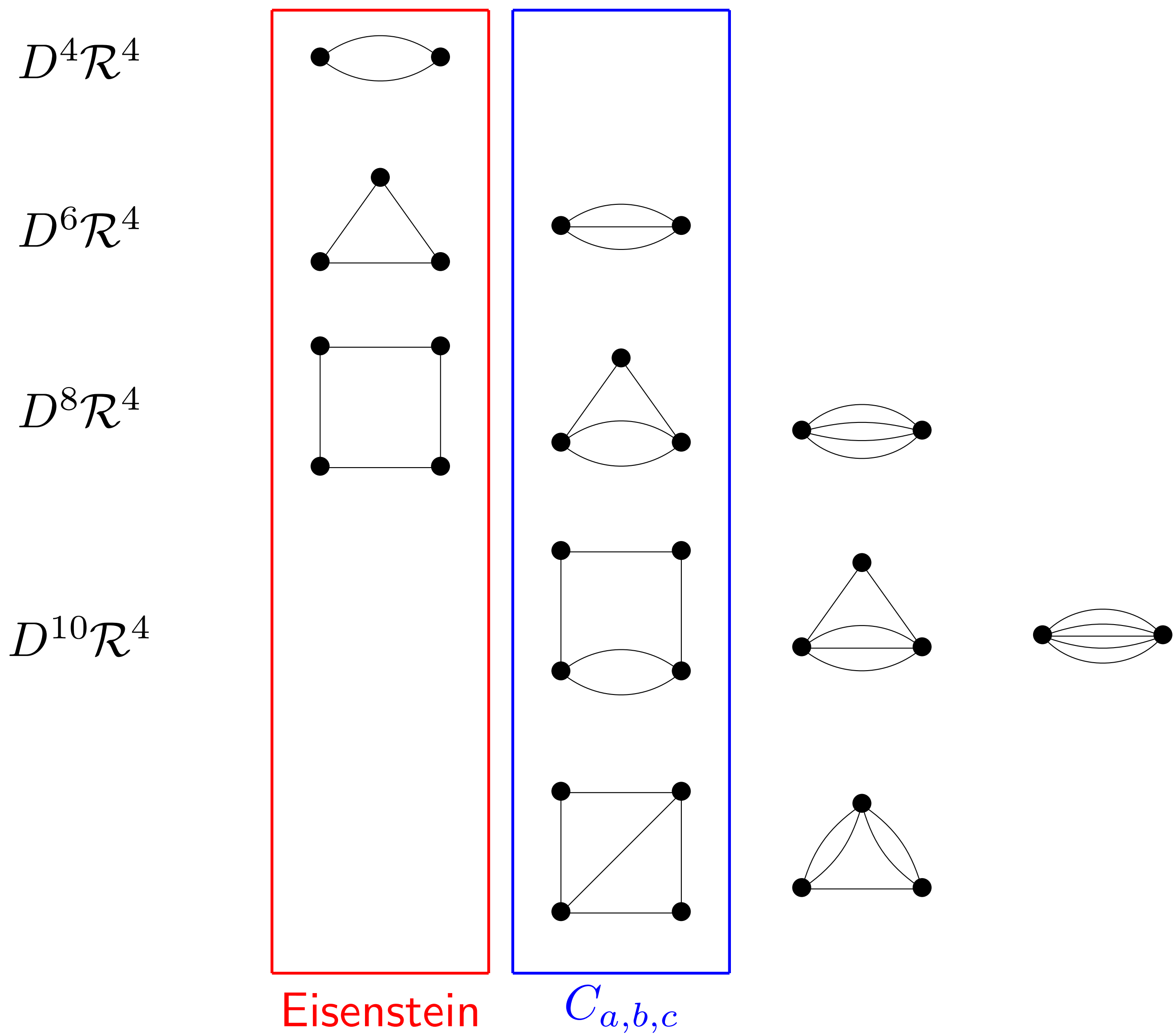


$D^{10}\mathcal{R}^4$





Worldsheet Feynman diagrams (connected)



Kronecker-Eisenstein series

- **One-loop worldsheet Feynman diagrams generate Eisenstein series.**
  - for example to order $s^2+t^2+u^2$

$$\int_\Sigma \frac{d^2z}{\tau_2} G(z|\tau)^2 = \sum_{(m,n)\neq(0,0)} \frac{\tau_2^2}{\pi^2|m\tau+n|^4} = E_2(\tau)$$

- **Two-loop Feynman diagrams generate "Kronecker-Eisenstein series".**

$$C_{a_1,a_2,a_3}(\tau) = \sum_{(m_r,n_r)\neq(0,0)} \delta_{m,0}\,\delta_{n,0} \prod_{r=1}^{3} \left(\frac{\tau_2}{\pi|m_r\tau+n_r|^2} \right)^{a_r}$$

- The total worldsheet momenta $m=m_1+m_2+m_3$, $n=n_1+n_2+n_3$ vanish;
- the *weight* is $w=a_1+a_2+a_3$;

- For our diagrams we have $a_r \geq 1$ and the sums converge;
- $C_{a_1,a_2,a_3}(\tau)$ is a modular function under $SL(2,\mathbb{Z})$.

Laplacian on moduli space

- **Structure of the space of Kronecker-Eisenstein series $C_{a,b,c}(\tau)$?**
- Tools : The Laplacian $\Delta = 4\tau_2^2 \partial_\tau \partial_{\bar\tau}$ acts algebraically on the space of $C_{a,b,c}$.

$$\Delta C_{a,b,c} = abC_{a+1,b-1,c} + \frac{1}{2} abC_{a+1,b+1,c-2} - 2abC_{a+1,b,c-1} + \frac{1}{2} a(a-1)C_{a,b,c} + \text{ 5 permutations of } (a,b,c)$$

– Δ preserves the "weight" $w = a + b + c$;
– proven by differentiating term by term and using algebraic rearrangements;

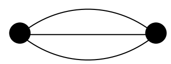
- **One of the subscript indices on the right side may equal 0 or -1,**

$$C_{a,b,0} = E_a E_b - E_{a+b} \qquad a + b \geq 3$$
$$C_{a,b,-1} = E_{a-1} E_b + E_a E_{b-1} \qquad a, b \geq 2$$

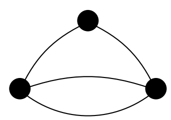
– all logarithmic divergences of the form E_1 cancel out.

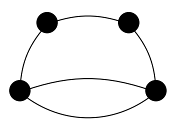
Examples at low weight w

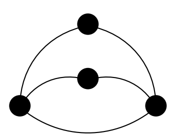
- **We find inhomogeneous Laplace-eigenvalue equations,**

$w = 3$ $\qquad C_{1,1,1} =$  $\qquad \Delta C_{1,1,1} = 6E_3$

– Use $\Delta E_3 = 6E_3$ to get $\Delta(C_{1,1,1} - E_3) = 0$;
– constant determined from asymptotics $C_{1,1,1} = E_3 + \zeta(3)$
(obtained earlier by Zagier using direct calculation of sums)

$w = 4$ $\qquad C_{2,1,1} =$  $\qquad (\Delta - 2)C_{2,1,1} = 9E_4 - E_2^2$

$w = 5$ $\qquad C_{3,1,1} =$  $\qquad (\Delta - 6)C_{3,1,1} = 3C_{2,2,1} + 16E_5 - 4E_2E_3$

$w = 5$ $\qquad C_{2,2,1} =$  $\qquad \Delta C_{2,2,1} = 8E_5$

– Note eigenvalues of the form $s(s-1)$ for $s = 1, 2, 3$;

Structure Theorem for $C_{a,b,c}$ modular functions

- $C_{a,b,c}(\tau)$ **are linear combinations of modular functions** $\mathfrak{C}_{w;s;\mathfrak{p}}(\tau)$ **which satisfy**

$$\big(\Delta - s(s-1)\big)\mathfrak{C}_{w;s;\mathfrak{p}} = \mathfrak{F}_{w;s;\mathfrak{p}}\big(E_{s'}, \zeta(s'')\big)$$

  - an inhomogeneous eigenvalue equation of weight $w = a+b+c$;
  - $\mathfrak{F}$ is a polynomial of degree 2 in $E_{s'}$ with $2 \leq s' \leq w$;
  - depends on $\zeta(s'')$ for s'' an odd integer $3 \leq s'' \leq w$;

$$s = w - 2\mathfrak{m} \qquad \mathfrak{m} = 1, \cdots, \left[\frac{w-1}{2}\right] \qquad \mathfrak{p} = 0, \cdots, \left[\frac{s-1}{3}\right]$$

- **Examples at low weight**

$w=3$	$s=1$	$0^{(1)}$
$w=4$	$s=2$	$2^{(1)}$
$w=5$	$s=1,3$	$0^{(1)} \oplus 6^{(1)}$
$w=6$	$s=2,4$	$2^{(1)} \oplus 12^{(2)}$
$w=7$	$s=1,3,5$	$0^{(1)} \oplus 6^{(1)} \oplus 20^{(2)}$
$w=8$	$s=2,4,6$	$2^{(1)} \oplus 12^{(2)} \oplus 30^{(2)}$

The generating function

- **There is a natural generating function,**

$$\mathcal{W}(t_1, t_2, t_2|\tau) = \sum_{a,b,c=1}^{\infty} t_1^{a-1} t_2^{b-1} t_3^{c-1} C_{a,b,c}(\tau)$$

Summing gives the sunset diagram for three scalars with masses $M_r^2 = -t_r \tau_2$,

$$\mathcal{W}(t_1, t_2, t_2|\tau) = \sum_{(m_r,n_r)\neq(0,0)} \delta_{m,0}\, \delta_{n,0} \prod_{r=1}^{3} \left(\frac{\tau_2}{\pi|m_r\tau + n_r|^2 - t_r\tau_2} \right)$$

- **Algebraic representation of Laplacian induces differential action on $\mathcal{W}$,**

$$\Delta \mathcal{W} - \mathfrak{L}^2 \mathcal{W} = \mathfrak{R}$$

$$\begin{aligned} \mathfrak{D} &= t_1\partial_1 + t_2\partial_2 + t_3\partial_3 \\ \mathfrak{L}^2 &= \mathfrak{D}^2 + \mathfrak{D} + (t_1^2 + t_2^2 + t_3^2 - 2t_1t_2 - 2t_2t_3 - 2t_3t_1)(\partial_1\partial_2 + \partial_2\partial_3 + \partial_3\partial_1) \\ \mathfrak{R} &= \text{quadratic polynomial in the Eisenstein series } E_s \end{aligned}$$

Proof via generating function

- **Permutations of (a, b, c) induces permutations of (t_1, t_2, t_3)**
  - $\mathfrak{S}_3$ adapted coordinates,

$$\begin{aligned} u &= t_1 + t_2 + t_3 & \varepsilon &= e^{2\pi i/3} \\ v/\sqrt{2} &= t_1 + \varepsilon t_2 + \varepsilon^2 t_3 & (t_1, t_3, t_2)(u, v, \bar v) &= (u, \bar v, v) \\ \bar v/\sqrt{2} &= t_1 + \varepsilon^2 t_2 + \varepsilon t_3 & (t_2, t_3, t_1)(u, v, \bar v) &= (u, \varepsilon^2 v, \varepsilon \bar v) \end{aligned}$$

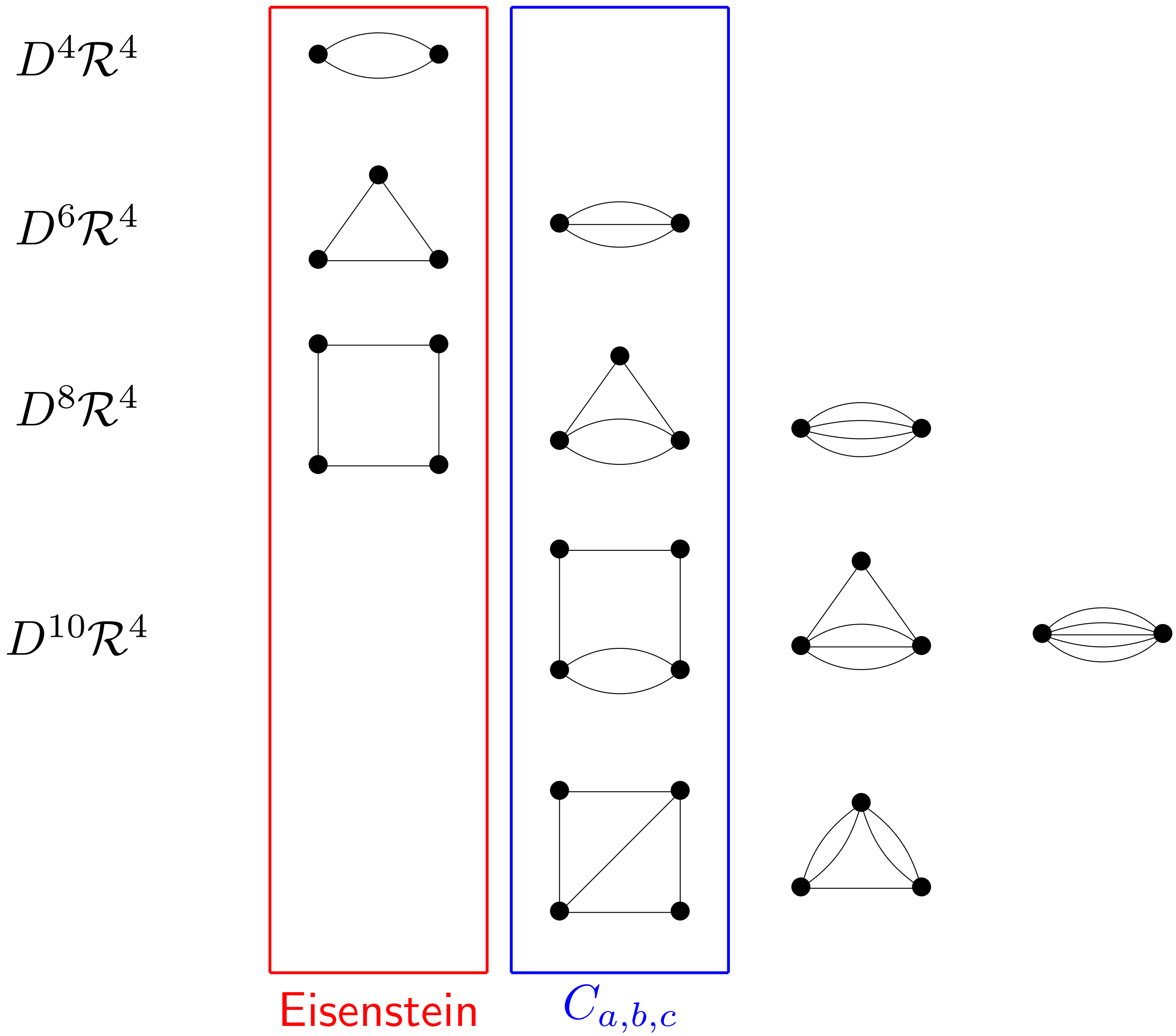
  - $\mathfrak{L}^2 = \mathfrak{L}_0^2 - \mathfrak{L}_1^2 - \mathfrak{L}_2^2$ Casimir of $SO(1,2)$ generated by $\mathfrak{L}_0, \mathfrak{L}_1, \mathfrak{L}_2$;
  - Simultaneously diagonalize the $\mathfrak{S}_3$-invariant operators $\mathfrak{D}$, $\mathfrak{L}_0^2$, and $\mathfrak{L}^2$

$$\begin{aligned} \mathfrak{D} \mathcal{W}_{w;s;\mathfrak{p}} &= w \mathcal{W}_{w;s;\mathfrak{p}} & \mathfrak{D} &= t_1 \partial_1 + t_2 \partial_2 + t_3 \partial_3 \\ \mathfrak{L}^2 \mathcal{W}_{w;s;\mathfrak{p}} &= s(s-1) \mathcal{W}_{w;s;\mathfrak{p}} & \mathfrak{L}^2 &= -(u^2 - 2v\bar v)(\partial_u^2 - 2\partial_v \partial_{\bar v}) \\ \mathfrak{L}_0^2 \mathcal{W}_{w;s;\mathfrak{p}} &= -9\mathfrak{p}^2 \mathcal{W}_{w;s;\mathfrak{p}} & \mathfrak{L}_0 &= iv\partial_v - i\bar v \partial_{\bar v} \end{aligned}$$

  - $\mathfrak{S}_3$-invariance of eigenfunctions requires $\mathfrak{p}$ to be integer;
  - which explains multiplicities $[(s-1)/3]$.

$\Longrightarrow$ constructive proof of Structure Theorem.

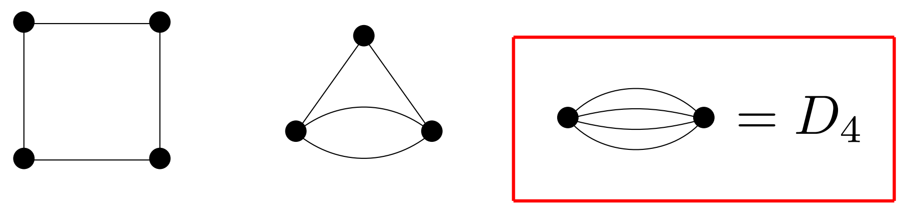
Recall ....



$D^4\mathcal{R}^4$
$D^6\mathcal{R}^4$
$D^8\mathcal{R}^4$
$D^{10}\mathcal{R}^4$
Eisenstein
$C_{a,b,c}$

Conjectured relation for modular functions in $D^8\mathcal{R}^4$

- $D^8\mathcal{R}^4$ requires



  - The modular function D_4 is *not* of the form $C_{a,b,c}$
  - no useful algebraic representation of the Laplacian is available (yet ?)

- **Tools: take an educated guess + check asymptotic behavior near cusp.**
  - Relations for $C_{a,b,c}$ involved linear combinations for given weight;
  - Consider combinations of D_4, $C_{2,1,1}$, E_4, and E_2^2

$$(\Delta - 2)\big(D_4 + \alpha C_{2,1,1} + \beta E_2^2 + \gamma E_4\big)$$

- **Inspection of asymptotics near the cusp $\tau_2 \to \infty$, leads us to conjecture,**

$$D_4 = 24 C_{2,1,1} + 3E_2^2 - 18E_4$$

  - as an *exact relation* between modular functions and Feynman diagrams

- **Additional support from direct numerical evaluation of the multiple sums.**

Structure of the asymptotics near the cusp

- **The expansion near the cusp $\tau_2 \to \infty$ takes the following form,**

$$D_4(\tau) = \sum_{k,\bar{k}=0}^{\infty} \mathcal{D}_4^{(k,\bar{k})}(\pi\tau_2)\, q^k \bar{q}^{\bar{k}} \qquad\qquad q = e^{2\pi i\tau}$$

- **We checked the following asymptotics (similarly for $C_{2,1,1}$, E_4, E_2^2)**

$$\begin{aligned}
\mathcal{D}_4^{(0,0)}(y) &= \frac{y^4}{945} + \frac{2\zeta(3)y}{3} + \frac{10\zeta(5)}{y} - \frac{3\zeta(3)^2}{y^2} + \frac{9\zeta(7)}{4y^3} \\
\mathcal{D}_4^{(0,1)}(y) &= \frac{4y^2}{15} + \frac{2y}{3} + 2 + \frac{4}{y} + \frac{12\zeta(3)}{y} - \frac{6\zeta(3)}{y^2} + \frac{9}{2y^2} + \frac{9}{4y^3} \\
\mathcal{D}_4^{(1,0)}(y) &= \mathcal{D}_4^{(0,1)}(y)
\end{aligned}$$

How could the conjecture fail ?

- Consider the difference $F = D_4 - 24C_{2,1,1} - 3E_2^2 + 18E_4$
  - the conjecture states $F = 0$

- If the conjecture were to fail, then $F \neq 0$ and its properties are,
  - modular function under $SL(2, \mathbb{Z})$;
  - its pure power part in the expansion near the cusp vanishes;
    $\Longrightarrow F$ is a *cuspidal function*
  - Vanishing of leading exponential restricts it further.

Progress towards a full proof

- **Inspired by a calculation of Zagier for C_{111}, we first perform n-sums**

$$D_4(\tau) = \sum_{(m_r,n_r)\neq(0,0)} \delta_{m,0}\,\delta_{n,0} \prod_{r=1}^{4} \frac{\tau_2}{\pi|m_r\tau+n_r|^2}$$

  - partition sum according to the number of vanishing m_r;
  - solve for n_4; decompose in partial fractions in n_3; sum over n_3,
  - for $m_3 \neq 0$, sum using the formula (and its derivatives in z)

$$\sum_{n_3\in\mathbb{Z}} \frac{1}{z+n_3} = -i\pi\, \frac{1+e^{2\pi i z}}{1-e^{2\pi i z}}$$

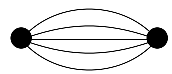
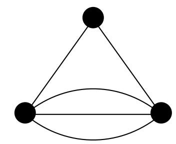
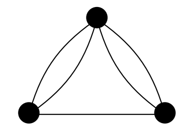
- **Explicit calculation (using MAPLE) shows the following structure**

$$D_4(\tau) = \sum_{k=-3}^{4} (\tau_2)^k\, \mathfrak{D}_4^{(k)}(q,\bar q) \qquad\qquad q = e^{2\pi i\tau}$$

  - where $\mathfrak{D}_4^{(k)}(q,\bar q)$ is an analytic function of $q,\bar q$ near the cusp
  - similar expansions for C_{211}, E_2^2, E_4 with *the same range for* k.

- **Conjecture proven for $k=-3,1,2,3,4$;**
  ... in progress for remaining values ...

Conjectured relation for modular functions in $D^{10}\mathcal{R}^4$

$D^{10}\mathcal{R}^4$ requires $D_5 =$  $D_{3,1,1} =$  $D_{2,2,1} =$ 

in addition to E_5, $C_{3,1,1}$, E_2E_3, and $E_2C_{1,1,1}$ functions of weight 5.

- **Educated guesses and asymptotics near the cusp lead us to conjecture,**

$$\begin{aligned}
D_5 &= 60C_{3,1,1} + 10E_2C_{1,1,1} - 48E_5 + 16\zeta(5) \\
40D_{3,1,1} &= 300C_{3,1,1} + 120E_2E_3 - 276E_5 + 7\zeta(5) \\
10D_{2,2,1} &= 20C_{3,1,1} - 4E_5 + 3\zeta(5)
\end{aligned}$$

- **Pattern expected to continue for higher $D^{2w}\mathcal{R}^4$ interactions with $w > 5$.**

Generalizations and Multi-zeta-values

- **Generalized infinite families entering genus-one diagrams** (but not all)

$$C_{a_1,\cdots,a_\rho}(\tau) = \sum_{(m_r,n_r)\neq(0,0)} \delta_{m,0}\,\delta_{n,0} \prod_{r=1}^{\rho} \left(\frac{\tau_2}{|m_r\tau + n_r|^2} \right)^{a_r}$$

  – for integers $a_r \geq 1$ enter diagrams of weight $w = a_1 + \cdots + a_\rho$

- **Multi-zeta-functions**

$$\zeta(s_1, \cdots, s_n) = \sum_{m_1 > \cdots > m_n \geq 1} \frac{1}{m_1^{s_1} \cdots m_n^{s_n}}$$

  – generalize the standard Riemann zeta-function for $n = 1$

- **The $C_{a_1,\cdots,a_\rho}$ provide a modular generalization of** $\zeta(s_1, \cdots, s_n)$
  – leading τ_2 behavior of $C_{a_1,\cdots,a_\rho}$ may be expressed as MZV.
  – MZV naturally enter into open string amplitudes (see Schlotterer's talk)

Genus-one coefficients of $D^{2w}\mathcal{R}^4$ for $w \geq 4$

- **Integration over moduli $\mathcal{M}_1$ produces non-analytic behavior in s, t, u;**
  - branch cuts due to loops with massless strings for $s, t, u, \ll 1$;
  - non-analytic parts may be isolated systematically,
    [ED, Phong 1993; Green, Russo, Vanhove 2008]
  - Analytic part is unique only after non-analytic part has been specified.

- **Partition fundamental domain $\mathcal{M}_1$ at fixed large $L \gg 1$;** [Maass; Selberg]
  - $\tau_2 > L$ gives non-analytic contributions in s, t, u;
  - $\tau_2 < L$ gives analytic contributions in s, t, u;

- **For compactifications on $\mathbb{T}^d$, for example,**

$$\mathcal{E}_{D^8\mathcal{R}^4}(\rho_d, L) = \frac{1}{2}\int_{\mathcal{M}_1}^{\tau_2<L} d\mu_1 \big(\Delta C_{2,1,1} - 5E_4 + E_2^2\big)\, \Gamma_{d,d,1}(\rho_d|\tau)$$

  - parts non-analytic at $s, t, u = 0$ cancel in comparing moduli ρ_d and ρ'_d;
  - "Differences" produce well-defined and unique effective interactions.

Summary and outlook

- **Low energy expansion of string theory has revealed a rich structure of**
  - non-holomorphic Kronecker-Eisenstein series on genus-one Riemann surfaces;
  - Zhang-Kawazumi modular invariant on genus-two Riemann surfaces;
  - differential and algebraic interrelations;
  - concrete analytic evaluation of local effective interactions beyond BPS.

- **Extensions at genus-one**
  - Understand general interrelations of Kronecker-Eisenstein series beyond $C_{a,b,c}$
  - Identify structure of the ring of all such non-holomorphic modular forms.
  - Equations obeyed by entire string integrand ? [ED, Green] ... in progress ...

- **Extensions at genus-two**
  - Lifts to toroidal compactifications [Pioline 2015]
  - Differential relations obeyed by higher order generalizations of Zhang-Kawazumi invariants [ED, Green, Vanhove] ... in progress ...

- **Significance for number theory ?**