Supermoduli and Superstring Amplitudes

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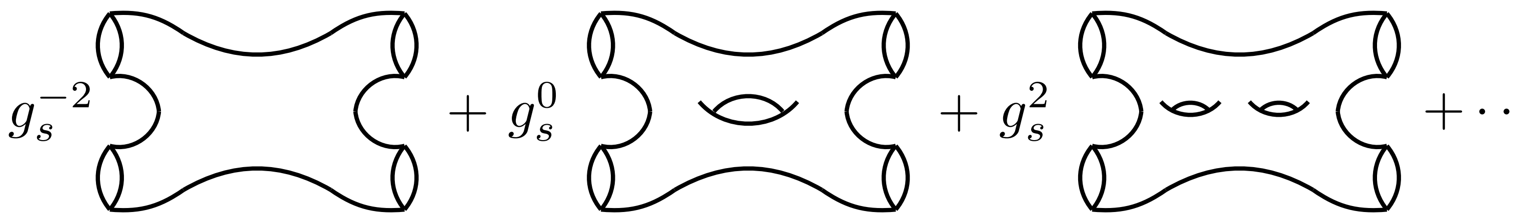
String Amplitudes

- **Quantum Mechanics predicts probabilities**

$$\text{probability} = |\text{ probability amplitude }|^2$$

- **Probability amplitude is given by summing over random surfaces**
  - governed by topological expansion in the genus h weighed by g_s^{2h-2}
  - g_s is the "string coupling"



  - orientable worldsheet supports a metric $\Longrightarrow$ Riemann surface

- **For each genus the sum reduces to an integral over moduli**
  - bosonic strings: moduli space of compact Riemann surfaces
  - superstrings: moduli space of compact super Riemann surfaces

(in the RNS formulation [Ramond 1971; Neveu, Schwarz 1971], [Gervais, Sakita 1971])

This talk

- **Highlights in constructing superstring amplitudes via supermoduli**
  - ⋆ role played by the super period matrix
  - ⋆ derive the genus-two amplitudes with four and five gravitons

- **Genus-two amplitudes with an arbitrary number N of gravitons**
  - ⋆ Summations over even spin structures for arbitrary N

    [ED, Hidding, Schlotterer 2022]
  - ⋆ Amplitude for 6 gravitons (even spin structure part only)

    [ED, Hidding, Schlotterer], in progress

- **Snowmass White Paper: String Perturbation Theory**

  [Berkovits, ED, Green, Johansson, Schlotterer, 2022]

Conformal structures

- **Orientable 2-dim manifold with Riemannian metric is a Riemann surface**
  - complex manifold (holomorphic transition functions)
  - complex = conformal structure $J : T(\Sigma) \to T(\Sigma), \; J^2 = -I$ is integrable
  - its tangents space splits $T(\Sigma) = T_{(1,0)}(\Sigma) \oplus T_{(0,1)}(\Sigma)$
  - Moduli space $\mathcal{M}_h = \{J\}/\mathrm{Diff}(\Sigma)$ of genus h Riemann surfaces

- **Moduli space itself is a complex orbifold**
  - Its tangent space splits $T(\mathcal{M}_h) = T_{(1,0)}(\mathcal{M}_h) \oplus T_{(0,1)}(\mathcal{M}_h)$

$$\dim_{\mathbb{C}} \mathcal{M}_h = \begin{cases} 0 & h = 0 \\ 1 & h = 1 \\ 3h-3 & h \geq 2 \end{cases}$$

Super conformal structures

- **Complex supermanifold of dim** $1|1$ (locally $\mathbb{C}^{1|1}$ with coordinates $z|\theta$)
  - vector fields $V(z,\theta)\partial_\theta + W(z,\theta)\partial_z$
    - $\Rightarrow$ produce $\mathcal{N}=2$ superconfomal transformations

- **Super Riemann surfaces (SRS) is a dim** $1|1$ **super manifold**
  - vector fields restricted to leave $D_\theta = \partial_\theta + \theta\partial_z$ invariant up to scaling
    - $\Rightarrow$ produce $\mathcal{N}=1$ superconformal transformations
  - Locally: SRS transition functions are $\mathcal{N}=1$ superconformal in $z|\theta$

- **Moduli space of SRS**: $\mathfrak{M}_h = \{\mathcal{J}\}/\mathrm{Diff}(\Sigma)$

  $=$ equivalence classes of $\mathcal{N}=1$ superconformal structures $\mathcal{J}$

$$\dim_{\mathbb{C}} \mathfrak{M}_h = \begin{cases} 0|0 & h=0 \\ 1|0 \text{ or } 1|1 & h=1 \quad \text{even or odd spin structure} \\ 3h-3|2h-2 & h\geq 2 \end{cases}$$

  - odd modulus at $h=1$ odd spin structure is a book keeping device
  - odd moduli appear non-trivially starting at genus 2

Worldsheet fields

- **Minkowski $M = \mathbb{R}^{10}$ with Lorentz group $SO(1,9)$**
  - x^μ scalar on Σ maps worldsheet Σ into space-time M
  - ψ^μ spinor on Σ but Lorentz vector under $SO(1,9)$
    - ⋆ Two sectors : NS bosons (tensor reps of $SO(1,9)$)
      R fermions (spinor reps of $SO(1,9)$)

- **With Minkowski signature Σ**
  - $\psi^\mu(\tau-\sigma)$ and $\tilde{\psi}^\mu(\tau+\sigma)$ are *independent* Majorana-Weyl spinors

- **With Euclidean signature Σ**
  - Left-movers $\tau+\sigma \to \tilde{z}$, right-movers $\tau-\sigma \to z$
  - $\psi^\mu(z)$ and $\tilde{\psi}^\mu(\tilde{z})$ are *independent* complex Weyl spinors
  - Globally, on a compact Riemann surface of genus h,
    - ⋆ all ψ^μ are sections of a the same spin bundle S (and $\tilde{\psi}^\mu$ of $\tilde{S}$)
    - ⋆ 2^{2h} distinct spin structures for S (and 2^{2h} independently for $\tilde{S}$)
  - GSO projection requires summations over spin structures,
    independently for δ and $\tilde{\delta}$ [Gliozzi, Scherk, Olive 1976]

Superstring worldsheets and moduli spaces

- **Consider Type II superstrings**
  - Heterotic and open superstrings are analogous

- **Independence of left and right chiralities**
  - Left : Super Riemann surface Σ_L with local coordinates $(\tilde z, \tilde \theta)$
    super moduli space $\mathfrak{M}_L$ with local coordinates $(\tilde m_i, \tilde \zeta_\alpha)$
  - Right : Super Riemann surface Σ_R with local coordinates (z, θ)
    super moduli space $\mathfrak{M}_R$ with local coordinates (m_i, ζ_α)

- **Pairing left and right chiralities** [Witten 2011]
  - fermionic variables remain independent: $(\theta, \tilde \theta)$ and $(\zeta, \tilde \zeta)$
  - bosonic variables are related as follows,
    - ⋆ Worldsheet: $\tilde z^* = z + \mathrm{nilpotent}$ defines cycle $\Sigma \subset \Sigma_L \times \Sigma_R$
      equivalently $\Sigma_{\rm red} = (\Sigma_L)_{\rm red} = (\Sigma_R)_{\rm red}$
    - ⋆ Moduli: $\tilde m_i^* = m_i + \mathrm{nilpotent}$ defines cycle $\mathfrak{M} \subset \mathfrak{M}_L \times \mathfrak{M}_R$
      equivalently $\mathfrak{M}_{\rm red} = (\mathfrak{M}_L)_{\rm red} = (\mathfrak{M}_R)_{\rm red}$
  - Stokes' theorem ensures independence of choice of cycles

Worldsheet action for Type II superstrings

- **Worldsheet is** $\Sigma \subset \Sigma_L \times \Sigma_R$
  - Σ_L has superconformal structure $\tilde{\mathcal{J}}$ with local coordinates $\tilde{z}|\tilde{\theta}$
  - Σ_R has superconformal structure $\mathcal{J}$ with local coordinates $z|\theta$

- **Superconformal invariant matter action** [Brink, Di Vecchia, Howe; Deser, Zumino 1976]

$$I_m[X^\mu, \tilde{\mathcal{J}}, \mathcal{J}] = \int_\Sigma [d\tilde{z}dz|d\tilde{\theta}d\theta] \tilde{D}_{\tilde{\theta}} X^\mu D_\theta X_\mu$$

  - worldsheet matter superfield in local coordinates ($D_\theta = \partial_\theta + \theta\partial_z$)

$$X^\mu(\tilde{z}, z|\tilde{\theta}, \theta) = x^\mu(\tilde{z}, z) + \theta\psi^\mu(\tilde{z}, z) + \tilde{\theta}\tilde{\psi}^\mu(\tilde{z}, z) + \tilde{\theta}\theta F^\mu(\tilde{z}, z)$$

  - Superconformal algebra acting on the fields is generated by

$$S_{z\theta} = \tfrac{1}{2}\psi^\mu\partial_z x_\mu \qquad T_{zz} = -\tfrac{1}{2}\partial_z x^\mu\partial_z x_\mu + \tfrac{1}{2}\psi^\mu\partial_z\psi_\mu$$

$$\tilde{S}_{\tilde{z}\tilde{\theta}} = \tfrac{1}{2}\tilde{\psi}^\mu\partial_{\tilde{z}} x_\mu \qquad \tilde{T}_{\tilde{z}\tilde{z}} = -\tfrac{1}{2}\partial_{\tilde{z}} x^\mu\partial_{\tilde{z}} x_\mu + \tfrac{1}{2}\tilde{\psi}^\mu\partial_{\tilde{z}}\tilde{\psi}_\mu$$

Parametrization of supermoduli

- **Local parametrization of bosonic moduli** (in conformal-invariant theory)
  - Complex structure J with metric $g = |dz|^2$ in local coordinates $(z, \tilde{z})$
  - deform complex structure by Beltrami differential to $g' = |dz + \mu d\tilde{z}|^2$
  - realized in CFT by insertion of $\int_\Sigma d\tilde{z}dz\, \mu_{\tilde{z}}{}^z\, T_{zz}$

- **Local parametrization of supermoduli** (in superconformal-invariant theory)
  - $\Sigma_{\rm red}$ and $\mathfrak{M}_{\rm red}$ are obtained by setting all odd Grassmann variables to zero
  - Start with $\Sigma_{\rm red}$ with complex structure given by $J \in \mathfrak{M}_{\rm red}$
  - Deformation of super conformal structure by inserting T and S

$$\int_{\Sigma_{\rm red}} d\tilde{z}dz \, \left(\mu_{\tilde{z}}{}^z T_{zz} + \chi_{\tilde{z}}{}^\theta S_{z\theta}\right)$$

  - local supersymmetry $\chi_{\bar{z}}{}^\theta \to \chi_{\bar{z}}{}^\theta + \partial_{\bar{z}} v^\theta$ and $\mu_{\bar{z}}{}^z \to \mu_{\bar{z}}{}^z + \partial_{\bar{z}}(v^\theta \chi_{\bar{z}}{}^\theta)$
  - χ and μ may be parametrized by local coordinates on $\mathfrak{M}_h$
  - nilpotency of χ guarantees expansion terminates at finite order $2h-2$

Higher genus Riemann surfaces

- **Compact Riemann surface Σ of genus h**
  - Homology group $H_1(\Sigma, \mathbb{Z})$ with intersection pairing $\mathfrak{J}$
  - Choose canonical homology basis $\mathfrak{A}_I, \mathfrak{B}_I$ for $H_1(\Sigma, \mathbb{Z}) \approx \mathbb{Z}^{2h}$
    with $\mathfrak{J}(\mathfrak{A}_I, \mathfrak{A}_J) = 0$, $\mathfrak{J}(\mathfrak{B}_I, \mathfrak{B}_J) = 0$, $\mathfrak{J}(\mathfrak{A}_I, \mathfrak{B}_J) = \delta_{IJ}$ for $I, J = 1, \cdots h$
  - and normalize dual holomorphic (1,0) forms ω_I by

$$\oint_{A_I} \omega_J = \delta_{IJ} \qquad\qquad \oint_{B_I} \omega_J = \Omega_{IJ}$$

  - Modular group $Sp(2h, \mathbb{Z})$ leaves $\mathfrak{J}$ invariant

$$\Omega \to (A\Omega + B)(C\Omega + D)^{-1} \qquad\qquad \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp(2h, \mathbb{Z})$$

- **The period matrix belongs to the Siegel upper half space $\mathcal{S}_h$**

$$\mathcal{S}_h = \{\Omega \quad h \times h \text{ complex matrices } \Omega^t = \Omega, \mathrm{Im}\,\Omega > 0\}$$

  - moduli space $\mathcal{M}_2 = \mathcal{S}_2/Sp(4, \mathbb{Z})$ (minus diagonal)
  - moduli space $\mathcal{M}_3 = \mathcal{S}_3/Sp(6, \mathbb{Z})$ (minus hyper-elliptic divisor)
  - For $h \geq 4$ characterizing $\mathcal{M}_h \subset \mathcal{S}_h/Sp(2h, \mathbb{Z})$ is the Schottky problem

The super period matrix

- **Compact super Riemann surface Σ of genus h** [ED, Phong 1988]
  - For even spin structure, there exist h superholomorphic forms $\hat\omega_I$ which produce a super period matrix $\hat\Omega$

$$\oint_{\mathfrak{A}_I} \hat\omega_J = \delta_{IJ} \qquad\qquad \oint_{\mathfrak{B}_I} \hat\omega_J = \hat\Omega_{IJ}$$

  - Explicit formulas in terms of the Szegö kernel S_δ give

$$\hat\Omega_{IJ} = \Omega_{IJ} - \frac{i}{8\pi} \int\!\!\int \omega_I(z) \chi_{\bar z}{}^\theta S_\delta(z,w) \chi_{\bar w}{}^\theta \omega_J(w) + \cdots$$

  - $\hat\Omega_{IJ}$ is locally supersymmetric with $\hat\Omega_{IJ} = \hat\Omega_{JI}$ and $\mathrm{Im}\,\hat\Omega > 0$

- **For genus $h=2$**
  - no additional terms above in $+\cdots$
  - Every $\hat\Omega$ corresponds to a Riemann surface (away from the diagonal)
  - Szegö kernel $S_\delta(z,w|\Omega)$ is non-singular on $\mathcal{M}_2$

$\Rightarrow$ **The super period matrix provides a holomorphic projection $\mathfrak{M}_2 \to \mathcal{M}_2$**

Singularities in the projection of $\overline{\mathfrak{M}}_2 \to \overline{\mathcal{M}}_2$

- **Projection $\mathfrak{M}_2 \to \mathcal{M}_2$ is holomorphic but $\overline{\mathfrak{M}}_2 \to \overline{\mathcal{M}}_2$ is not** [Witten 2013]
  - There may be singularities on the boundary of $\mathfrak{M}_2$

$$\Omega = \begin{pmatrix} \tau & u \\ u & \sigma \end{pmatrix} \qquad \begin{matrix} u \to 0 & \text{separating node} \\ \sigma \to i\infty & \text{non-separating node} \end{matrix}$$

  - Key ingredient in $\hat{\Omega}$ is the Szegö kernel

$$S_\delta(z, w|\Omega) = \frac{\vartheta[\delta](z-w|\Omega)}{\vartheta[\delta](0|\Omega)E(z,w)}$$

  - As $u \to 0$ we have $\vartheta[\delta](0|\Omega) \to \vartheta[\delta_1](0|\tau)\,\vartheta[\delta_2](0|\tau)$
  - Even $\delta = [\delta_1, \delta_2]$ with δ_1, δ_2 odd produces a singularity in S_δ and $\hat{\Omega}$

- **Physical effects**
  - in flat space-time $\mathbb{R}^{10}$ singularity absent thanks to ψ-zero modes
  - contribution when susy is broken by radiative corrections [Witten 2013]
  - two-loop vacuum energy in Heterotic strings on CY orbifold $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$
    - ⋆ zero for $E_8 \times E_8 \to E_6 \times E_8$ with unbroken susy
    - ⋆ non-zero for $Spin(32)/\mathbb{Z}_2 \to SO(26) \times U(3)$ with broken susy

[ED, Phong 2013; Berkovits, Witten 2014]

Singularities in the projection $\mathfrak{M}_3 \to \mathcal{M}_3$

- **Some basic structure**
  - A hyper-elliptic surface is a branched double cover of the sphere $\hat{\mathbb{C}} = S^2$
  - All genus 1 and all genus 2 surfaces are hyper-elliptic
  - $\mathcal{M}_3$ contains the divisor of hyper-elliptic surfaces

- **The period matrix for genus 3 (even spin structure)**

$$\hat{\Omega}_{IJ} = \Omega_{IJ} - \frac{i}{8\pi} \int\!\!\int \omega_I(z) \chi_{\bar{z}}{}^{\theta} S_\delta(z,w) \chi_{\bar{w}}{}^{\theta} \omega_J(w) + \mathcal{O}(\chi^4)$$

  - The hyper-elliptic divisor crosses the interior of $\mathfrak{M}_3$ along subvarieties characterized by $\vartheta[\delta](0|\Omega) = 0$ for some even spin structure δ where the Szegö kernel S_δ diverges
  - The projection $\mathfrak{M}_3 \to \mathcal{M}_3$ is not globally holomorphic [Witten 2015]
  - Predicted by global results on projectedness [Donagi, Witten 2014]

Super period matrix for Ramond punctures

- **Moduli space with n punctures** $\dim \mathcal{M}_{h;n} = 3h-3+n > 0$
- **Punctures on super Riemann surfaces**
  - NS puncture at $z=0$ satisfies $\psi(e^{2\pi i} z) = +\psi(z)$
  - R puncture at $z=0$ satisfies $\psi(e^{2\pi i} z) = -\psi(z)$
  - dimension $\dim \mathfrak{M}_{h;n,2r} = (3h-3+n+2r|2h-2+n+r)$
- **R-punctures occur in pairs** (p_η, q_η) **with R-divisor** $\mathcal{F} = \sum_{\eta=1}^r (p_\eta + q_\eta)$
  - R-fields are sections of $\mathcal{R}^{-1}$ where $\mathcal{R}^2 = K^{-1} \otimes \mathcal{O}(-\mathcal{F})$
  - $H^0(\Sigma, \mathcal{R}^{-1}) = r$ giving r holomorphic $(\frac{1}{2}, 0)$ forms ρ_η
- **Period for Ramond punctures** [Witten 2015; ED, Phong 2015]
  - $\hat\omega_I$ and $\hat\rho_\eta$ resp. odd and even superholomorphic forms
  - R-periods $\hat\Omega_{I\eta}$ extracted from $\hat\omega_I$; $\hat\Omega_{\zeta\eta}$ extracted from $\hat\rho_\eta$ on $\mathcal{F}$

$$\hat\Omega = \begin{pmatrix} \hat\Omega_{IJ} & \hat\Omega_{I\eta} \\ \hat\Omega_{\zeta I} & \hat\Omega_{\zeta\eta} \end{pmatrix} = \begin{pmatrix} \hat\Omega_{JI} & \hat\Omega_{\eta I} \\ \hat\Omega_{I\zeta} & -\hat\Omega_{\eta\zeta} \end{pmatrix}$$

  - $\hat\Omega$ with R-punctures may also be derived from non-separating degeneration of $\hat\Omega$ at higher genus *without R-punctures*

Evaluating genus 2 superstring amplitudes

- **Specialize to genus $h = 2$ and even spin structures**
  - Natural parametrization of $\mathfrak{M}_2$ using holomorphic projection

$$\begin{aligned} m^A &= (\hat\Omega_{IJ}, \zeta^1, \zeta^2) \\ \chi_{\bar z}{}^\theta &= \zeta^1 \delta(z, q_1) + \zeta^2 \delta(z, q_2) \end{aligned}$$

  - Deformations of complex structure by Beltrami diff. $\hat\mu = \mathcal{O}(\zeta^1 \zeta^2)$

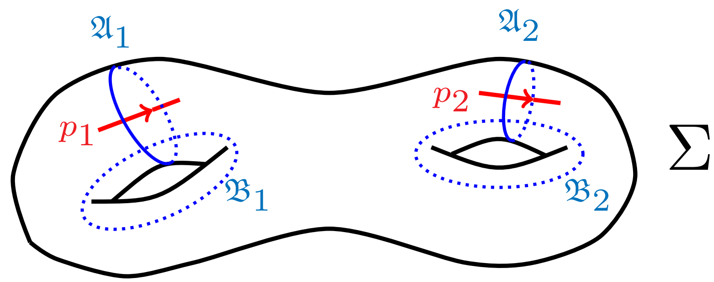
$$\Omega_{IJ} \to \hat\Omega_{IJ} \quad \begin{cases} g & \to \quad \hat g = g + \hat\mu \\ \partial_{\bar z} & \to \quad \hat\partial_{\bar z} = \partial_{\bar z} + \hat\mu \partial_z \\ \langle \cdots \rangle(g) & \to \quad \langle \cdots \rangle(\hat g) + \int \hat\mu \langle T \cdots \rangle(\hat g) \end{cases}$$

  - supersymmetry guarantees independence of choice of points q_1, q_2
  - in particular, absence of earlier ambiguities pointed out by [Atick, Rabin, Sen 1989]

Chiral splitting and loop momenta

- **Loop momenta** [Verlinde, Verlinde 1988; ED, Phong 1988]
  - ⋆ Momentum flowing through a curve is the line integral of the $\partial_z x^\mu$
  - ⋆ A canonical homology basis gives a natural choice of loop momenta

$$p^I_\mu = \frac{1}{2\pi} \oint_{\mathfrak{A}_I} \partial_z x_\mu$$



- **Chiral splitting into chiral blocks** [ED, Phong 1989]
  - ⋆ Amplitude is given by pairing left and right chiral blocks

$$\mathcal{A}_N^{(2)} = \int_{\mathbb{R}^{20}} dp \int_{\mathfrak{M}_2} \int_{\Sigma^N} \mathcal{F}[\delta](\varepsilon, k, p|z, \theta, \hat{\Omega}, \zeta) \tilde{\mathcal{F}}[\tilde{\delta}](\tilde{\varepsilon}, k, p|\tilde{z}, \tilde{\theta}, \hat{\Omega}, \tilde{\zeta})$$

- $\mathcal{F}[\delta]$ **is locally holomorphic on** Σ_R **and** $\mathfrak{M}_R$
  - ⋆ Bosonic moduli paired with the same super period matrix $\hat{\Omega}$
  - ⋆ Integral over $\mathfrak{M}_2$ includes summation over spin structures $\delta, \tilde{\delta}$

Properties of chiral blocks

- **Universal monodromy = "homology shift invariance "** [ED, Phong 1988-89]

  ⋆ Moving a vertex point z_j around an $\mathfrak{A}_J$ or $\mathfrak{B}_J$ cycle

$$\begin{aligned} \mathcal{F}[\delta](\varepsilon,k,p|z_i+\delta_{ij}\,\mathfrak{A}_J,\theta_i,\hat\Omega,\zeta) &= e^{2\pi i k_j\cdot p^J}\mathcal{F}[\delta](\varepsilon,k_i,p|z_i,\theta_i,\hat\Omega,\zeta) \\ \mathcal{F}[\delta](\varepsilon,k,p^I|z_i+\delta_{ij}\,\mathfrak{B}_J,\theta_i,\hat\Omega,\zeta) &= \mathcal{F}[\delta](\varepsilon,k,p^I+\delta^I_J k_j|z_i,\theta_i,\hat\Omega,\zeta) \end{aligned}$$

  ⋆ Physical amplitude $\mathcal{A}_N^{(2)}$ is invariant by translation invariance of $\int dp$

  ⋆ Monodromy is universal, valid for bosonic, Type II, Heterotic strings

The chiral measure in terms of ϑ-constants

- **Chiral measure on $\mathfrak{M}_2$ for superstrings in $M = \mathbf{R}^{10}$** [ED, Phong 2001]

$$d\mu[\delta](\hat\Omega,\zeta) = \left[\mathcal{Z}[\delta](\hat\Omega) + \zeta^1\zeta^2 \frac{\Xi_6[\delta](\hat\Omega)\ \vartheta[\delta]^4(0,\hat\Omega)}{16\pi^6\ \Psi_{10}(\hat\Omega)}\right] d^2\zeta d^3\hat\Omega$$

  - Spin structures represented by half-integer characteristic $\delta \in \{0, 1/2\}^4$
  - $\Psi_{10}(\hat\Omega)$ = Igusa's unique cusp modular form of weight 10
  - The factor $\mathcal{Z}[\delta]$ may be evaluated as well, but will not be given here.

- **The modular form $\Xi_6[\delta](\hat\Omega)$ may be defined, for genus 2,**
  - Each even spin structure δ uniquely maps to a partition of the six odd spin structures ν_i. Let $\delta \equiv \nu_1 + \nu_2 + \nu_3 \equiv \nu_4 + \nu_5 + \nu_6$

$$\Xi_6[\delta](\hat\Omega) = \sum_{1\le i<j\le 3} \langle \nu_i|\nu_j\rangle \prod_{k=4,5,6} \vartheta[\nu_i+\nu_j+\nu_k](0,\hat\Omega)^4$$

  - Symplectic pairing signature: $\langle \nu_i|\nu_j\rangle \equiv \exp 4\pi i(\nu_i'\nu_j'' - \nu_i''\nu_j') \in \{\pm 1\}$

- **$\Xi_6[\delta](\hat\Omega)$ admits a natural generalization to genus 3 and 4**

  [Cacciatori, Della Piazza, Van Geemen 2008; Matone, Volpato 2008; Grushevsky, Salvatti-Mann 2008; Morozov 2008]

Chiral Amplitudes

- **Chiral Amplitudes on $\mathfrak{M}_2$**
  - involve correlation functions $\mathcal{F}_0, \mathcal{F}_2$ which depend on $\hat\Omega$ and on ζ

$$\mathcal{F}[\delta](\hat\Omega,\zeta) = d\mu[\delta](\hat\Omega,\zeta)\left(\mathcal{F}_0[\delta](\hat\Omega) + \zeta^1\zeta^2\mathcal{F}_2[\delta](\hat\Omega)\right)$$

- **Projection to chiral amplitudes on $\mathcal{M}_2$**

  by integrating over odd moduli ζ at fixed δ and fixed $\hat\Omega$

$$\mathcal{L}[\delta](\hat\Omega) = \int_\zeta \mathcal{F}[\delta](\hat\Omega,\zeta) = \left(\mathcal{Z}[\delta]\mathcal{F}_2[\delta] + \frac{\Xi_6[\delta]\ \vartheta[\delta]^4}{16\pi^6\ \Psi_{10}}\mathcal{F}_0[\delta]\right) d^3\hat\Omega$$

- **Gliozzi-Scherk-Olive projection**
  - realized by summation over spin structures with constant phases;
  - separately in left and right chiral amplitudes for Type II and Heterotic;
  - phases to be determined from requirement of modular covariance.

Four-graviton amplitude in Type II

- **Projected to $\mathcal{M}_2$ and summed over spin structures** [ED, Phong 2005]

$$\mathcal{A}_4^{(2)} = g_s^2 \, \mathcal{R}^4 \int_{\mathcal{M}_2} \frac{|d\Omega^3|^2}{(\det \mathrm{Im}\,\Omega)^3} \int_{\Sigma^4} \frac{\mathcal{Y} \wedge \bar{\mathcal{Y}}}{(\det \mathrm{Im}\,\Omega)^2} \exp \left\{ \sum_{i<j} s_{ij} \, G(z_i, z_j|\Omega) \right\}$$

  - ⋆ string momenta k_i with $s_{ij} = -\alpha' k_i \cdot k_j/2$
  - ⋆ $\mathcal{R}^4 = t_8 \tilde{t}_8$ scalar contraction of four linearized Riemann tensors $\mathcal{R}$
  - ⋆ Measure on Σ^4 interlaces kinematic and worldsheet data

$$\begin{aligned} \mathcal{Y} &= s_{14}\Delta(z_1,z_2)\Delta(z_3,z_4) - s_{12}\Delta(z_1,z_4)\Delta(z_2,z_3) \\ \Delta(z,w) &= \omega_1(z)\omega_2(w) - \omega_2(z)\omega_1(w) \end{aligned}$$

  - ⋆ Standard string Green function in terms of the prime form $E(z,w|\Omega)$

$$G(z,w|\Omega) = -\ln |E(z,w|\Omega)|^2 + 2\pi (\mathrm{Im}\,\Omega)^{-1}_{IJ} \, \mathrm{Im} \int_w^z \omega_I \, \mathrm{Im} \int_w^z \omega_J$$

- **α' expansion matches S-duality predictions in Type IIB for BPS operators**
  - ⋆ coefficient of $D^4\mathcal{R}^4$ [ED, Gutperle, Phong 2005; Gomez, Mafra 2010]
  - ⋆ coefficient of $D^6\mathcal{R}^4$ [ED, Green, Pioline, Russo 2014]

Five-graviton amplitude in Type II

- **Massless NS states; even spin structures** [ED, Schlotterer 2021] (152 pages !)

$$\mathcal{F}_5 = \mathcal{I}_5 \sum_i \Big\{ \mathfrak{P}^I(z_i) \cdot \big(\varepsilon_i \, \mathrm{t}_i \, \mathcal{Y}_I + k_i \, \mathfrak{T}_{iI}\big) - \sum_{j \neq i} \mathcal{Y}_I \, \mathrm{t}_{ij} \, g^I_{i,j} \Big\}$$

- **Universal chiral Koba-Nielsen factor, common to all string theories**

$$\mathcal{I}_5 = \exp \Big\{ i\pi \Omega_{IJ} \, p^I \cdot p^J + 2\pi i \sum_j k_j \cdot p^I \int_{z_0}^{z_j} \omega_I - \sum_{i \neq j} s_{ij} \ln E(z_i, z_j | \Omega) \Big\}$$

- **Kinematic combinations of** $f_i^{\mu\nu} = k_i^\mu \varepsilon_i^\nu - k_i^\nu \varepsilon_i^\nu$

$$\mathrm{t}_1 = t_8(f_2, f_3, f_4, f_5) \qquad\qquad \mathrm{t}_{12} = t_8([f_1, f_2], f_3, f_4, f_5)$$

- **Remaining ingredients**

$$\mathfrak{P}^I = 2\pi i p^I + \sum_{j \neq i} g^I_{i,j} k_j \qquad\qquad g^I_{i,j} = \partial^I \ln \vartheta[\nu](z_j - z_i | \Omega)$$

$$\mathcal{Y}_I = 4 s_{12} \, \omega_I(4) \Delta(5,1) \Delta(2,3) + \text{ cycl}(1,2,3,4,5)$$

$$\mathfrak{T}_{1I} = (\mathrm{t}_{12} - 2\mathrm{t}_1 \varepsilon_1 \cdot k_2) \Big\{ \omega_I(3) \Delta(1,5) \Delta(2,4) + \text{ cycl}(3,4,5) \Big\} + \text{ cycl}(2,3,4,5)$$

- **Remarkably simple !**
  - ⋆ Also obtained by amalgam of chiral splitting and pure spinor strings

[ED, Mafra, Pioline, Schlotterer 2020] using [Gomez, Mafra, Schlotterer 2015]

Spin structure sums for higher multiplicity

- **Major effort goes into evaluating sums over spin structures**
  - ⋆ required for GSO projection onto supersymmetric amplitudes
  - ⋆ required to project to $E_8 \times E_8$ or $Spin(32/\mathbb{Z}_2)$ in Heterotic strings

- **Even spin structures and NS external states**
  - ⋆ Correlator of chiral fermions for spin structure δ given by Szegö kernel

$$\langle \psi(z)\psi(w)\rangle = S_\delta(z,w) = \frac{\vartheta[\delta](\int_w^z \omega|\Omega)}{\vartheta[\delta](0|\Omega)E(z,w)}$$

  - ⋆ String amplitude integrands involve cyclic products of Szegö kernels

$$C_\delta(z_1,\cdots,z_n) = S_\delta(z_1,z_2)S_\delta(z_2,z_3)\cdots S_\delta(z_{n-1},z_n)S_\delta(z_n,z_1)$$

    (they also involve other products that may be treated similarly)

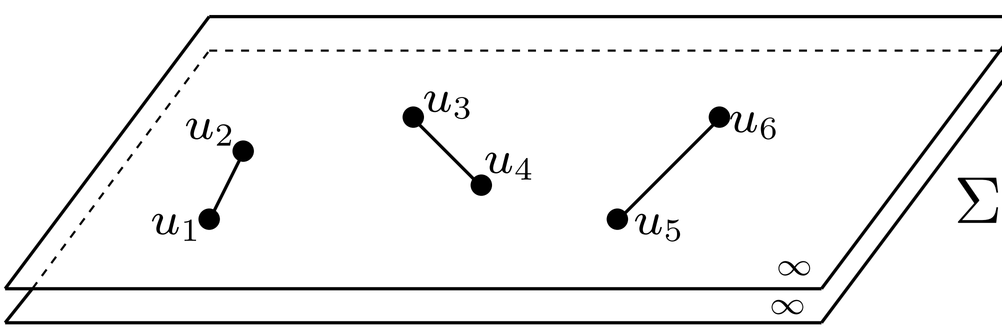
  - ⋆ Evaluating the spin structure sums for $n=4,5$ point amplitudes involved
    - – Riemann identities; Fay trisecant identity (cfr bosonization)
    - – and every other trick we could think of
  - $\Rightarrow$ those methods show no promising generalization to $n \geq 6$
    - – the problem was also considered in [Tsuchiya 2012; 2017; 2022]

Reduction of higher multiplicity spin structure sums

- **Theorem** [ED, Hidding, Schlotterer 2022]
  The spin structure sum of $C_\delta(z_1, \cdots, z_n)$ for arbitrary n reduces to the spin structure sums for the cases $n = 0, 2, 3, 4$

  ⋆ The proof is constructive in the hyper-elliptic formulation
  ⋆ The result will be translated into the ϑ-function formulation

- **Every genus two surface Σ is hyper-elliptic**
  ⋆ namely a double cover of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$
  ⋆ ramified over 6 branch points $u_1, \cdots, u_6$
  ⋆ points $z \in \Sigma$ parametrized by $z = (x, s)$ where $s^2 = (x - u_1) \cdots (x - u_6)$
  ⋆ Moduli space $\mathcal{M}_2$ isomorphic to $\{u_1, \cdots, u_6\}/(SL(2, \mathbb{C}) \times \mathfrak{S}_6)$



Sketch of proof of the Theorem

⋆ An even spin structure δ is isomorphic to a $3+3$ partition of branch points

$$\{u_1, \cdots, u_6\} = A \cup B \qquad A \cap B = \emptyset \qquad |A| = |B| = 3$$

⋆ The Szegö kernel is given in terms of this partition by

$$S_\delta(z_1, z_2) = \frac{s_A(x_1)s_B(x_2) + s_B(x_1)s_A(x_2)}{2(x_1 - x_2)} \left[\frac{dx_1\, dx_2}{s(x_1)\, s(x_2)}\right]^{\frac{1}{2}}$$

where $s_A(x)s_B(x) = s(x)$ and $s_A(x)^2$ and $s_B(x)^2$ are polynomials given by

$$s_A(x)^2 = \prod_{r \in A}(x - u_r) \qquad s_B(x)^2 = \prod_{r \in B}(x - u_r)$$

⋆ The cyclic product of Szegö kernels is thus given by (using $x_{n+1} = x_1$)

$$C_\delta(z_1, \cdots z_n) = \frac{\prod_{i=1}^n (s_A(x_i)s_B(x_{i+1}) + s_B(x_i)s_A(x_{i+1}))}{2^n\, x_{12}x_{23} \cdots x_{n1}} \frac{dx_1 \cdots dx_n}{s(x_1) \cdots s(x_n)}$$

- **Lemma 1**

  All spin structure dependence is contained in polynomials with $2m \leq n$

$$Q_\delta(i_1, \cdots, i_m | j_1, \cdots, j_m) = \prod_{\alpha=1}^m s_A(x_{i_\alpha})^2 s_B(x_{j_\alpha})^2 + (A \leftrightarrow B)$$

Sketch of proof of the Theorem (cont'd)

- **Lemma 2**

  All spin structure dependence of Q_δ *is polynomial in* $\ell_\delta^{11}, \ell_\delta^{12} = \ell_\delta^{21}, \ell_\delta^{22}$

$$\ell_\delta^{11} = \tfrac{1}{4}\alpha_2\beta_2 - \tfrac{3}{20}\mu_4 \qquad s_A(x)^2 = x^3 - \alpha_1 x^2 + \alpha_2 x - \alpha_3$$

$$\ell_\delta^{12} = \tfrac{1}{4}(\alpha_1\beta_2 + \alpha_2\beta_1) - \tfrac{9}{40}\mu_3 \qquad s_B(x)^2 = x^3 - \beta_1 x^2 + \beta_2 x - \beta_3$$

$$\ell_\delta^{22} = \tfrac{1}{4}\alpha_1\beta_1 - \tfrac{3}{20}\mu_2 \qquad s(x)^2 = x^6 - \mu_1 x^5 + \cdots - \mu_5 x + \mu_6$$

- **Lemma 3: The trilinear relations**

  Every trilinear $\ell_\delta^{a_1a_2}\,\ell_\delta^{a_3a_4}\,\ell_\delta^{a_5a_6}$ *may be expressed as a polynomial of total degree two in the combinations* $\ell_\delta^{11}, \ell_\delta^{12}$ *and* ℓ_δ^{22} *whose coefficients are polynomials in* $\mu_1, \cdots, \mu_6$

- *Combining Lemmas 1, 2 and 3 implies that all spin structure dependence* of C_δ is given by a quadratic polynomial in $\ell_\delta^{11}, \ell_\delta^{12}, \ell_\delta^{22}$ with coefficients that depend only on μ_i.

- *The spin structure sums of the linears* $\ell_\delta^{a_1a_2}$ *and of the bilinears* $\ell_\delta^{a_1a_2}\ell_\delta^{a_3a_4}$ *are determined by* N*-point functions with* $N \leq 4$, *which concludes the proof of the Theorem.*

$SL(2,\mathbb{C})$ tensorial structure of the trilinear relations

- **Component form of the trilinear relations** e.g.

$$(\boldsymbol{\ell}_\delta^{11})^3 = \frac{\mu_4(\boldsymbol{\ell}_\delta^{11})^2}{20} - \frac{\mu_5\boldsymbol{\ell}_\delta^{11}\boldsymbol{\ell}_\delta^{12}}{4} + \mu_6\boldsymbol{\ell}_\delta^{11}\boldsymbol{\ell}_\delta^{22} - \frac{\mu_6(\boldsymbol{\ell}_\delta^{12})^2}{4} + \frac{\mu_4^2\boldsymbol{\ell}_\delta^{11}}{50} - \frac{9\mu_3\mu_5\boldsymbol{\ell}_\delta^{11}}{160} + \frac{3\mu_2\mu_6\boldsymbol{\ell}_\delta^{11}}{20} + \frac{\mu_4\mu_5\boldsymbol{\ell}_\delta^{12}}{40}$$
$$- \frac{9\mu_3\mu_6\boldsymbol{\ell}_\delta^{12}}{80} - \frac{\mu_5^2\boldsymbol{\ell}_\delta^{22}}{16} + \frac{3\mu_4\mu_6\boldsymbol{\ell}_\delta^{22}}{20} - \frac{3\mu_4^3}{2000} + \frac{9\mu_3\mu_4\mu_5}{1600} - \frac{3\mu_2\mu_5^2}{320} - \frac{81\mu_3^2\mu_6}{6400} + \frac{9\mu_2\mu_4\mu_6}{400}$$

- **The $\boldsymbol{\ell}_\delta^{ab}$ transform under the 3-dimensional irrep of $SL(2,\mathbb{C})$ by**

$$\boldsymbol{\ell}_\delta^{ab} \to J\, g_c^a\, g_d^b\, \boldsymbol{\ell}_\delta^{cd} \qquad g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{C}) \qquad J = \prod_{j=1}^{6} (\gamma u_j + \delta)^{-1}$$

- **The trilinear relations in $SL(2,\mathbb{C})$ tensorial form**

$$\mathbf{7} \qquad \boldsymbol{\ell}_\delta^{(a_1a_2}\,\boldsymbol{\ell}_\delta^{a_3a_4}\,\boldsymbol{\ell}_\delta^{a_5a_6)} = \mathrm{M}_1^{b_1b_2(a_1\cdots a_4}\,\boldsymbol{\ell}_\delta^{a_5a_6)}\,\boldsymbol{\ell}_\delta^{c_1c_2}\,\varepsilon_{b_1c_1}\,\varepsilon_{b_2c_2} + \cdots$$

$$\mathbf{3} \qquad (\det \boldsymbol{\ell}_\delta)\,\boldsymbol{\ell}_\delta^{a_1a_2} = \tfrac{3}{2}\,\mathrm{M}_1^{a_1a_2b_1\cdots b_4}\,\boldsymbol{\ell}_\delta^{c_1c_2}\boldsymbol{\ell}_\delta^{c_3c_4}\varepsilon_{b_1c_1}\cdots\varepsilon_{b_4c_4} + \cdots$$

⋆ where $\mathbf{M}_1$ is the symmetric rank 6 tensor under $SL(2,\mathbb{C})$ with components

$$\mathrm{M}_1^{111111} = \mu_6 \qquad \mathrm{M}_1^{111112} = \tfrac{\mu_5}{6} \qquad \mathrm{M}_1^{111122} = \tfrac{\mu_4}{15} \ \cdots$$

$Sp(4,\mathbb{Z})$ tensorial structure of the trilinear relations

- **Correspondence between hyper-elliptic and ϑ-function formulations**
  - ⋆ via standard Thomae formulas and holomorphic 1-forms ω_I

$$\varpi_1 = \frac{dx}{s(x)} \qquad \varpi_2 = -\frac{x\, dx}{s(x)} \qquad \omega_I(z) = \varpi_a(z)\, \sigma^a{}_I$$

  - ⋆ we obtain the *modular tensors* $\mathfrak{L}_\delta$ and $\mathfrak{M}_1$ (δ transforms)

$$\boldsymbol{\ell}_\delta^{ab} \quad \to \quad \mathfrak{L}_\delta^{IJ} = \frac{\pi}{5i} \partial^{IJ} \ln \left\{ \frac{\vartheta[\delta](0)^{20}}{\Psi_{10}} \right\}$$

$$\mathbf{M}_1^{a_1 \cdots a_6} \quad \to \quad \mathfrak{M}_1^{I_1 \cdots I_6} = \Psi_{10}^{-\frac{1}{2}} \partial^{(I_1} \vartheta[\nu_1](0) \cdots \partial^{I_6)} \vartheta[\nu_6](0)$$

  - ⋆ where $\nu_1, \cdots, \nu_6$ are the six (distinct) odd spin structures

- **Trilinear relations are between $Sp(4,\mathbb{Z})$ modular tensors**

$$\mathfrak{L}_\delta^{(I_1 I_2}\, \mathfrak{L}_\delta^{I_3 I_4}\, \mathfrak{L}_\delta^{I_5 I_6)} \quad = \quad \mathfrak{M}_1^{J_1 J_2 (I_1 \cdots I_4}\, \mathfrak{L}_\delta^{I_5 I_6)}\, \mathfrak{L}_\delta^{K_1 K_2}\, \varepsilon_{J_1 K_1}\, \varepsilon_{J_2 K_2} + \cdots$$

$$(\det \mathfrak{L}_\delta)\, \mathfrak{L}_\delta^{I_1 I_2} \quad = \quad \tfrac{3}{2}\, \mathfrak{M}_1^{I_1 I_2 J_1 \cdots J_4}\, \mathfrak{L}_\delta^{K_1 K_2}\, \mathfrak{L}_\delta^{K_3 K_4}\, \varepsilon_{J_1 K_1} \cdots \varepsilon_{J_4 K_4} + \cdots$$

Modular tensors

- **Structure of (locally holomorphic) modular tensors under $Sp(4,\mathbb{Z})$**
  - ⋆ totally symmetric tensor $\mathfrak{T}$ of rank r and weight w
  - ⋆ transforms under $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp(4,\mathbb{Z})$ by

$$\mathfrak{T}^{I_1 \cdots I_r} \to \det\left(C\Omega + D\right)^w \left(C\Omega + D\right)^{I_1}{}_{J_1} \cdots \left(C\Omega + D\right)^{I_r}{}_{J_r} \mathfrak{T}^{J_1 \cdots J_r}$$

- **Classic Siegel modular forms of weight w correspond to rank $r = 0$**
  - ⋆ $Sp(4,\mathbb{Z})$: polynomial ring generated by $\Psi_4, \Psi_6, \Psi_{10}, \Psi_{12}, \Psi_{35}$ [Igusa]

- **Non-holomorphic modular tensors arise in the α' expansion**
  - ⋆ related to the Kawazumi-Zhang invariant [Kawazumi 2008;Zhang 2008; Kawazumi 2016]
  - ⋆ higher genus modular graph functions [ED, Green, Pioline, Schlotterer 2020; ED, Schlotterer 2021]

Summary and outlook

- **Four- and Five-point 2-loop amplitudes**
  - ⋆ direct calculation for even spin structure using supermoduli
  - ⋆ remarkable simplicity of the final amplitude

- **Higher point 2-loop amplitudes**
  - ⋆ all even spin structures sums available via modular tensors

- **Ring structure for modular tensors of arbitrary rank and weight**
  - ⋆ what are the generators ?
  - ⋆ which subspace is needed for string amplitudes ?

- **What is the structure of string amplitudes ?**
  - ⋆ the expansion in α' showed extensive structure
    - – modular graph functions and relation to polylogarithms cfr Schlotterer's talk
  - ⋆ finite α' involves modular tensors and intertwined kinematic dependence
    - – **can one build an efficient library for these structures** ?