

Quantum Criticality via Magnetic Branes

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- ED, Per Kraus, *Magnetic Branes Solutions in AdS*, arXiv:0908.3875;
- ED, Per Kraus, *Holographic Metamagnetism, Quantum Criticality ...*, arXiv:1003.1302;
- ED, Per Kraus, *Magnetic Field Induced Quantum Criticality ...*, arXiv:1006.2573;
- ED, Per Kraus, Akhil Shah, *RG Flow of Magnetic Brane Correlators*, arXiv:1012.5072;
- ED, Per Kraus, *Charged Magnetic Brane Correlators and Twisted ...*, arXiv:1105.3998;
- ED, Per Kraus, *Holographic Quantum Criticality in the Plane*, arXiv:1202.xxxx.

The Problem

- Explore strong coupling dynamics via holography
 - look for universal behavior in “interesting” systems
 - obtain thermodynamics, correlators, and transport properties
 - try to make contact with experiments, e.g. in condensed matter
- Investigate 3+1-dim gauge theory (There is a wealth of results in 2+1 dim.)
 - with constant external magnetic field
 - and finite electric charge density
 - supersymmetric or not supersymmetric
- Rich structure, exhibiting e.g.
 - quantum critical behavior
 - emergent 1+1-dimensional CFT in IR correlators

Universal Gravity Dual

- Concentrate on universal operators;
 - stress tensor $T^{\mu\nu}$ (dual to metric g_{MN})
 - current J^μ (dual to gauge field A_M)
- Einstein-Maxwell-Chern-Simons in 5-dimensions,

$$S = -\frac{1}{16\pi G_5} \int \sqrt{g} (R - 12 + F_{MN}F^{MN}) + \frac{k}{12\pi G_5} \int A \wedge F \wedge F$$

- We will consider general values of k , choosing $k \geq 0$ by parity;
 - k only dimensionless coupling: $\partial_\mu J^\mu \sim k \vec{E} \cdot \vec{B}$ gives chiral anomaly;
 - $k = \sqrt{4/3}$ includes consistent truncation of all susy asymptotic AdS_5 compactifications of Type IIB and M-theory (Gauntlett and Varela)
 \Rightarrow Our approach is “top-down” for this value.

Outline

- Qualitative gauge dynamics in the presence of a magnetic field;
- Thermodynamics and IR behavior of correlators,
 - Purely magnetic brane (zero charge density)
 - Charged magnetic brane (non-zero charge density;)
- In progress: **Quantum critical behavior in 2+1 dims.**

Basic Gauge Theory Dynamics

- Charges q in a constant magnetic field $B \Rightarrow$ Landau levels

$$\text{Bosons} \quad E_{n,p}^B = \sqrt{p^2 + (2n+1)|qB|} \quad n = 0, 1, 2, \dots$$

$$\text{Fermions} \quad E_{n,p}^F = \sqrt{p^2 + 2n|qB|} \quad n = 0, 1, 2, \dots$$

- where p is the momentum component parallel to B .
- breaks susy; modifications with susy exist (Almuhairi, Polchinski 2011)

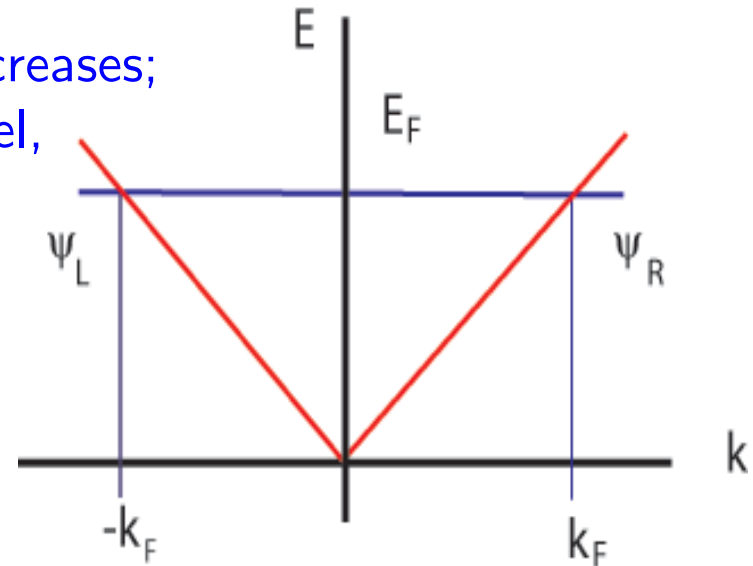
- For large B , all levels, except LLL fermions, acquire large mass,

\Rightarrow Interacting Weyl fermions in 1+1 dimensions

- $q > 0$ right-movers with $p > 0$ and field ψ_R
- $q < 0$ left-movers with $p < 0$ and field ψ_L

\Rightarrow Expect a family of 1+1 dimensional CFTs.

- At finite density, fermion levels fill up to Fermi surface energy E_F ;
 - As charge density ρ increases, E_F increases; When E_F reaches next Landau level, or energies of 3-dim excitations,
 - \Rightarrow possible transition at $\rho \sim B^{3/2}$ (on dimensional grounds)



- QFT approach to 1+1 dim fermions: **Luttinger Liquid theory**
 - Linearize excitations around Fermi “surface”;
 - include all possible four-Fermi interactions;
 - \Rightarrow e.g. entropy density, for excitation with velocity v ,

$$s = \frac{\pi T}{3v}$$

Holographic Dual set-up

- 5-dim EMCS theory with CS coupling k

$$0 = dF$$

$$0 = d * F + kF \wedge F$$

$$R_{MN} = 4g_{MN} - 2F_M^P F_{NP} + \frac{1}{3}g_{MN}F^2$$

- Constant B -field at the asympt AdS_5 boundary,

$$F = Bdx^1 \wedge dx^2 + \dots$$

- Finite electric charge density ρ ,

$$F \sim \frac{\rho}{r^2} dr \wedge dt + \dots$$

Zero charge density: purely magnetic brane

- Invariance under translations in x^μ , and Lorentz in $x^\pm = x^3 \pm t$ implies,

$$F = B dx^1 \wedge dx^2$$

$$ds^2 = \frac{dr^2}{L^2} + 2L dx^+ dx^- + K ((dx^1)^2 + (dx^2)^2)$$

- where K, L depend only on r . The field equations reduce to,

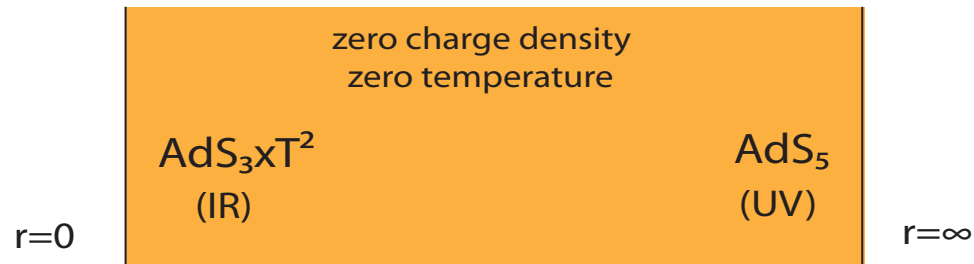
$$0 = (L^2 K)'' - 24K$$

$$-4B^2 = L^2 (K')^2 + 4KLK'L' + K^2 (L')^2 - 24K^2$$

- get L from K by quadrature of the first equation;
- solve for K numerically, subject to boundary conditions;

RG flow of purely magnetic brane

- The purely magnetic brane solution produces an RG flows;



- The magnetic brane links AdS_5 and $\text{AdS}_3 \times T^2$ asymptotics,

$$\begin{array}{lll}
 r \rightarrow \infty & K(r) \sim c_V r & L(r) \sim 2r \\
 r \rightarrow 0 & K(r) \sim B/\sqrt{3} & L(r) \sim 2\sqrt{3}r
 \end{array}$$

- derive value of constant numerically, $c_V \approx 2.797$.

- Dual gauge theory RG flow is from 3+1 YM to 1+1-dim CFT
 - Consistent with $\rho \ll B^{3/2}$ where the LLL fermions dominate.

Thermodynamics

- Low temperature thermodynamics governed by Cardy's formula,
 - Entropy density is given in terms of the central charge;
 - Central charge for IR AdS_3 is given by Brown-Henneaux formula,

$$s_{\text{grav}} = \frac{\pi c}{3} T \qquad c = \frac{3\ell_3}{2G_3} \qquad \frac{1}{G_3} = \frac{\text{Vol}(T^2)}{G_5}$$

- Comparison with large N free LLL fermions of gauge dual,

$$s_{\text{grav}} = \sqrt{\frac{4}{3}} s_{\text{gauge}} \sim BT$$

- All temperatures
 - Replace IR near-horizon AdS_3 by BTZ (need to extend Ansatz)
 - Construct (numerically) interpolating solution asymptotic to AdS_5 ,
 - Entropy density connects to (known) high T behavior,

$$s_{\text{grav}} = \frac{3}{4} s_{\text{gauge}} \sim T^3$$

Correlators: sources, expectation values

- Probe IR by correlators of stress tensor $T_{\mu\nu}$, and current J_μ
 - Extract expectation values of \mathcal{J}, \mathcal{T} by Fefferman-Graham expansion,

$$ds^2 = \frac{dr^2}{4r^2} + \left(4r g_{\mu\nu}^{(0)}(x) + g_{\mu\nu}^{(2)}(x) + \frac{1}{4r} g_{\mu\nu}^{(4)}(x) + \dots \right) dx^\mu dx^\nu$$

$$A = \left(A_\mu^{(0)}(x) + \frac{1}{4r} A_\mu^{(2)}(x) + \dots \right) dx^\mu$$

- $g_{\mu\nu}^{(0)}(x), A_\mu^{(0)}(x)$ are sources; expectation values $T_{\mu\nu}, J_\mu$ are given by

$$4\pi G_5 T_{\mu\nu}(x) = g_{\mu\nu}^{(4)}(x)$$

$$2\pi G_5 J_\mu(x) = A_\mu^{(2)}(x)$$

- Compute correlators by linear response,

$$J_\mu(x) = i \int d^4 y \sqrt{g^{(0)}} \langle \mathcal{J}_\mu(x) \mathcal{J}^\nu(y) \rangle \delta A_\nu^{(0)}(y)$$

Current-current correlators: set-up

- Zero charge density: $\langle \mathcal{J}T \rangle = 0$; correlators $\langle \mathcal{J}\mathcal{J} \rangle$ decouple from $\langle TT \rangle$.
- Linearize Maxwell-Chern-Simons around purely magnetic brane solution
 - expand in plane waves around magnetic brane solution A_B
 - set transverse momenta $p_1 = p_2 = 0$,

$$A = A_B + a_\mu(r, p_\pm) dx^\mu e^{ip_+ x^+ + ip_- x^-}$$

- Decouple reduced equations via $\varepsilon_\pm = p_- a_+ \pm p_+ a_-$ and $p^2 = p_+ p_-$

$$0 = KL\varepsilon'_+ - 2kB\varepsilon_-$$

$$0 = KL^2(KL\varepsilon'_-)' - 4k^2B^2L\varepsilon_- - 2p^2K^2\varepsilon_-$$

- General p^2 requires numerical solution (K, L known only numerically)

Overlapping Expansions

- Interested in IR behavior, so we specialize to $0 < p^2 \ll 1$.
- Make use of overlapping expansions,
 - ★ Near region $r \ll 1$
 - approximate K, L by small r form: $K \sim 1, L \sim 2\sqrt{3}r$
 - Impose regularity as $r \rightarrow 0$
 - solved by $\varepsilon_-(r) \sim K_{2k}(p/\sqrt{r})$
 - ★ Far region $p^2 \ll r$
 - Terms suppressed by p^2/r may be neglected;
 - solved by $a_{\pm}(r) \sim A_{\pm}^{(0)} + A_{\pm}^{(2)} \left(A_0(r)^{\pm 1} - \mu^{\pm 1} \right)$
 - Using $A_0(r) \sim \mu - \rho/r$ for $r \rightarrow \infty$ identify source and vev.
 - ★ Overlap region $p^2 \ll r \ll 1$
 - Both expansions valid simultaneously, in finite region;
 - Match solutions using,
 - $A_0(r) \sim r^k$ for $r \ll 1$
 - $K_{2k}(z) \sim z^{2k} + z^{-2k}$

Current-current correlators

- Current-current correlators are (up to contact terms)

$$\langle \mathcal{J}_+(p) \mathcal{J}_+(-p) \rangle = \frac{kc}{2\pi} \frac{p_+}{p_-} \frac{1}{1 - \zeta p^{4k}} \quad c = \frac{3\ell_3}{2G_3}$$

$$\langle \mathcal{J}_-(p) \mathcal{J}_-(-p) \rangle = \frac{kc}{2\pi} \frac{p_-}{p_+} \frac{\zeta p^{4k}}{1 - \zeta p^{4k}} \quad \zeta = \zeta(k)$$

$$\langle \mathcal{J}_+(p) \mathcal{J}_-(-p) \rangle = -\frac{kc}{2\pi} \frac{\zeta p^{4k}}{1 - \zeta p^{4k}}$$

- Check the chiral anomaly equation,

$$p_+ \mathcal{J}_- + p_- \mathcal{J}_+ = \frac{kc}{\pi} (p_+ A_-^{(0)} - p_- A_+^{(0)})$$

- In the IR limit, only the $+$ chirality correlator survives,

$$\langle \mathcal{J}_+(x) \mathcal{J}_+(0) \rangle \sim -\frac{kc}{2\pi (x^+)^2}$$

\Rightarrow a single **unitary** chiral current algebra, which saturates the anomaly.

Maxwell-Chern-Simons holography in AdS_3

- Taking p^2 large, only the $-$ chirality correlator survives,

$$\langle \mathcal{J}_-(x) \mathcal{J}_-(0) \rangle \sim + \frac{kc}{2\pi(x^-)^2}$$

- Again saturates the anomaly;
- But now has **non-unitary** sign and is, by itself, inconsistent.
- MCS holography directly in terms of AdS_3 solution is problematic:
 - take large r behavior of near region

$$a_{\pm}(r) = a_{\pm}^0 + a_{\pm}^1 r^{\pm k}$$

- How does one identify source and expectation value modes ?

$\Rightarrow \text{AdS}_5$ RG flow solution provides consistent MCS holography for AdS_3 .

Stress-tensor correlators

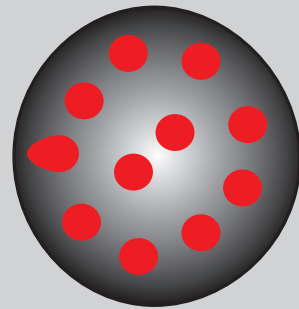
- Extract stress tensor correlators from overlapping expansions method,

$$\langle T_{\pm\pm}(x)T_{\pm\pm}(0) \rangle = \frac{c}{8\pi^2(x^\pm)^4} \quad c = \frac{3\ell_3}{2G_3}$$

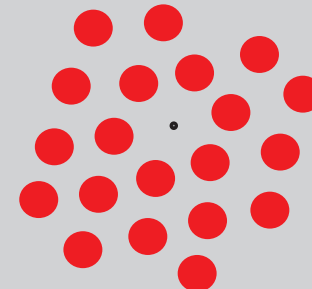
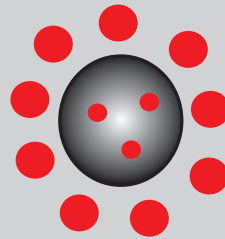
- T defined in terms of the AdS_5 asymptotics;
 - Central charge of IR CFT coincides with c of near-horizon AdS_3 .
 - Brown-Henneaux coord transf at near-horizon AdS_3 form ∞ -dim Virasoro
 - Asympt symmetry group of AdS_5 does not support this.
 - BH coord transf extend to full asympt AdS_5 RG flow solution.
- \Rightarrow example of emergent Virasoro symmetry in IR

Charged magnetic brane solution

- For $B = 0$, Reissner-Nordstrom has all charge inside the horizon;
 - also has non-zero entropy at $T = 0$.
- For $B \neq 0$, an effective 2+1-dim CS term is generated;
 - this makes the gauge field carry its charge outside the horizon.
- For $B \rightarrow \infty$, all charge is carried by the gauge field outside of horizon.

 $B=0$ 

all charge inside
the black hole



all charge outside
the black hole

 $B = \infty$

- Charge requires turning on gauge field $A(r)$, and extending Ansatz,

$$F = B dx^1 \wedge dx^2 + A' dr \wedge dx^+ + C' dr \wedge dx^-$$

$$ds^2 = \frac{dr^2}{L^2 - MN} + M(dx^+)^2 + 2L dx^+ dx^- + N(dx^-)^2 + K dx^i dx^i$$

- There exist solutions with
 - $C = N = 0$, and K, L *unmodified* from purely magnetic solution

$$A(r) = \mu \exp\left(2kB \int_{\infty}^r \frac{dr'}{KL}\right)$$

$$M(r) = L(r) \left(-\alpha_{\infty} - 4kB \int_{\infty}^r \frac{dr' A^2}{KL^2} \right)$$

- $\mu = A(\infty)$ is the chemical potential,
- α_{∞} tilts the UV space-time coordinates,

- Near-boundary geometry as $r \rightarrow \infty$,

$$A(r) \sim \mu - \rho \frac{\sqrt{\alpha_\infty}}{r} \qquad M(r) \sim -2r\alpha_\infty$$

- Near-horizon geometry as $r \rightarrow 0$ is a Schrödinger: $A \sim \frac{q}{k} r^k$,

$$ds^2 \sim \frac{dr^2}{12r^2} + 4\sqrt{3}r dx^+ dx^- - \left(\alpha_0 r + \frac{2q^2 r^{2k}}{k(2k-1)} \right) (dx^+)^2 + \frac{B}{\sqrt{3}} dx^i dx^i$$

- Constants α_0, q related to α_∞, μ via purely magnetic brane data;

$$\alpha_\infty - \alpha_0 = \alpha_c = 24k \int_0^\infty dr \frac{A^2}{KL^2}$$

$$\ln \frac{q}{k\mu} = 2kb \lim_{r \rightarrow 0} \left(-\frac{\ln r}{2b} + \int_\infty^r \frac{dr'}{Le^{2V}} \right)$$

Regularity of solutions

- View solutions as $T \rightarrow 0$ limit of finite temperature black branes;
 - Choose time direction $t = x^+$, $x^3 = x^-$, horizon at $r = 0$;
- Regularity requires $M(r) \leq 0$
 - can be realized only for $k > \frac{1}{2}$
 - requires $\alpha_0 \geq 0$
 - but α_0 and α_∞ are related by,
- There exists a **critical magnetic field** \hat{B}_c , since we must have,

$$\frac{\alpha_\infty}{\alpha_c} = \frac{\hat{B}^3}{\hat{B}_c^3} \geq 1 \quad \text{where} \quad \hat{B} \equiv \frac{B}{\rho^{2/3}}$$

(recall that only dimensionless ratios like \hat{B} have physical meaning)

Low T thermodynamics for $\hat{B} > \hat{B}_c$

- Match overlapping expansions to construct interpolating solution between
 - Asymptotically AdS_5 as $r \rightarrow \infty$
 - near-horizon BTZ as $r \rightarrow 0$
- Low $T = 0$ limit gives **exact** formula for the entropy density s ,

$$\frac{\hat{s}}{\hat{T}} = \frac{\pi}{6} \frac{\hat{B}^3}{\hat{B}^3 - \hat{B}_c^3} \quad \hat{T} \equiv \frac{T}{\sqrt{B}} \quad \hat{s} \equiv \frac{s}{\sqrt{B^3}}$$

- cfr Luttinger liquids behavior; note: vanishing entropy density at $T = 0$
- Divergence of \hat{s}/\hat{T} as $\hat{B} \rightarrow \hat{B}_c$ signals a *quantum critical point* at \hat{B}_c .
 - Expect *quantum critical behavior* near $\hat{B} = \hat{B}_c$ and $T = 0$.
(see also Jensen, Karch, Thompson)
- The phase for $\hat{B} < \hat{B}_c$ has *non-zero entropy* at $T = 0$.
 - continues into the Reissner-Nordstrom solution for $B = 0$.

Low T thermodynamics for $\hat{B} \sim \hat{B}_c$

- Set $\alpha_0 = 0$, $\alpha_\infty = \alpha_c$. Near horizon geometry metric,

$$ds^2 = \frac{dr^2}{12r^2} + 4\sqrt{3}rdtdx_3 - \frac{q^2 r^{2k}}{k(k - \frac{1}{2})} dt^2 + dx_1^2 + dx_2^2$$

- Invariant under $r \rightarrow \lambda r$, $t \rightarrow \lambda^{-k} t$, $x_3 \rightarrow \lambda^{k-1} x_3$
- with dynamical scaling exponent $z = k/(1 - k)$
- z and dim d determine scaling of entropy density

$$\hat{s} \sim \hat{T}^{\frac{d}{z}} = \hat{T}^{\frac{1-k}{k}}$$

- Note: variable dynamical exponents arise also with a dilaton
(Goldstein, Kachru, Prakash, Trivedi)

- Numerical analysis shows this behavior realized only when $\frac{1}{2} < k \leq \frac{3}{4}$
 - Near horizon region has electrically charged black hole \neq BTZ,

- But this is not the only scale transformation that matters;
 - Take a transformation that leaves AdS_3 part invariant,

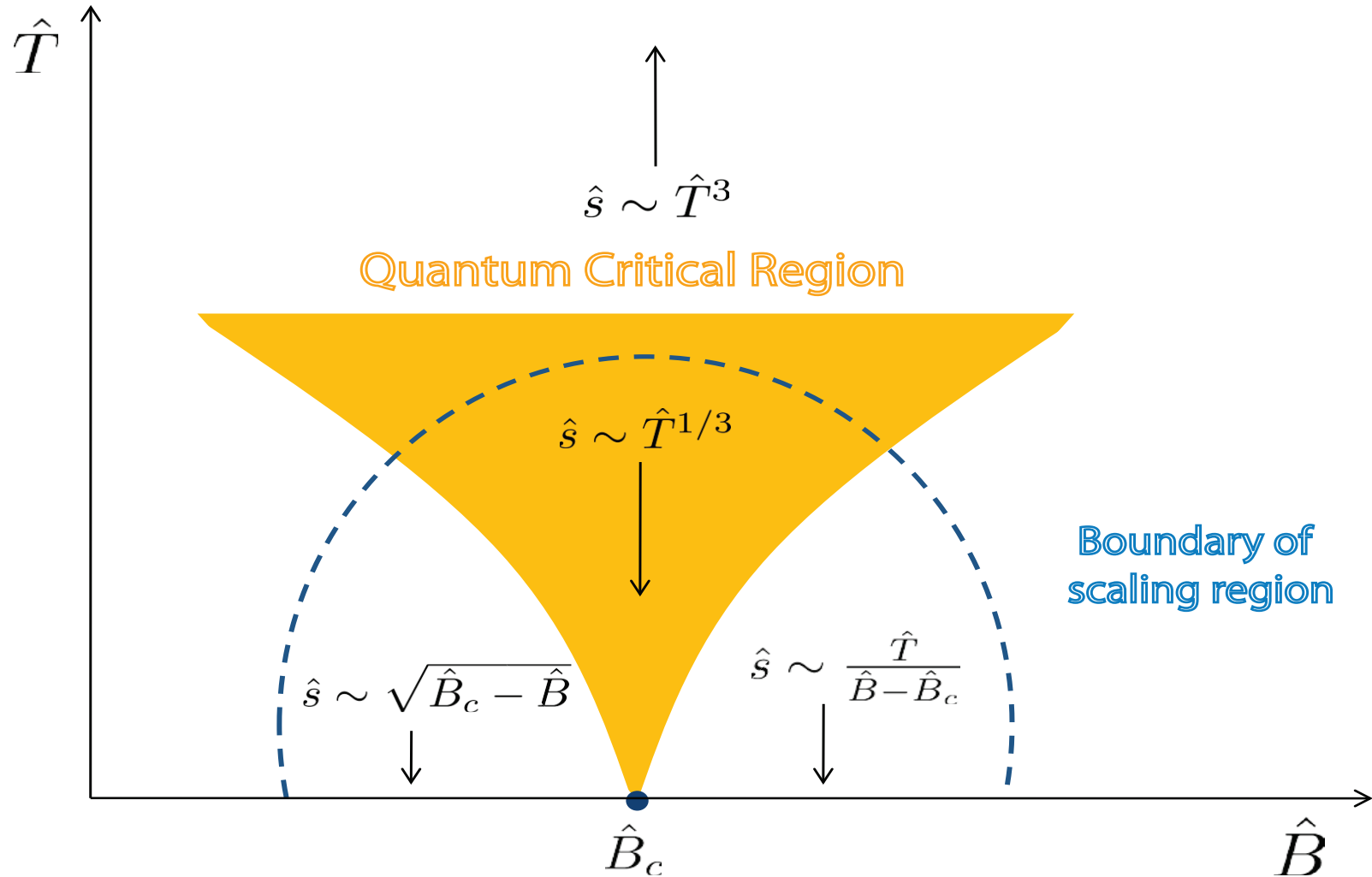
$$r \rightarrow \lambda r \quad t \rightarrow t/\sqrt{\lambda} \quad x_3 \rightarrow x_3/\sqrt{\lambda}$$

- then the $r^{2k} dt^2$ term transforms as follows,

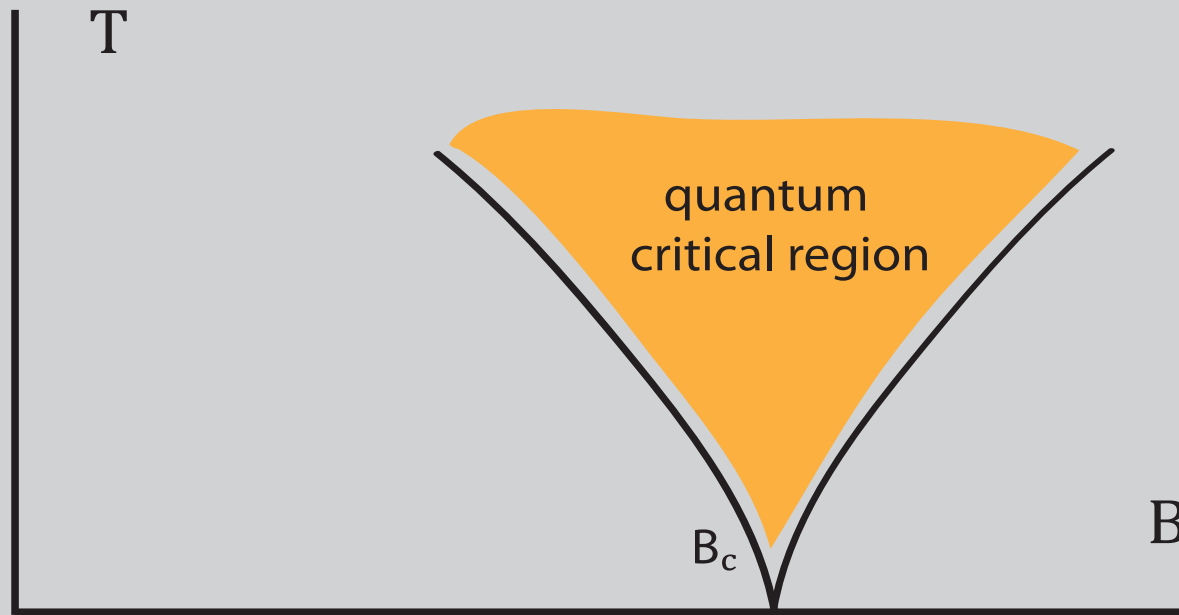
$$r^{2k} dt^2 \rightarrow \lambda^{2k-1} r^{2k} dt^2$$

- For $k > \frac{1}{2}$ this scales to 0 as $\lambda \rightarrow 0$, and is **irrelevant** in the IR.
- When $k \geq \frac{3}{4}$, we find instead $\hat{s} \sim \hat{T}^{\frac{1}{3}}$ at fixed \hat{B}
 - analytically: matching near-horizon BTZ onto far region solution
 - numerically: exponent verified to 0.1% accuracy;

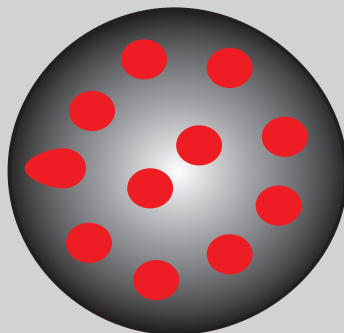
Phase diagram ($k \geq 3/4$)



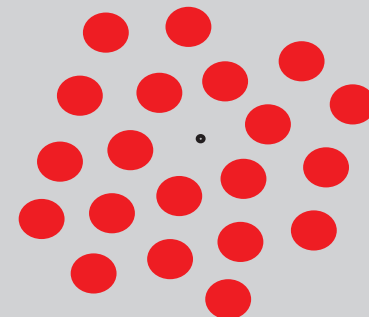
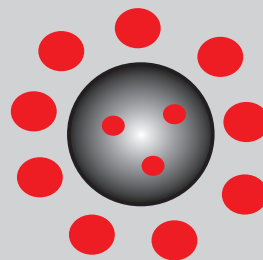
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all charge inside
the black hole



all charge outside
the black hole

Non-zero charge density – correlators

- Turn on charge density by turning on chemical potential μ , set $T = 0$;
 - **Minus** chirality correlators are unmodified from zero charge density,
 - **Plus** chirality correlators are modified to **twisted Virasoro**,

$$\langle \mathcal{J}_+(x) \mathcal{J}_+(0) \rangle \sim -\frac{kc}{2\pi(x^+)^2}$$

$$\langle \mathcal{J}_+(x) \mathcal{T}_{++}(0) \rangle \sim +\frac{kc\mu}{2\pi^2(x^+)^2}$$

$$\langle \mathcal{T}_{++}(x) \mathcal{T}_{++}(0) \rangle \sim -\frac{kc\mu^2}{2\pi^2(x^+)^2} + \frac{c}{8\pi^2(x^+)^4}$$

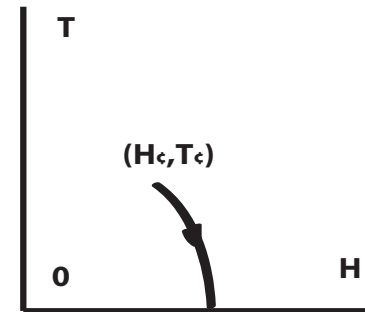
- Simple re-interpretation of μ -dependence via operator mixing,

$$\mathcal{J}_+^{(\mu)} = \mathcal{J}_+^{(\mu=0)} \quad \mathcal{T}_{++}^{(\mu)} = \mathcal{T}_{++}^{(\mu=0)} - \mu \mathcal{J}_+^{(\mu=0)}$$

- \implies For $\hat{B} > \hat{B}_c$ (and $k > 1$),
the underlying CFT, including its central charge, is unmodified.

Relation to Condensed Matter Systems ?

- Phase transition across $B \neq 0$ is *metamagnetic*.
 - no symmetry change;
 - achieved experimentally in layered materials;
 - with a line of first order phase transitions;
 - dial end point (H_c, T_c) to zero temperature;
 - by varying magnetic field H in plane of layers.
- Specifically, $\text{Sr}_3\text{Ru}_2\text{O}_7$ (Rost, Perry, Mercure, Mackenzie, Grigera, 2009)
 - At low temperatures ($\sim 1\text{K}$)
specific heat capacity exhibits critical behavior across $H \sim 8\text{ T}$



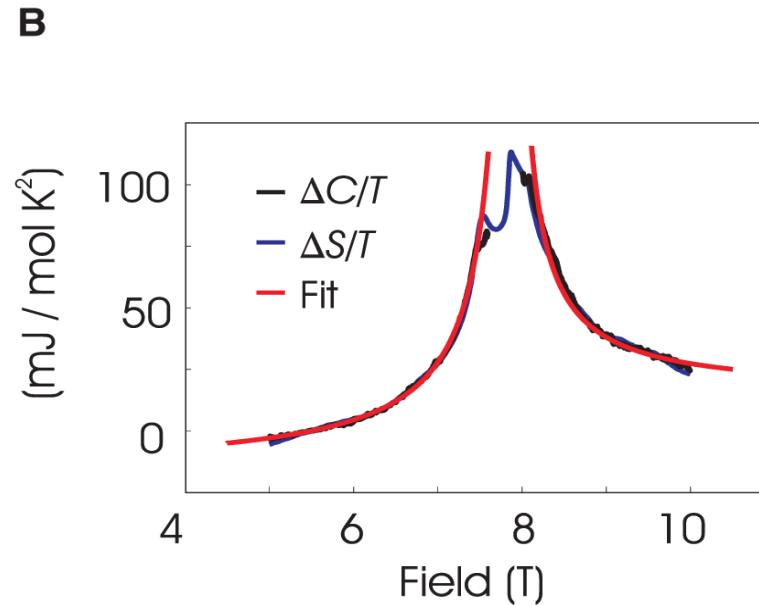


Fig. 4. (A) Entropy divided by temperature as a function of applied field and temperature for $\text{Sr}_3\text{Ru}_2\text{O}_7$. **(B)** Both $\Delta S/T$ (blue) and the specific heat, $\Delta C/T$ (black), diverge as $[(H - H_c)/H_c]^{-1}$ outside the cut off region (red lines). $\Delta C/T$ is fixed relative to $\Delta S/T$ at 5.5 T on the low field side and 9.5 T on the high field side. **(C)** Isoentropes dip to lower and lower temperatures as a function of applied magnetic field in another signature of quantum criticality, until that dip is cut off by the phase formation that truncates the peak in S/T in (A).

- in remarkable agreement with holographic predictions
- divergence cut off by formation of novel “electronic nematic phase”
see e.g. (Fradkin, Kivelson, Lawler, Eisenstein, Mackenzie, 2009)

Work in Progress – Future Directions

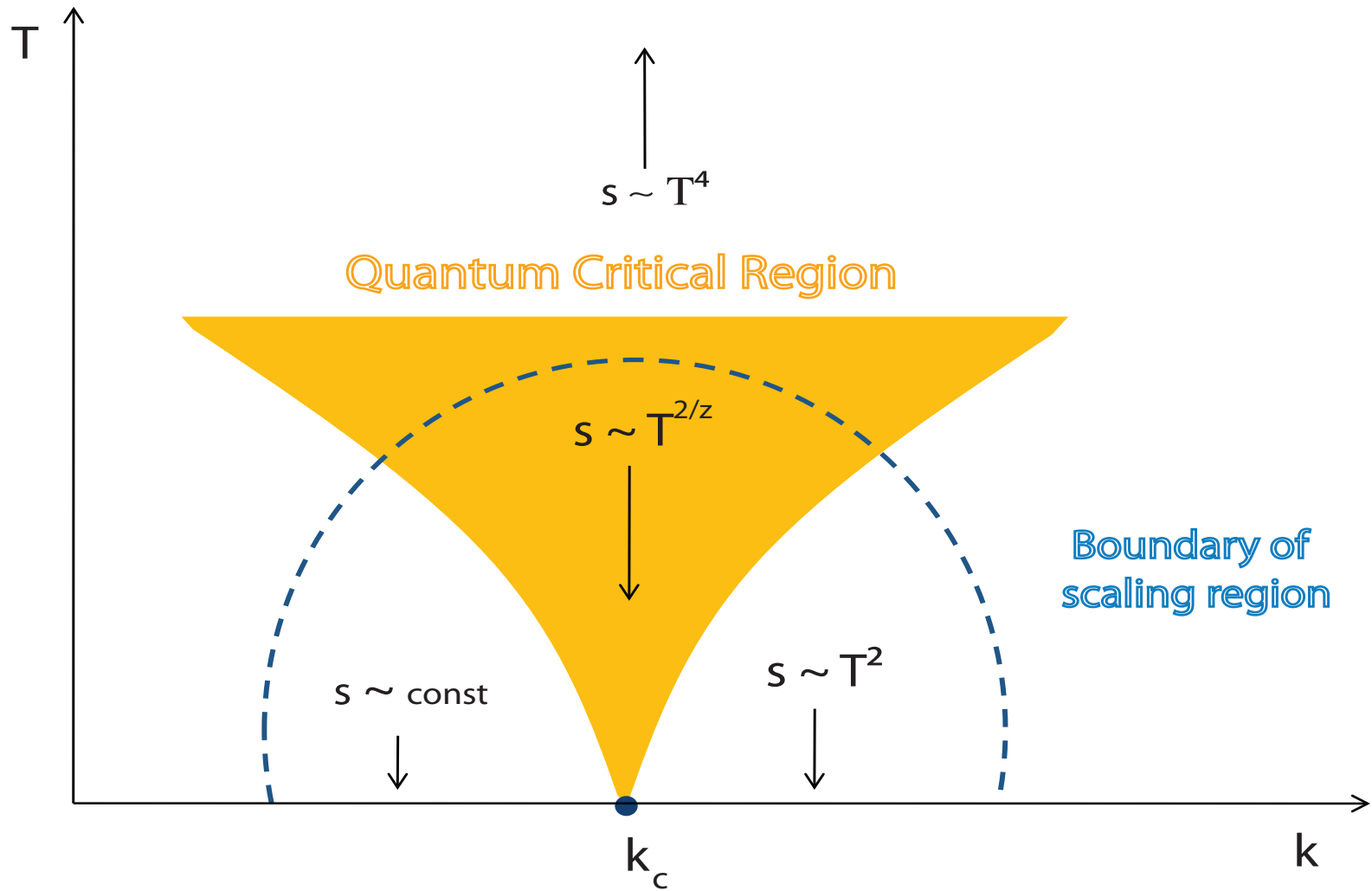
- Dynamical exponent $1/3$
 - arises in Hertz/Millis theory of metamagnetic quantum phase transitions
(based on a non-local effective theory; thus open to controversy)
 - no sign of exponent in IR correlators calculated here (T strictly zero)
 - can holography produce relevant correlators/transport properties ?
- Holographic description of electronic nematic phase ?
 - Relation with possible smectic/striped phases ?
Nakamura, Ooguri, Park; Donos, Gauntlett, Panteleidou

In progress: Quantum Criticality in 2+1 dims

- Holó dual of 2+1-dim Quantum Criticality via magnetic branes.
 - purely magnetic brane gives flow $AdS_6 \rightarrow AdS_4$
 - simplest system has gauge field $F = dA$ and 3-form $G = dC$.

$$S \sim \int d^6x \sqrt{g} \left(R - \Lambda + |F|^2 + |G|^2 \right) + 4k \int C \wedge F \wedge F$$
 - Might arise from string theory. But here: “bottom-up construction”.
- Turning on magnetic field B and charge density ρ turns on C -field,
 - $B = 0$ is Reissner-Nordstrom, with non-zero entropy density at $T = 0$;
 - $\rho = 0$ is the magnetic brane, $AdS_6 \rightarrow AdS_4$, and $s \sim T^2$
 - at finite ρ, B , the flow depends on the value of k
- A quantum critical point emerges at $k = k_c = 1/\sqrt{3}$
 - For $k < k_c$, flows to IR $AdS_2 \times \mathbf{R}^4$ (RN) near-horizon
 - For $k > k_c$, flows to IR $AdS_4 \times \mathbf{R}^2$ near-horizon, with $s = e^{C(k,B)} T^2$
 - At $k = k_c$, flows to IR **Lifshytz** geometry, $s \sim T^{\frac{2}{z(B)}}$

The Phase Diagram for 2+1 dimensions



Dynamical Exponent $z(B)$

