Flat connections, Fay identities and Polylogarithms on arbitrary Riemann surfaces

Eric D'Hoker
Mani L. Bhaumik Institute for Theoretical Physics
Department of Physics and Astronomy, UCLA

Amplitudes and Algebraic Geometry
Erwin Schrödinger International Institute, Vienna
March 2026

Motivation

- **Key goals of this workshop include**
  - ⋆ Understanding the function spaces in which quantum field theory and string theory amplitudes take their values
  - ⋆ Via algebras of functions that close under differentiation and integration
  - ⋆ Rendering integrations "algorithmic"

- **Key themes towards these goals include**
  - ⋆ periods / polylogarithms / elliptic / Calabi-Yau
  - ⋆ Grassmannians / Amplituhedra / other ——hedra
  - ⋆ cluster algebras

- **Several speakers have advocated the need for a "larger structure"**
  - ⋆ more symmetry
  - ⋆ larger algebras

This talk

- **Polylogarithms on arbitrary Riemann surfaces**
  - ⋆ in an arbitrary number of variables *(motivated by string theory and QFT)*
  - ⋆ closure under integration requires "Fay identities"
    = quadratic relations between integration kernels

- **Key results of this talk** arXiv:2602.01461 and 2602.09108 with Oliver Schlotterer
  - ⋆ Polylogarithms will be defined in terms of **flat connections**
    valued in infinite-dimensional Lie algebras
  - ⋆ Fay identities are equivalent to flatness of a connection
  - ⋆ Flatness ensures integrability of an associated linear problem

This talk

- **Polylogarithms on arbitrary Riemann surfaces**
  - ⋆ in an arbitrary number of variables *(motivated by string theory and QFT)*
  - ⋆ closure under integration requires "Fay identities"
    = quadratic relations between integration kernels
- **Key results of this talk** arXiv:2602.01461 and 2602.09108 with Oliver Schlotterer
  - ⋆ Polylogarithms will be defined in terms of **flat connections**
    valued in infinite-dimensional Lie algebras
  - ⋆ Fay identities are equivalent to flatness of a connection
  - ⋆ Flatness ensures integrability of an associated linear problem
- **May remind you of a famous** *non-linear* ⇔ *linear* **map** [Lax 1968]
  - ⋆ Solution to non-linear differential eqs by Lax pairs and inverse scattering
  - ⋆ The KdV equation may be recast in the form of a Lax pair

$$\partial_t V + \partial_x(\partial_x^2 V - 3V^2) = 0 \qquad \Leftrightarrow \qquad \partial_t L = [L, A]$$

and is solved by inverse scattering of the Schrödinger equation

$$\begin{aligned} L\psi(x) &= -\partial_x^2\psi(x) + V(x)\psi(x) = E\psi(x) \\ A &= 4\partial_x^3 - 6V\partial_x - 3(\partial_x V) \end{aligned}$$

Polylogarithms from flat connections

- **Flat connection $\mathcal{J}(x)$**
  - valued in Lie algebra $\mathfrak{g}$ on a Riemann surface Σ (possibly with punctures)

$$d_x \mathcal{J}(x) - \mathcal{J}(x) \wedge \mathcal{J}(x) = 0$$

- **Flatness of $\mathcal{J}(x)$ guarantees integrability of the equation**

$$d_x \Gamma(x,y) = \mathcal{J}(x) \Gamma(x,y)$$

- **The integral along a path γ_{xy} from y to x is homotopy-invariant**
  - i.e. depends only on the homotopy class $[\gamma_{xy}]$ of the path γ_{xy}
  - and may be obtained by iterated integrals over $\mathcal{J}(x)$

$$\Gamma(x,y) = 1 + \int_y^x \mathcal{J}(z_1) + \int_y^x \mathcal{J}(z_1) \int_y^{z_1} \mathcal{J}(z_2) + \cdots$$

where each integral is along the path γ_{xy}
  - The result is, generally, multiple-valued in $x \in \Sigma$

Polylogarithms from flat connections (cont'd)

- **Consider a Lie algebra $\mathfrak{g}$ that is freely generated by $c = \{c_1, \cdots, c_n\}$**
  - A *word* is an ordered sequence of letters (e.g. $\mathfrak{w} = c_3 c_1 c_2 c_2$ when $n \geq 3$) and takes values in the universal enveloping algebra of $\mathfrak{g}$
  - Taylor expanding $\Gamma(x, y)$ in words $\mathfrak{w}$ produces generalized polylogarithms

$$\Gamma(x, y) = \sum_{\mathfrak{w} \in \mathcal{W}(c)} \mathfrak{w} \, \Gamma(x, y; \mathfrak{w})$$

where $\mathcal{W}(c)$ is the set of words formed from the letters $c_1, \cdots, c_n$.

- **The homotopy-invariant $\Gamma(x, y; \mathfrak{w})$ are** *generalized polylogarithms*
  - Definition extended to the *algebra of words* by linearity

$$\Gamma(x, y; \lambda \, \mathfrak{w}_1 + \mu \, \mathfrak{w}_2) = \lambda \, \Gamma(x, y; \mathfrak{w}_1) + \mu \, \Gamma(x, y; \mathfrak{w}_2)$$

  - Close under multiplication

$$\Gamma(x, y; \mathfrak{w}_1) \Gamma(x, y; \mathfrak{w}_2) = \sum_{\mathfrak{w} \in \mathfrak{w}_1 \sqcup\!\sqcup \, \mathfrak{w}_2} \Gamma(x, y; \mathfrak{w})$$

  - Close under differentiation

Flat connections on the sphere

- **The algebra of rational functions on the sphere**
  - ⋆ is closed under differentiation, but not under integration
- **Standard polylogarithms on the sphere form an algebra**
  - ⋆ that closes under differentiation and integration
  - ⋆ and may be generated with the help of a flat connection
- **KZ connection in one variable $x \in S^2$ with punctures p_i** [Knizhnik, Zamolodchikov 1984]

$$\mathcal{J}(x) = \sum_{i=1}^{n} \frac{c_i \, dx}{x - p_i}$$

  - ⋆ Meromorphic and single-valued on Σ, flat on $\Sigma \setminus \{p_1, \cdots, p_n\}$
  - ⋆ For freely generated $\mathfrak{g} = \{c_1, \cdots, c_n\}$ gives polylogarithms [Goncharov]
- **Closure under integration in x requires various identities**
  - ⋆ e.g. express products of simple poles by partial fraction decomposition

$$\frac{1}{(x-p_i)(x-p_j)} = \frac{1}{p_i - p_j} \left(\frac{1}{x-p_i} - \frac{1}{x-p_j} \right)$$

Flat connections on the sphere (cont'd)

- **Trade punctures for additional "variables" = multivariable connection**
  - ⋆ on configuration space $\boldsymbol{x} = (x_1, \cdots, x_n) \in \mathrm{Cf}_n(\Sigma) = \Sigma^n \setminus \{\text{diagonals}\}$

$$\mathcal{J}(\boldsymbol{x}) = \sum_{i \neq j} \frac{dx_i - dx_j}{x_i - x_j} t_{ij} \qquad\qquad t_{ij} = t_{ji}$$

  - ⋆ holomorphic and thus closed $d\mathcal{J} = 0$ on $\mathrm{Cf}_n(\Sigma)$, so flatness reads

$$d\mathcal{J} - \mathcal{J} \wedge \mathcal{J} = 0 \qquad \Leftrightarrow \qquad [\mathcal{J}, \mathcal{J}] = 0$$

  - ⋆ The algebra $\mathfrak{g}$ is now no longer freely generated by the t_{ij}
    instead, it corresponds to the braid algebra

$$[t_{ij}, t_{k\ell}] = [t_{ij} + t_{ik}, t_{jk}] = 0$$

    for i, j, k, ℓ mutually distinct

Flat connections on the torus

- **Torus $\Sigma = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ with modulus τ satisfying $\mathrm{Im}\,(\tau) > 0$**
  - ⋆ Elliptic functions on the torus play the role of rational functions on the sphere
  - ⋆ Algebra of elliptic functions closes under differentiation but not integration
  - ⋆ Elliptic polylogarithms may be generated by flat connections

- **The Lie algebra $\mathfrak{g}$ is freely generated by a, b**
  - ⋆ allow one simple puncture p (higher order punctures see [Enriquez-Zerbini 2022])

- **Meromorphic flat connection** [Levin, Racinet 2007; Calaque, Enriquez, Etingof 2009]

$$\mathcal{J}_(z, p) = dz\, F(z-p; \mathrm{ad}_b)\, \mathrm{ad}_b\, a \qquad\qquad F(z;\alpha) = \frac{\vartheta_1'(0)\vartheta_1(z+\alpha)}{\vartheta_1(z)\vartheta_1(\alpha)}$$

  - ⋆ multiple-valued $\mathcal{J}(z+m+n\tau;p) = e^{-2\pi i n\,\mathrm{ad}_b}\mathcal{J}(z;p)$ for $m,n \in \mathbb{Z}$

- **Analog of partial fraction decomposition to integrate products**
  - $=$ Fay trisecant identity [Fay 1973] for the torus

$$\begin{aligned} F(z-p;\alpha)F(z-q;\beta) \quad &= \quad F(p-q;\beta)F(z-p;\alpha+\beta) \\ &\quad\; +F(q-p;\alpha)F(z-q;\alpha+\beta) \end{aligned}$$

Flat connections on the torus (cont'd)

- **Meromorphicity, simple poles and single-valuedness are in conflict**

- **Single-valued non-meromorphic flat connection** [Brown, Levin 2011]

$$\mathcal{J}_{\mathsf{BL}}(z,p) = 2\pi i \frac{dz - d\bar z}{\tau - \bar\tau}\, b + e^{2\pi i (z-\bar z)/(\tau-\bar\tau)\mathrm{ad}_b}\, \mathcal{J}(z,p)$$

- **Integration kernels for the construction of polylogarithms**
  - ⋆ obtained by Taylor expanding in b

$$\mathcal{J}(z,p) \quad = \quad dz\, a + dz \sum_{r=1}^\infty g^{(r)}(z-p)\, \mathrm{ad}_b^r a$$

$$\mathcal{J}_{\mathsf{BL}}(z,p) \quad = \quad 2\pi i \frac{dz - d\bar z}{\tau - \bar\tau}\, b + dz\, a + dz \sum_{r=1}^\infty f^{(r)}(z-p)\, \mathrm{ad}_b^r a$$

  - ⋆ The kernels $g^{(r)}(z)$ are *meromorphic* but multiple-valued on Σ
  - ⋆ The kernels $f^{(r)}(z)$ are *single-valued on* Σ but non-meromorphic

- **Both connections generalize to several variables and/or punctures**
  - ⋆ in terms of the same integration kernels

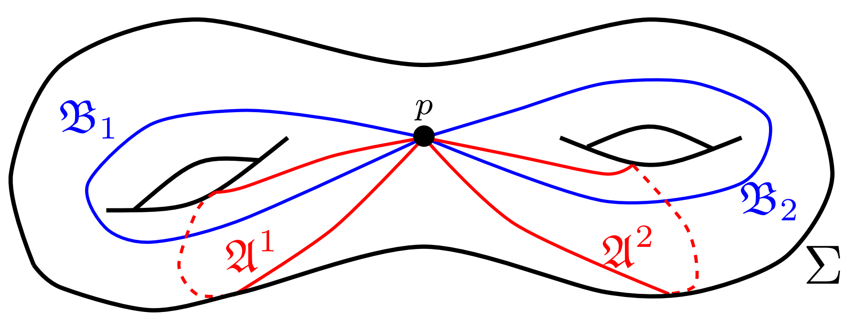
[Calaque, Enriquez, Etingof 2009; Brown, Levin 2011]

Riemann surfaces of arbitrary genus

- **Compact Riemann surface Σ of arbitrary genus $h \geq 1$**
  - ⋆ Homology group $H_1(\Sigma, \mathbb{Z})$ admits non-degenerate intersection pairing $\mathfrak{J}$ canonical basis $\mathfrak{A}^I, \mathfrak{B}_J$ has $\mathfrak{J}(\mathfrak{A}^I, \mathfrak{A}^J) = \mathfrak{J}(\mathfrak{B}_I, \mathfrak{B}_J) = 0$, $\mathfrak{J}(\mathfrak{A}^I, \mathfrak{B}_J) = \delta^I_J \quad 1 \leq I, J \leq h$
  - ⋆ Canonical bases preserved by modular transformations $M \in Sp(2h, \mathbb{Z})$

$$\begin{pmatrix} \mathfrak{B} \\ \mathfrak{A} \end{pmatrix} \to M \begin{pmatrix} \mathfrak{B} \\ \mathfrak{A} \end{pmatrix} \qquad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad M^t \mathfrak{J} M = \mathfrak{J}$$

  - ⋆ Homotopy group $\pi_1(\Sigma, p)$ generated by $\mathfrak{A}^I, \mathfrak{B}_J$ sharing a point p
  - ⋆ We shall denote the universal covering space $\tilde{\Sigma}$



  - ⋆ Holomorphic $(1,0)$ forms ω_I, normalized on $\mathfrak{A}$ cycles, give the periods Ω_{IJ}

$$\oint_{\mathfrak{A}^I} \omega_J = \delta^I_J \qquad \oint_{\mathfrak{B}_I} \omega_J = \Omega_{IJ}$$

  - ⋆ Riemann relations imply that $Y = \mathrm{Im}\, \Omega$ is positive definite = metric

Lie algebra for multivariable flat connections

- **The generators**
  - ⋆ On the torus with one variable the Lie algebra was generated by a pair (a, b)
  - ⋆ Braiding generators $t_{ij} = t_{ji}$ carry over from the sphere

$$(a,\, b) \xrightarrow{\text{genus } h} (a^I,\, b_I) \xrightarrow{n \text{ variables}} (a_i^I,\, b_{iI},\, t_{ij})$$

for $I = 1, \cdots, h$ and $i, j = 1, \cdots, n$

- **The Lie algebra $\mathfrak{g} = \mathfrak{t}_{h,n}$ is not freely generated** (e.g. for $h \geq 2$)
  - ⋆ defining structure relations with i, j, k, ℓ mutually distinct

$$[a_i^I, a_j^J] = [a_i^I, t_{jk}] = 0 \qquad\qquad [b_{iI}, a_j^J] = \delta_I^J t_{ij}$$

$$[b_{iI}, b_{jJ}] = [b_{iI}, t_{jk}] = 0 \qquad\qquad [a_i^I, b_{iI}] = \sum_{j \neq i} t_{ij}$$

  - ⋆ $\mathfrak{t}_{h,n}$ admits a bi-grading $|a| = (1,0)$, $|b| = (0,1)$, $|t| = (1,1)$
  - ⋆ considering infinite series requires degree completing $\mathfrak{t}_{h,n} \to \hat{\mathfrak{t}}_{h,n}$

- **Imply the structure relations** $[t_{ij}, t_{k\ell}] = [t_{ij} + t_{ik}, t_{jk}] = 0$

Flat connections for arbitrary h and n

- **Arbitrary genus & multiple variables motivated by QFT & string theory**
  - ⋆ where genus is loop order and variables are vertex operator insertion points

- **Connections valued in $\hat{\mathfrak{t}}_{h,n}$ in n variables $\boldsymbol{x} = (x_1, \cdots, x_n) \in \Sigma^n$**
  - ⋆ multiple-valued on Σ^n and meromorphic on $\tilde{\Sigma}^n$ [Enriquez 2011]
  - ⋆ single-valued $n = 1$ on Σ and non-meromorphic [ED, Hidding, Schlotterer 2023]
  - ⋆ single-valued on Σ^n and non-meromorphic [ED, Schlotterer 2026]

- **Require flatness on configuration space**

$$\boldsymbol{x} \in \mathrm{Cf}_n(\Sigma) = \Sigma^n \setminus \text{diagonals}$$

The Enriquez connection

- **Theorem 1** [Enriquez 2011] (alternative proof [ED, Schlotterer 2025])

There exists a unique flat connection $\mathcal{K}_\mathsf{E}$ with values in $\hat{\mathfrak{t}}_{h,n}$ in n variables such that

  - ⋆ $\mathcal{K}_\mathsf{E}$ is a meromorphic $(1,0)$ form on $\tilde\Sigma^n$ of a-degree one;
  - ⋆ whose only poles are simple in x_i at x_j with residue t_{ij};
  - ⋆ whose monodromy around $\mathfrak{A}^I$ cycles is trivial and around $\mathfrak{B}_I$ cycles is

$$K_\mathsf{E}(\mathfrak{B}_{iI}\cdot \boldsymbol{x}) = e^{-2\pi i B_{iI}} K_\mathsf{E}(\boldsymbol{x}) \qquad\qquad B_{iI} \equiv \mathrm{ad}_{b_{iI}}$$

  - ⋆ Comment: modular properties not manifest

- **Taylor expanding in b defines the Enriquez kernels $\varpi^{I_1\cdots I_r}{}_J$ and $\chi^{I_1\cdots I_r}$**

$$K_\mathsf{E}(\boldsymbol{x}) = \sum_{i=1}^n dx_i \sum_{r=0}^\infty B_{iI_1}\cdots B_{iI_r}\Big(\varpi^{I_1\cdots I_r}{}_J(x_i)a_i^J + \sum_{j\neq i}\chi^{I_1\cdots I_r}(x_i,x_j)t_{ij}\Big)$$

  - ⋆ They generalize the genus one meromorphic kernels $(\varpi^\emptyset{}_J(x) = \omega_J(x))$

$$g^{I_1\cdots I_r}{}_J(x,y) = \varpi^{I_1\cdots I_r}{}_J(x) - \chi^{I_1\cdots I_{r-1}}(x,y)\delta_J^{I_r} \quad\longrightarrow\quad g^{(r)}(x-y)$$

  - ⋆ Formulas for the Enriquez kernels
    - – via meromorphic Abelian differentials [Enriquez, Zerbini 2022; ED, Schlotterer 2025]
    - – via Schottky parametrization for hyperelliptic Σ [Baune, Broedel, Lisitsyn, Zerbini 2024]

DHS kernels

- **Arakelov Green function $\mathcal{G}(x,y)$ on a compact Riemann surface Σ**
  - ⋆ is a smooth single-valued symmetric scalar in $x \neq y \in \Sigma$ defined by

$$\partial_{\bar{x}}\partial_x \mathcal{G}(x,y) = \pi\kappa(x) - \pi\delta(x,y) \qquad \int_\Sigma d^2x\, \kappa(x)\mathcal{G}(x,y) = 0$$

where $d^2x\,\kappa(x)$ is the pull-back of the Kähler form on the Jacobian $J(\Sigma)$

$$\kappa(x) = \frac{1}{2h}\omega_I(x)\bar{\omega}^I(x) \qquad \bar{\omega}^I = Y^{IJ}\bar{\omega}_J \qquad \int_\Sigma d^2x\,\kappa(x) = 1$$

  - ⋆ both κ and $\mathcal{G}$ are conformal and modular invariant

- **Define iterated integrals over Σ starting with $\mathcal{G}^\emptyset(x,y) = \mathcal{G}(x,y)$**
  - ⋆ as smooth single-valued scalars in $x \neq y \in \Sigma$ [ED, Hidding, Schlotterer 2023]

$$\mathcal{G}^{I_1\cdots I_r}(x,y) = \int_\Sigma d^2z\, \mathcal{G}(x,z)\,\bar{\omega}^{I_1}(z)\,\partial_z\mathcal{G}^{I_2\cdots I_r}(z,y)$$

$$\Phi^{I_1\cdots I_r}{}_J(x) = \int_\Sigma d^2z\, \mathcal{G}^{I_1\cdots I_{r-1}}(x,z)\,\bar{\omega}^{I_r}(z)\,\omega_J(z) \qquad r \geq 1$$

  - ⋆ they generalize the genus one Brown-Levin kernels

$$f^{I_1\cdots I_r}{}_J(x,y) = \partial_x\Phi^{I_1\cdots I_r}{}_J(x) - \partial_x\mathcal{G}^{I_1\cdots I_{r-1}}(x,y)\delta^{I_r}_J \quad \longrightarrow \quad f^{(r)}(x-y)$$

The DHS connection

- **The DHS connection with values in $\hat{\mathfrak{t}}_{h,n}$ in n variables $\boldsymbol{x} \in \Sigma^n$**
  - ⋆ is a single-valued $(1,0) \oplus (0,1)$ form on Σ^n given by

$$\mathcal{J}_{\mathsf{DHS}}(\boldsymbol{x}) = \sum_{i=1}^{n} J_i(\boldsymbol{x}) dx_i - \pi \sum_{i=1}^{n} \bar{\omega}^I(x_i)\, b_{iI}\, d\bar{x}_i$$

whose components J_i have a-degree one and Taylor expand into DHS kernels

$$J_i(\boldsymbol{x}) = \sum_{r=0}^{\infty} B_{iI_1} \cdots B_{iI_r} \Big(\partial_i \Phi^{I_1 \cdots I_r}{}_J(x_i) a_i^J + \sum_{j \neq i} \partial_i \mathcal{G}^{I_1 \cdots I_r}(x_i, x_j) t_{ij} \Big)$$

where $\partial \Phi^{\emptyset}{}_J(x) = \omega_J(x)$ while $\mathcal{G}^{I_1 \cdots I_r}$ and $\Phi^{I_1 \cdots I_r}{}_J$ were defined above

- **Theorem 2** *The connection $\mathcal{J}_{\mathrm{DHS}}$ has the following properties* [ED, Schlotterer 2026]
  - ⋆ single-valued and smooth on $\mathrm{Cf}_n(\Sigma)$
  - ⋆ whose only poles are simple in x_i at x_j with residue t_{ij};
  - ⋆ satisfies Maurer-Cartan $d\mathcal{J}_{\mathsf{DHS}} - \mathcal{J}_{\mathsf{DHS}} \wedge \mathcal{J}_{\mathsf{DHS}} = 0$ on $\mathrm{Cf}_n(\Sigma)$
  - ⋆ invariant under the modular group $Sp(2h, \mathbb{Z})$

Modular tensors

- **Holomorphic Abelian differentials transform under $M \in Sp(2h,\mathbb{Z})$**
  - ⋆ via non-linear representations $L(M,\Omega)$ and $R(M,\Omega)$ valued in $G\ell(h,\mathbb{C})$

$$\begin{aligned} \omega_I &\to \omega_J \, R^J{}_I(M,\Omega) & \qquad R(M,\Omega) &= (C\Omega+D)^{-1} \\ \bar\omega^I &\to L^I{}_J(M,\Omega)\,\bar\omega^J & L(M,\Omega) &= C\Omega+D \end{aligned} \qquad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

- **Modular tensors are sections of (holomorphic) vector bundles over $\mathcal{M}_h$**

[Cléry, van der Geer 2014; Kawazumi 2017; ED, Green, Pioline 2017; ED, Schlotterer 2020]

$$\mathcal{T}^{I_1\cdots I_r}{}_{J_1\cdots J_s} \to L^{I_1}{}_{I'_1}\cdots L^{I_r}{}_{I'_r}\,\mathcal{T}^{I'_1\cdots I'_r}{}_{J'_1\cdots J'_s}\,R^{J'_1}{}_{J_1}\cdots R^{J'_s}{}_{J_s}$$

  - ⋆ DHS kernels $\mathcal{G}^{I_1\cdots I_r}$ and $\Phi^{I_1\cdots I_r}{}_J$ are modular tensors [ED, Hidding, Schlotterer 2023]

$$\begin{aligned} \mathcal{G}^{I_1\cdots I_r}(x,y) &\to L^{I_1}{}_{I'_1}\cdots L^{I_r}{}_{I'_r}\,\mathcal{G}^{I'_1\cdots I'_r}(x,y) \\ \Phi^{I_1\cdots I_r}{}_J(x) &\to L^{I_1}{}_{I'_1}\cdots L^{I_r}{}_{I'_r}\,\Phi^{I'_1\cdots I'_r}{}_{J'}(x)\,R^{J'}{}_J \end{aligned}$$

- **The DHS connection is modular invariant**
  - ⋆ provided the generators transform as follows under $M \in Sp(2h,\mathbb{Z})$

$$a_i^I \to L^I{}_{I'}\,a_i^{I'} \qquad\qquad b_{iI} \to b_{iI'}\,R^{I'}{}_I \qquad\qquad t_{ij} \to t_{ij}$$

  - ⋆ which is an automorphism of the Lie algebra $\hat{\mathfrak{t}}_{h,n}$

Relating the Enriquez and DHS connections

- **Theorem 3** *The connections $\mathcal{K}_{\rm E}$ and $\mathcal{J}_{\rm DHS}$ are related*
  - ⋆ by the composition of a gauge transformation $\mathcal{U}(\boldsymbol{x},\boldsymbol{y})$

$$d - \mathcal{K}_{\sf E}(\boldsymbol{x}; a, b, t) = \mathcal{U}(\boldsymbol{x},\boldsymbol{y})^{-1}\big(d - \mathcal{J}_{\sf DHS}(\boldsymbol{x}; \hat a, \hat b, \hat t)\big)\,\mathcal{U}(\boldsymbol{x},\boldsymbol{y})$$

  and a Lie algebra automorphism $\hat{\mathfrak{t}}_{h,n} \to \hat{\mathfrak{t}}_{h,n} : (\hat a, \hat b, \hat t) \to (a, b, t)$
  - ⋆ the gauge transformation is smooth and constructed out of $\mathcal{J}_{\sf DHS}$

$$\mathcal{U}(\boldsymbol{x},\boldsymbol{y}) = \mathrm{P}\exp\int_{\boldsymbol{y}}^{\boldsymbol{x}} \mathcal{J}_{\sf DHS}(\bullet; \xi, \eta, 0)$$

  Single-variable case [ED, Enriquez, Schlotterer, Zerbini 2025]

  Multi-variable case [ED, Schlotterer 2026]

- **The relation allows one to convert**
  - ⋆ between connections

    (meromorphic, multiple-valued) ⇔ (single-valued, modular invariant)
  - ⋆ and between associated polylogarithms (see Federico Zerbini's talk)

Interchange identities

- **DHS kernels satisfy interchange identities** (similarly for Enriquez kernels)
  - ⋆ The simplest case in terms of $\mathcal{G}$ and Φ is as follows

$$\omega_I(x)\partial_y \mathcal{G}(y,x) + \omega_I(y)\partial_x \mathcal{G}(x,y) - \omega_K(x)\partial_y {\Phi^K}_I(y) - \omega_K(y)\partial_x {\Phi^K}_I(x) = 0$$

The interchange identities help to *decouple* integrations over x and y

  - ⋆ General identities involve two points $x, y \in \Sigma$ for each $\vec{I} = I_1 \cdots I_r$

$$\sum_{\vec{I}=\vec{P}\vec{Q}} \Big\{ {f^{\theta(\vec{Q})L}}_K(x,y)\, {f^{\vec{P}}}_L(y,x) + {f^{\theta(\vec{Q})}}_L(x,y)\, {f^{\vec{P}L}}_K(y,x) \Big\} = 0$$

– where $\vec{I} = \vec{P}\vec{Q}$ sums over all deconcatenations of $\vec{I}$

– the antipode θ acts by $\theta(I_1 \cdots I_r) = (-)^r I_r \cdots I_1$ and we recall

$${f^{\vec{I}}}_J(x,y) = {f^{I_1\cdots I_r}}_J(x,y) = \partial_x {\Phi^{I_1\cdots I_r}}_J(x) - \partial_x \mathcal{G}^{I_1\cdots I_{r-1}}(x,y)\delta_J^{I_r}$$

⋆ Special cases proven for DHS [ED, Schlotterer 2020]

⋆ General case proven for DHS and Enriquez [ED, Schlotterer 2024]

Fay identities

- **DHS kernels satisfy Fay identities** (similarly for Enriquez kernels)
  - ⋆ The simplest case in terms of $\mathcal{G}$ and Φ is as follows

$$\begin{aligned}\partial_x \mathcal{G}(x,z)\partial_y \mathcal{G}(y,z) \;&=\; \partial_x \mathcal{G}(x,y)\partial_y \mathcal{G}(y,z) + \partial_y \mathcal{G}(y,x)\partial_x \mathcal{G}(x,z) \\ &\quad -\omega_I(x)\partial_y \mathcal{G}^I(y,z) - \omega_I(y)\partial_x \mathcal{G}^I(x,z) + \partial_x\partial_y \mathcal{G}_2(x,y)\end{aligned}$$

  - ⋆ General Fay identities on three points $x, y, z \in \Sigma$

$$\begin{aligned}& f^{\vec{I}L}{}_K(x,z)\Big(f^{\vec{J}M}{}_L(y,z) - f^{\vec{J}M}{}_L(y,x)\Big) - \sum_{\vec{I}=\vec{P}\vec{Q}} f^{\vec{P}}{}_L(x,y) f^{(\vec{J}\sqcup L\vec{Q})M}{}_K(y,z) \\ & - f^{\vec{I}M}{}_L(x,y)\Big(f^{\vec{J}L}{}_K(y,z) - f^{\vec{J}L}{}_K(y,x)\Big) - \sum_{\vec{J}=\vec{R}\vec{S}} f^{\vec{R}}{}_L(y,x) f^{(\vec{I}\sqcup L\vec{S})M}{}_K(x,z) \\ & - \sum_{\vec{J}=\vec{R}\vec{S}} \Big(f^{\vec{R}}{}_L(y,x) f^{\vec{I}M\theta(\vec{S})L}{}_K(x,y) + f^{\vec{I}M\theta(\vec{S})}{}_L(x,y) f^{\vec{R}L}{}_K(y,x)\Big) = 0\end{aligned}$$

    The Fay identities *decouple* integration in z (cfr partial fraction)

  - ⋆ Proven for DHS and conjectured for Enriquez ($f \to g$) in [ED, Schlotterer 2024]
  - ⋆ Proven for Enriquez and shown to be exhaustive in [Baune, Broedel, Im, Lisitsyn, Moeckli 2024]

Interchange and Fay identities $\Longleftrightarrow$ flatness

- **Recall the flatness conditions for DHS and Enriquez**

DHS	$\partial_i J_j(\boldsymbol{x}) - \partial_j J_i(\boldsymbol{x}) = 0$	$[J_i(\boldsymbol{x}), J_j(\boldsymbol{x})] = 0$
Enriquez	$\partial_i K_j(\boldsymbol{x}) - \partial_j K_i(\boldsymbol{x}) = 0$	$[K_i(\boldsymbol{x}), K_j(\boldsymbol{x})] = 0$

  - ⋆ The differential equations are solved by the properties of the kernels

- **Theorem 4** $\{$*Interchange & Fay identities*$\}$ $\Longleftrightarrow$ *flatness of* $\mathcal{J}_{\mathrm{DHS}}$
  - ⋆ The interchange and Fay identities arise from the commutators

$$[J_i(\boldsymbol{x}), J_j(\boldsymbol{x})] = 0 \quad \Longleftrightarrow \quad \text{all interchange \& Fay for DHS}$$

$$[K_i(\boldsymbol{x}),\ K_j(\boldsymbol{x})\,] = 0 \quad \Longrightarrow \quad \text{all interchange \& Fay for Enriquez}$$

  - ⋆ Proven by full use of the structure relations of $\mathfrak{t}_{h,n}$ [ED, Schlotterer 2026]

- **Path ordered exponentials are homotopy invariant**

$$\Gamma(\boldsymbol{x}, \boldsymbol{y}) = \mathrm{Pexp} \int_{\boldsymbol{y}}^{\boldsymbol{x}} \mathcal{J}_{\mathrm{DHS}}$$

  - ⋆ Taylor expansion in powers of b produces polylogarithms
  - ⋆ Since $\mathfrak{t}_{h,n}$ is not freely generated, there are relations between different words
  - ⋆ These relations are equivalent to the Fay identities

The connections in the presence of punctures

- **DHS for variables** $\boldsymbol{x} = (x_1, \cdots, x_n)$ **and punctures** $\boldsymbol{y} = (x_{n+1}, \cdots, x_{n+p})$
  - ⋆ DHS connection for $n+p$ variables $\boldsymbol{z} = \boldsymbol{x} \cup \boldsymbol{y}$ no punctures

$$\mathcal{J}_{\mathsf{DHS}}(\boldsymbol{z}) = \sum_{i=1}^{n} \Big(J_i(\boldsymbol{z}) dx_i - \pi \bar\omega^I(x_i) b_{iI} d\bar x_i \Big) + \sum_{\alpha=n+1}^{n+p} \Big(J_\alpha(\boldsymbol{z}) dx_\alpha - \pi \bar\omega^I(x_\alpha) b_{\alpha I} d\bar x_\alpha \Big)$$

  - ⋆ Setting $dx_\alpha = d\bar x_\alpha = 0$ for $\alpha = n+1, \cdots, n+p$ produces a reduced connection

$$J_i(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\vec I} B_{i \vec I} \Big(\partial_i \Phi^{\vec I}{}_J(x_i) a_i^J + \sum_{j \neq i} \partial_i \mathcal{G}^{\vec I}(x_i, x_j) t_{ij} + \sum_{\alpha=n+1}^{n+p} \partial_i \mathcal{G}^{\vec I}(x_i, y_\alpha) t_{i\alpha} \Big)$$

  - ⋆ All dependence on the generators $a^I_\alpha, b_{\alpha I}, t_{\alpha\beta}$ has disappeared
  - ⋆ The remaining generators satisfy the algebra $\mathfrak{t}_{h,n,p}$

$$[a_i^I, a_j^J] = [a_i^I, t_{jk}] = 0 \qquad [b_{iI}, b_{jJ}] = [b_{iI}, t_{jk}] = 0 \qquad [t_{i\alpha}, t_{j\beta}] = [t_{i\alpha}, t_{jk}] = 0$$

$$[a_i^I, t_{j\alpha}] = [a_i^I + a_j^I, t_{ij}] = 0 \qquad [b_{iI}, t_{j\alpha}] = [b_{iI} + b_{jI}, t_{ij}] = 0 \qquad [t_{ij}, t_{k\ell}] = [t_{i\alpha} + t_{j\alpha}, t_{ij}] = 0$$

$$[t_{i\alpha}, t_{j\alpha} + t_{ij}] = 0 \qquad [t_{ik} + t_{jk}, t_{ij}] = 0 \qquad [b_{iI}, a_j^J] - \delta_I^J t_{ij} = 0$$

$$[b_{iI}, a_i^I] + \sum_{j \neq i} t_{ij} + \sum_\alpha t_{i\alpha} = 0$$

Application to string theory

- **Goal: string integrands in terms of DHS integration kernels $\times$ Koba-Nielsen**
  - $\star$ e.g. for even spin structures and NS external states in the RNS formulation

- **The RNS formulation appeals to worldsheet fermions**
  - $\star$ For even spin structure δ, generic moduli, the fermion propagator satisfies

$$\partial_{\bar{x}} S_\delta(x,y) = \pi \delta(x,y)$$

  - $\star$ Fermion correlators include cyclic products, with $\boldsymbol{x} = (x_1, \cdots, x_n)$

$$C_\delta(\boldsymbol{x}) = S_\delta(x_1,x_2) S_\delta(x_2,x_3) \cdots S_\delta(x_{n-1},x_n) S_\delta(x_n,x_1)$$

- **Theorem 5: Cyclic products decompose** [ED, Hidding, Schlotterer 2023; ED, Schlotterer 2025]

$$C_\delta(\boldsymbol{x}) = \sum_{r=0}^{n} C_\delta^{I_1 \cdots I_r} \, \mathcal{V}_{I_1 \cdots I_r}(\boldsymbol{x})$$

  - $\star$ $\mathcal{V}_{I_1 \cdots I_r}$ are independent of δ, single-valued on Σ^n, meromorphic in $\boldsymbol{x}$ and entirely expressible in terms of DHS kernels
  - $\star$ $C_\delta^{I_1 \cdots I_r}$ are independent of $\boldsymbol{x}$ and contain all δ-dependence
  - $\star$ both are tensors under the congruence subgroup $\Gamma_h(2) \subset Sp(2h, \mathbb{Z})$

(Note: in terms of Enriquez kernels the coefficients $\tilde{C}_\delta^{I_1 \cdots I_r}$ are not modular tensors)

Summary and Outlook

Flatness of $\mathcal{J}_{\mathrm{DHS}}$ $[J_i^{(1,0)}, J_j^{(1,0)}] = 0$	$\Longleftrightarrow$	Flatness of $\mathcal{K}_{\mathrm{E}}$ $[K_i, K_j] = 0$
$\Updownarrow$		$\Downarrow$
Interchange and Fay identities for DHS kernels	$\Longrightarrow$	Interchange and Fay identities for Enriquez kernels

- **Ongoing projects**

  ⋆ Extend flat connections to include moduli variations [ED, Enriquez, Schlotterer, Zerbini 2026]

  ⋆ Relations between polylogarithms induced by Fay identities [ED, Schlotterer, Sohnle 2026]

  ⋆ Structure of spaces of modular tensors at higher genus [ED, Schlotterer, $\cdots$]

  (strongly motivated by string theory)

  ⋆ Embedding Calabi-Yau into universal curves [ED, Schlotterer, $\cdots$]