Mathematical adventures of a theoretical physicist

From string theory to modular graph forms, modular tensors and polylogarithms

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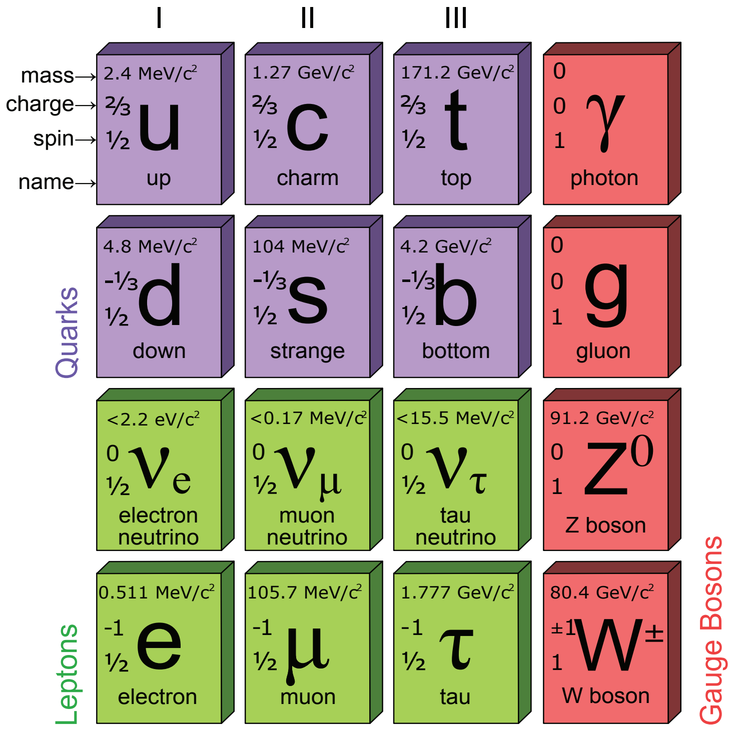
The Standard Model of Elementary Particles

- **Golden decade of elementary particle physics: 1968-1978**
  - ⋆ experimental discovery of quarks
  - ⋆ weak and strong interactions via Yang-Mills theory
  - ⋆ predictive “renormalizable” quantum field theory



I	II	III	
2.4 MeV/c^2 ⅔ ½ u up	1.27 GeV/c^2 ⅔ ½ c charm	171.2 GeV/c^2 ⅔ ½ t top	0 0 1 γ photon
4.8 MeV/c^2 -⅓ ½ d down	104 MeV/c^2 -⅓ ½ s strange	4.2 GeV/c^2 -⅓ ½ b bottom	0 0 1 g gluon
<2.2 eV/c^2 0 ½ ν_e electron neutrino	<0.17 MeV/c^2 0 ½ ν_μ muon neutrino	<15.5 MeV/c^2 0 ½ ν_τ tau neutrino	91.2 GeV/c^2 0 1 Z^0 Z boson
0.511 MeV/c^2 -1 ½ e electron	105.7 MeV/c^2 -1 ½ μ muon	1.777 GeV/c^2 -1 ½ τ tau	80.4 GeV/c^2 ±1 1 $W^\pm$ W boson

- **Further elaborated and confirmed after 1978**
  - ⋆ W and Z found in 1983
  - ⋆ top quark found in 1995
  - ⋆ Higgs particle found in 2011-13

- **Remarkable agreement with experiment**

  e.g. Electron magnetic moment $\mu_e/\mu_{\rm Bohr}$

  expmt: $1.001\,159\,652\,180\,59(13)$

  theory: $1.001\,159\,652\,180\,46(18)$

What lies beyond the Standard Model ?

- **Theoretical physicists thrive on contradictions and discrepancies**
  - ⋆ which often open a window into new physics
  - ⋆ remarkable agreement of SM is both a blessing and a curse
- **Can one combine quantum field theory with general relativity ?**
  - ⋆ not predictive: "non-renormalizable"
  - ⋆ also, there appear some serious discrepancies
    - Cosmological vacuum energy density

      expmt: $\sim 10^{-12}(eV)^4$

      theory: $> 10^{+45}(eV)^4$ (based on the largest scale in the SM)
- **Several beautiful ideas**
  - ⋆ supersymmetry/super Yang-Mills theory/supergravity
  - ⋆ superstring theory provides a unified quantum theory of gravity

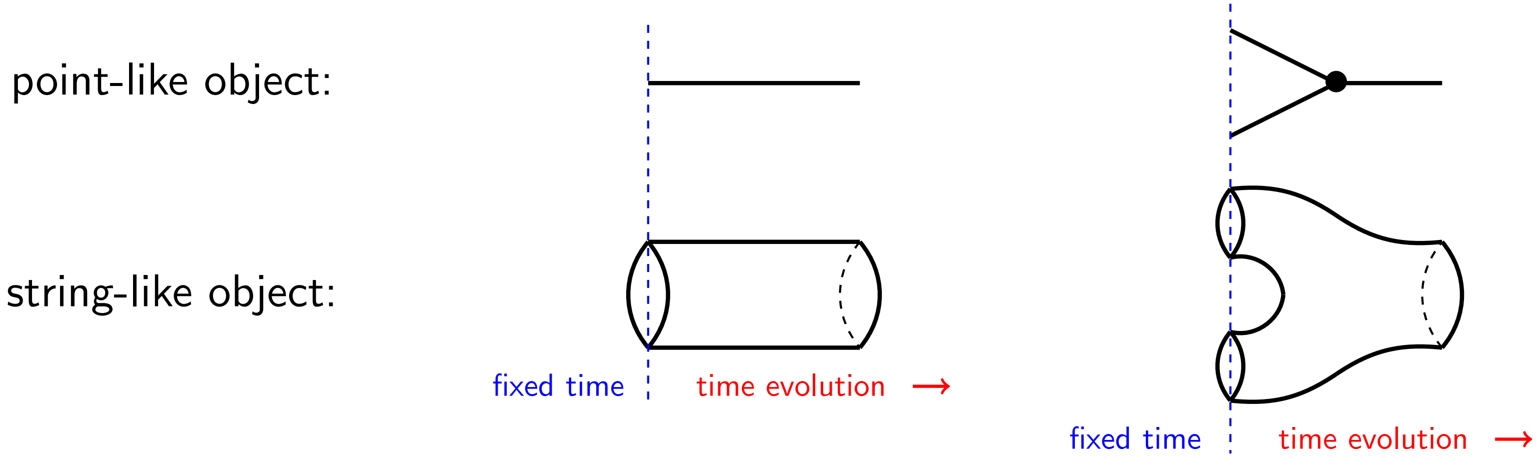
String Theory

- **Elementary constituents are one-dimensional objects in space**

  by contrast with QFT where they are point-like objects

- **Under time evolution a string sweeps out a two-dimensional surface**

  by contrast with QFT where a point sweeps out a curve



- **Interactions of strings are topological via joining and splitting**

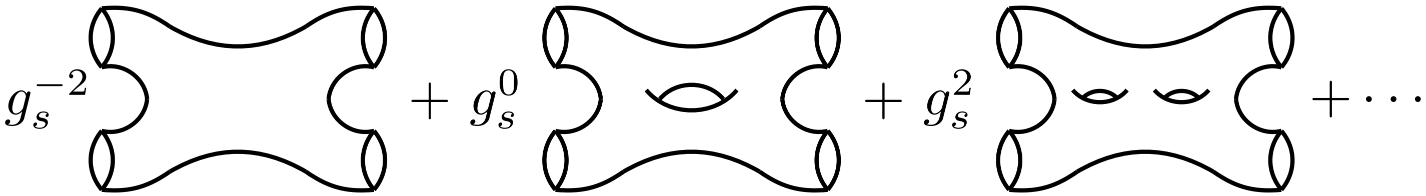
  by contrast with QFT where they require additional input

Sums over Riemann surfaces

- **Quantum mechanics predicts**

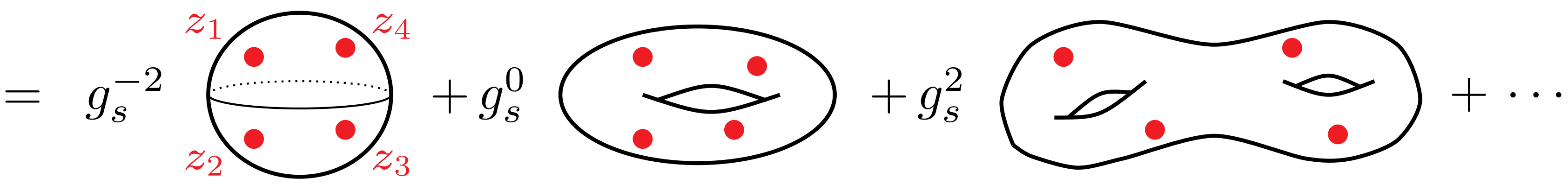
$$\textit{probability} = |\ \textit{probability amplitude}\ |^2$$

- **Probability amplitude in string theory**



⋆ topological (genus) expansion in the string coupling $g_s \in \mathbb{R}$

- **Starting in far past and observed in the far future** *"S-matrix"*



⋆ each marked point z_i represents an incoming or outgoing string

⋆ integrate over marked points and complex structure moduli

PROGRAM : elucidating structure of string amplitudes

- **Intimate relation with quantum field theory amplitudes**
    - ⋆ strings observed at long distances (low energy) are point-like
    - ⋆ importance of understanding the space of functions
    - ⋆ role of polylogarithms, integrability, periods, Calabi-Yau spaces, ...

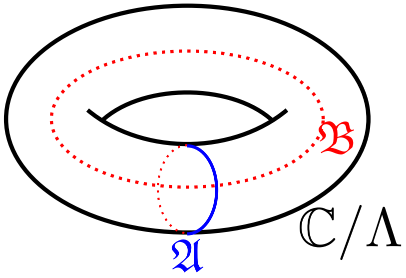
This talk

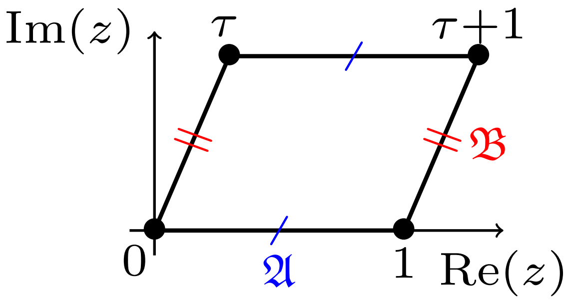
- **Modular graph functions and forms for genus one (the torus)**
    - ⋆ map graphs to $SL(2,\mathbb{Z})$ modular forms
    - ⋆ naturally generalize holomorphic and real-analytic Eisenstein series
- **Modular graph tensors on higher genus Riemann surfaces**
    - ⋆ map graphs to $Sp(2h,\mathbb{Z})$ tensors on moduli space of Riemann surfaces
    - ⋆ generalize number-theoretic invariants of Kawazumi and Zhang
- **Polylogarithms on Riemann surfaces of arbitrary genus**
    - ⋆ generalize polylogarithms of Brown and Levin

MODULAR GRAPH FUNCTIONS AND FORMS

Genus-one: the torus

- **Represent the torus by $\Sigma = \mathbb{C}/\Lambda$ for a lattice $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$**





  - ⋆ modulus $\tau \in \mathcal{H}_1 = \{\tau_1 + i\tau_2;\ \tau_1, \tau_2 \in \mathbb{R};\ \tau_2 > 0\}$
  - ⋆ automorphisms of the torus form the modular group $SL(2,\mathbb{Z})$

$$\begin{pmatrix} \mathfrak{B} \\ \mathfrak{A} \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \mathfrak{B} \\ \mathfrak{A} \end{pmatrix} \qquad \tau \to \frac{a\tau + b}{c\tau + d} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

- **Essential ingredient: the scalar Green function $g(z|\tau)$ in Σ**

$$\partial_{\bar z}\partial_z g(z-y|\tau) = -\pi\delta(z-y) + \tfrac{\pi}{\tau_2} \qquad \int_\Sigma d^2z\, g(z-y|\tau) = 0$$

  - ⋆ Solved by Kronecker-Eisenstein series $(z = u + \tau v,\ u, v \in \mathbb{R}/\mathbb{Z})$

$$g(z|\tau) = \sum_{m,n\in\mathbb{Z}}{}' \frac{\tau_2}{\pi|m\tau+n|^2} e^{2\pi i(mu-nv)}$$

Genus-one amplitudes

- **Consider the following integral over N points on the torus**

$$\mathcal{B}_N^{(1)}(s_{ij}|\tau) = \prod_{k=1}^{N} \int_\Sigma \frac{d^2 z_k}{\tau_2} \, \exp\left\{ \sum_{1 \leq i < j \leq N} s_{ij} \, g(z_i - z_j|\tau) \right\}$$

  - ⋆ absolutely convergent and holomorphic in parameters s_{ij} for $|s_{ij}| < 1$
  - ⋆ simple poles at $s_{ij} \in \mathbb{N}$
  - ⋆ invariant under the modular group $SL(2,\mathbb{Z})$ using invariance of g

$$g\left(\frac{z}{c\tau+d}\Big|\frac{a\tau+b}{c\tau+d}\right) = g(z|\tau) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

- **String amplitude for $N = 4$ gravitons** (Green, Schwarz, 1982)

$$\mathcal{A}_4^{(1)}(s_{ij}) = \int_{\mathcal{M}_1} \frac{d^2\tau}{\tau_2^2} \mathcal{B}_4^{(1)}(s_{ij}|\tau) \qquad \qquad \mathcal{M}_1 = \mathcal{H}_1 / SL(2,\mathbb{Z})$$

  - ⋆ s_{ij} parametrize momenta of incoming/outgoing strings
  - ⋆ Analytic continuation in s_{ij} exhibits physical poles and branch cuts
  - ⋆ Quotient by $SL(2,\mathbb{Z})$ gives a *finite* amplitude (Shapiro 1972)

    free of the obstacles to quantizing general relativity (Scherk, Schwarz 1974; Yonega 1974)

Graphical Representation of Taylor series of $\mathcal{B}_N^{(1)}(s_{ij}|\tau)$

- **Absolute convergence of the integral $\mathcal{B}_N^{(1)}(s_{ij}|\tau)$ for $|s_{ij}| < 1$ and fixed τ**
  - ⋆ allows for Taylor expansion in the variables s_{ij}
    - $=$ physically corresponds to the "low energy expansion"

- **Represented by Feynman graphs**
  - ⋆ Each integration point z_i on Σ is represented by a vertex $\bullet$
  - ⋆ Each Green function by an edge between vertices z_i and z_j

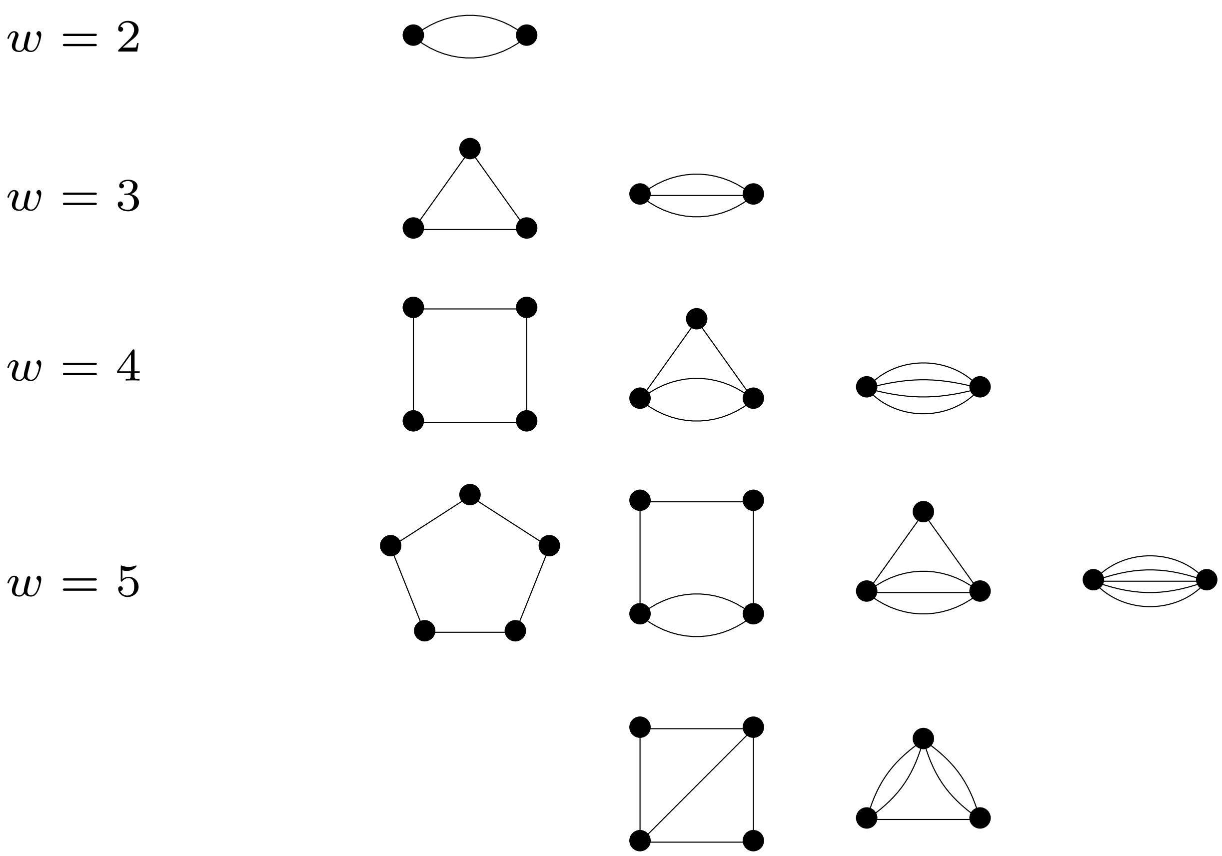
$= \quad g(z_i - z_j|\tau)$

  - ⋆ Each vertex is integrated over Σ
  - ⋆ To a graph with w edges we assign *weight* w

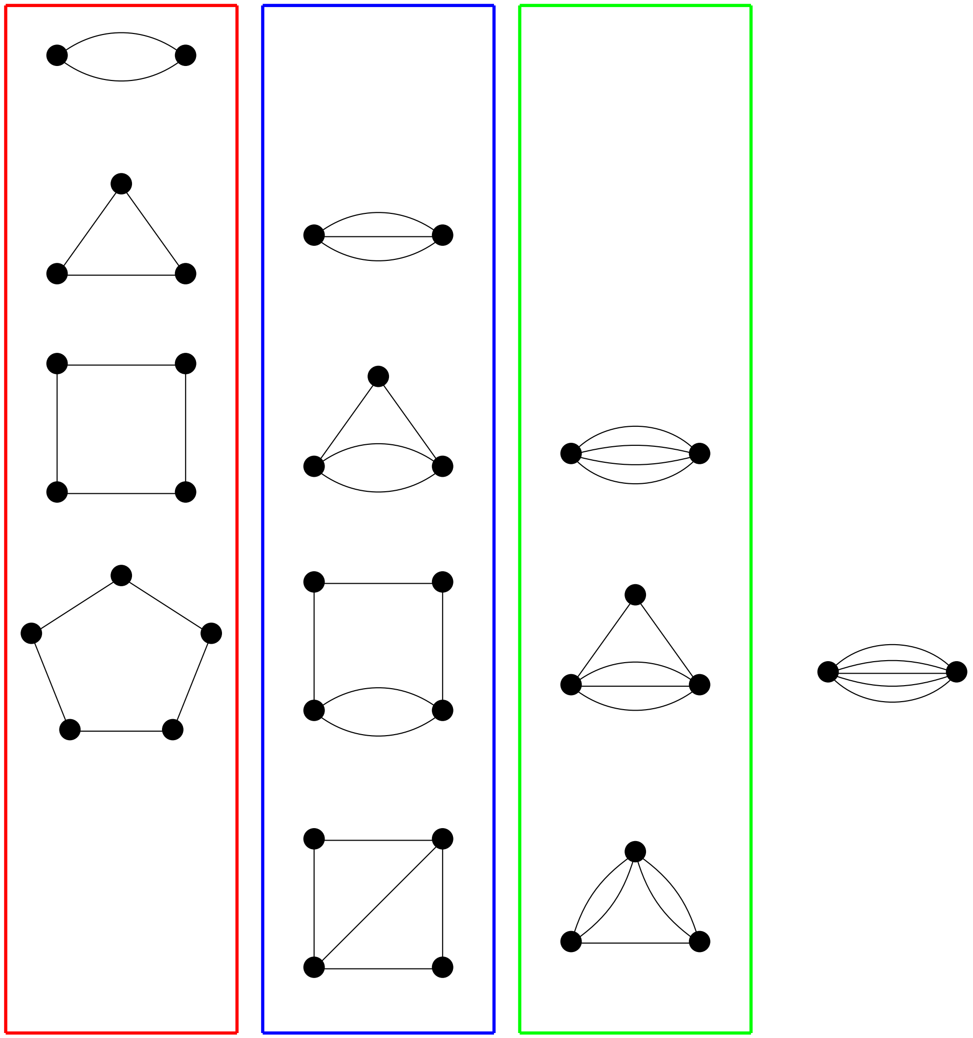
- **Reducibility** : A graph which becomes disconnected
  - ⋆ upon cutting one edge vanishes by $\int_\Sigma g = 0$
  - ⋆ upon removing one vertex factorizes into the product of its components

Modular graph functions

- **Since $\mathcal{B}_N^{(1)}(s_{ij}|\tau)$ is a modular function, so are its Taylor coefficients**
  - ⋆ Actually, each individual graph produces a *modular graph function*
  - ⋆ $\mathcal{B}_N^{(1)}(s_{ij}|\tau)$ is a generating function for modular graph functions



Modular graph functions organized by loop order



$w = 2$

$w = 3$

$w = 4$

$w = 5$

one-loop two-loops three-loops

One-loop modular graph functions: Eisenstein series

- **One-loop weight w graph has w edges and w vertices**

$$\prod_{i=1}^{w} \int_\Sigma \frac{d^2 z_i}{\tau_2} g(z_i - z_{i+1}|\tau) = \sum_{p\in\Lambda'} \frac{\tau_2^w}{\pi^w |p|^{2w}} = E_w(\tau)$$

  - ⋆ lattice of momenta on the torus $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ and $\Lambda' = \Lambda \setminus \{0\}$
  - ⋆ invariant under the modular group $SL(2,\mathbb{Z})$

- **Real-analytic Eisenstein series $E_s(\tau)$**
  - ⋆ Expansion near the cusp $\tau \to i\infty$

$$E_s = \frac{2\zeta(2s)}{\pi^s}\tau_2^s + \frac{2\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)\,\pi^{s-\frac{1}{2}}}\,\tau_2^{1-s} + \mathcal{O}(e^{-2\pi\tau_2})$$

  - ⋆ Eigenfunction of the Laplace-Beltrami operator $\Delta = 4\tau_2^2 \partial_\tau \partial_{\bar\tau}$ on $\mathcal{H}_1$

$$\Delta E_s = s(s-1)E_s$$

Two-loop modular graph functions

- **Two-loop graphs evaluate to a multiple Kronecker-Eisenstein series**

$$C_{a,b,c}(\tau) = \sum_{p_1,p_2,p_3 \in \Lambda'} \delta(p_1+p_2+p_3) \frac{\tau_2^w}{\pi^w |p_1|^{2a} \, |p_2|^{2b} \, |p_3|^{2c}}$$

  ⋆ Absolutely convergent for $a, b, c \in \mathbb{N}$, of weight $w = a+b+c$
  ⋆ invariant under $SL(2,\mathbb{Z})$

- **Theorem 1** (ED & Bill Duke 2017)

  Expansion near cusp $\tau \to i\infty$: Laurent polynomial in τ_2 of degree $(w, 1-w)$

$$C_{a,b,c}(\tau) = c_w (\pi\tau_2)^w + \frac{c_{2-w}}{(\pi\tau_2)^{w-2}} + \sum_{k=1}^{w-1} \frac{c_{w-2k-1}\,\zeta(2k+1)}{(\pi\tau_2)^{2k+1-w}} + \mathcal{O}(e^{-2\pi\tau_2})$$

  where $c_w, c_{w-2k-1} \in \mathbb{Q}$ are explicitly known with

$$c_{2-w} = \sum_{k=1}^{w-2} \gamma_k \, \zeta(2k+1)\zeta(2w-2k-3) \qquad \gamma_k \in \mathbb{Z}$$

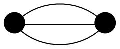
System of differential identities

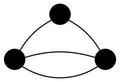
- **Two-loop modular graph functions** $C_{a,b,c}$ **of weight** $w = a+b+c$**,** $a,b,c \in \mathbb{N}$
  - ⋆ obey a system of differential equations of uniform weight w

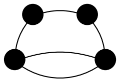
$$\begin{aligned} 2\Delta C_{a,b,c} \quad &= \quad 2ab\, C_{a+1,b-1,c} + ab\, C_{a+1,b+1,c-2} - 4ab\, C_{a+1,b,c-1} \\ &\quad\; + a(a-1)\, C_{a,b,c} + \text{ 5 permutations of } (a,b,c) \end{aligned}$$

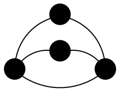
  - ⋆ where $C_{a,b,0} = E_a E_b - E_{a+b}$ and $C_{a,b,-1} = E_{a-1}E_b + E_a E_{b-1}$

- **Examples suggest "eigenvalues" of the form** $s(s-1)$ **and** $s \in \mathbb{N}$

$C_{1,1,1} =$  $(\Delta - 0)C_{1,1,1} = 6E_3$

$C_{2,1,1} =$  $(\Delta - 2)C_{2,1,1} = 9E_4 - E_2^2$

$C_{3,1,1} =$  $(\Delta - 6)C_{3,1,1} = 3C_{2,2,1} + 16E_5 - 4E_2E_3$

$C_{2,2,1} =$  $(\Delta - 0)C_{2,2,1} = 8E_5$

Differential and algebraic identities

- **Theorem 2** (ED, Green, Vanhove 2015)

  For every weight $w = a + b + c \geq 3$ *the two-loop modular graph functions* $C_{a,b,c}(\tau)$ *are linear combinations of modular functions* $\mathfrak{C}_{w;s;\mathfrak{p}}(\tau)$ *satisfying*

$$\big(\Delta - s(s-1)\big)\mathfrak{C}_{w;s;\mathfrak{p}} = \mathfrak{F}_{w;s;\mathfrak{p}}$$

$$s = w - 2\mathfrak{m} \qquad \mathfrak{m} = 1, \cdots, [\tfrac{w-1}{2}] \qquad \mathfrak{p} = 0, \cdots, [\tfrac{s-1}{3}]$$

  and $\mathfrak{F}_{w;s;\mathfrak{p}}$ *is a polynomial of degree 2 and weight* w *in* $E_{s'}$ *with* $2 \leq s' \leq w$.

- **Differential identities for zero "eigenvalue" imply algebraic identities**

$$\Delta C_{1,1,1} = 6E_3 \qquad C_{1,1,1} = E_3 + \zeta(3)$$

$$\Delta C_{2,2,1} = 8E_5 \qquad C_{2,2,1} = \tfrac{2}{5}E_5 + \tfrac{\zeta(5)}{30}$$

$$\Delta(C_{3,3,1} + C_{3,2,2}) = 18E_7 \qquad C_{3,3,1} + C_{3,2,2} = \tfrac{3}{7}E_7 + \tfrac{\zeta(7)}{252}$$

  ⋆ fix integration constant using asymptotics at cusp of Theorem 1

Modular graph functions at higher loop order

- **Expansion near the cusp** $\tau \to i\infty$
  - ⋆ Laurent polynomial in τ_2 of degree $(w, 1-w)$ $+\mathcal{O}(e^{-2\pi\tau_2})$
  - ⋆ coefficients contain multiple zeta-values (Zerbini 2017)

- **Laplace-Beltrami operator for 3 loops and higher**
  - ⋆ no longer maps the space of modular graph functions into itself

- **Need to generalize** (ED & Green 2016)
  - ⋆ modular graph functions: identical exponents of $p = m + \tau n$ and $\bar p$
  - ⋆ generalize by allowing for unequal exponents as well

$$\frac{1}{|p|^{2a}} \longrightarrow \frac{1}{p^a \, \bar p^b}$$

  - ⋆ but retain $a - b \in \mathbb{Z}$ for good modular properties

Modular graph forms

- **A modular graph form associated with a decorated graph** (Γ, A, B)
  - ⋆ *with vertices* $v = 1, \cdots, V$ *and edges* $r = 1, \cdots, R$ *consists of*
  - ⋆ *connectivity matrix* Γ *with components* $\Gamma_{v\, r} \in \{0, \pm 1\}$
  - ⋆ *decoration of edges by "exponents"* $A = [a_1, \cdots, a_R]$ *and* $B = [b_1, \cdots, b_R]$

To the decorated graph (Γ, A, B) *we associate a function on* $\mathcal{H}_1$

$$\mathcal{C}_\Gamma \begin{bmatrix} A \\ B \end{bmatrix} (\tau) = \sum_{p_1, \cdots, p_R \in \Lambda'} \left(\prod_{r=1}^{R} \frac{(\tau_2/\pi)^{b_r}}{(p_r)^{a_r} \, (\bar p_r)^{b_r}} \right) \prod_{v=1}^{V} \delta \left(\sum_{r=1}^{R} \Gamma_{v\, r} p_r \right)$$

- **Transform under** $SL(2,\mathbb{Z})$ **as modular forms**

$$\mathcal{C}_\Gamma \begin{bmatrix} A \\ B \end{bmatrix} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta} \right) = (\gamma\tau + \delta)^{\mu} \, \mathcal{C}_\Gamma \begin{bmatrix} A \\ B \end{bmatrix} (\tau) \qquad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{Z})$$

⋆ modular weight $(\mu, 0)$ with $\mu = \sum_r (a_r - b_r)$: "modular graph form"
⋆ when $\mu \neq 0$ there is no canonical normalization for powers of τ_2
⋆ $A = B \Rightarrow \mu = 0$ recover modular graph functions for Green function g
⋆ $B = 0$ recover holomorphic modular forms

Closure of the space of modular graph forms

- **The Maass operator $\overline{\nabla} = -2i\tau_2^2\partial_{\bar\tau}$ acts by**

$$\overline{\nabla}\,\mathcal{C}_\Gamma\begin{bmatrix}A\\B\end{bmatrix} = \sum_{r=1}^{R} b_r\,\mathcal{C}_\Gamma\begin{bmatrix}A-S_r\\B+S_r\end{bmatrix}$$

  - ⋆ where $A=[a_1\ \cdots\ a_R]$, $B=[b_1\ \cdots\ b_R]$ and $S_r=[0_{r-1}\ 1\ 0_{R-r}]$
  - ⋆ further momentum conservation identities
  - ⋆ imply closure under differentiation of the space of modular graph forms

- **Applications**
  - ⋆ Systematic construction of modular graph forms and their identities

    (ED, Green 2016; ED, Kaidi 2016; Basu 2016-2019; Kaidi, Gerken 2016; Gerken, Kleinschmidt, Schlotterer 2018-2020; Kleinschmidt, Verschinnin 2017; Dorigoni, Doroudiani, Drewitt, Hidding, Kleinschmidt, Matthes, Schlotterer, Verbeek 2022)
  - ⋆ Integration over τ gives low energy effective interactions (ED, Green 2019)
  - ⋆ Algebraic geometry setting (Brown 2018)

MODULAR GRAPH TENSORS

on higher genus Riemann surfaces

Compact Riemann surfaces Σ of arbitrary genus h

- **Homology and cohomology**
  - ⋆ One-cycles $H_1(\Sigma, \mathbb{Z}) \approx \mathbb{Z}^{2h}$ with intersection pairing $\mathfrak{J}(\cdot, \cdot) \to \mathbb{Z}$
  - ⋆ Canonical basis $\mathfrak{J}(\mathfrak{A}_I, \mathfrak{A}_J) = \mathfrak{J}(\mathfrak{B}_I, \mathfrak{B}_J) = 0$, $\mathfrak{J}(\mathfrak{A}_I, \mathfrak{B}_J) = \delta_{IJ}$ for $1 \leq I, J \leq h$
  - ⋆ Canonical dual basis of holomorphic one-forms ω_I in $H^{(1,0)}(\Sigma)$

$$\oint_{\mathfrak{A}_I} \omega_J = \delta_{IJ} \qquad\qquad \oint_{\mathfrak{B}_I} \omega_J = \Omega_{IJ}$$

  - ⋆ Period matrix Ω obeys Riemann relations $\Omega^t = \Omega, \ \mathrm{Im}\,(\Omega) > 0$

- **Modular group** $Sp(2h, \mathbb{Z}) : H_1(\Sigma, \mathbb{Z}) \to H_1(\Sigma, \mathbb{Z})$ **leaves** $\mathfrak{J}(\cdot, \cdot)$ **invariant**

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad\qquad M^t \mathfrak{J} M = \mathfrak{J} \qquad\qquad \begin{pmatrix} \mathfrak{B} \\ \mathfrak{A} \end{pmatrix} \to M \begin{pmatrix} \mathfrak{B} \\ \mathfrak{A} \end{pmatrix}$$

  - – action on 1-forms ω and period matrix Ω given by

$$\begin{aligned} \omega &\to \omega\,(C\Omega + D)^{-1} \\ \Omega &\to (A\Omega + B)\,(C\Omega + D)^{-1} \end{aligned}$$

Modular graph functions for arbitrary genus

- **Canonical modular invariant volume form κ on Σ** (Einstein convention for summing indices)

$$\kappa = \frac{i}{2h}\omega_I \wedge \overline{\omega}^I \qquad \int_\Sigma \kappa = 1 \qquad \overline{\omega}^I = \left[(\mathrm{Im}\,\Omega)^{-1}\right]^{IJ}\overline{\omega}_J$$

- **The Arakelov Green function $\mathcal{G}(z,w|\Sigma)$ generalizes $g(z-w|\tau)$ of $h=1$**

$$\partial_{\bar w}\partial_w \mathcal{G}(w,z|\Sigma) = -\pi\delta(w,z) + \pi\kappa(w) \qquad \int_\Sigma \kappa\,\mathcal{G} = 0$$

- **Natural $Sp(2h,\mathbb{Z})$ invariant generating function** (ED, Green, Pioline 2017)

$$\mathcal{C}_N^{(h)}(s_{ij}|\Sigma) = \int_{\Sigma^N} \prod_{i=1}^N \kappa(z_i) \exp\left\{ \sum_{1\leq i<j\leq N} s_{ij}\,\mathcal{G}(z_i,z_j|\Sigma)\right\}$$

  - ⋆ Integrals absolutely convergent for $|s_{ij}|<1$
  - ⋆ Taylor coeffs in s_{ij} give *higher genus modular graph functions*
  - ⋆ Admit finite degree Laurent polynomial degenerations of Σ
  - ⋆ Differential equations *remain an open problem*

Genus-two string amplitude

- **Actually, the genus 2 string amplitude does NOT correspond to** $\mathcal{C}_4^{(2)}(s_{ij}|\Sigma)$
  - ⋆ volume form κ is unique on Σ
  - ⋆ but κ^N is not unique on Σ^N for $N \geq 2$

- **Instead, the four-graviton string amplitude is** (ED & Phong 2005; Berkovits 2005)

$$\mathcal{B}_4^{(2)}(s_{ij}|\Sigma) = \int_{\Sigma^4} \mathcal{Y}\overline{\mathcal{Y}} \exp\left\{ \sum_{1\leq i<j\leq 4} s_{ij}\, \mathcal{G}(z_i, z_j|\Sigma) \right\}$$

- ⋆ Measure given by a holomorphic $(1,0)^{\otimes 4}$ form $\mathcal{Y}$ on Σ^4

$$\mathcal{Y} = (s_{14} - s_{13})\Delta(z_1, z_2)\Delta(z_3, z_4)/(\det Y) + 2 \text{ cycl perms of } (2,3,4)$$

where Δ is a holomorphic $(1,0)^{\otimes 2}$ form on Σ^2

$$\Delta(z_i, z_j) = \omega_1(z_i)\omega_2(z_j) - \omega_2(z_i)\omega_1(z_j)$$

- ⋆ The measure $\mathcal{Y}\overline{\mathcal{Y}}$ and $\mathcal{B}_4^{(2)}(s_{ij}|\Sigma)$ are $Sp(4,\mathbb{Z})$-invariant.

Kawazumi-Zhang invariant

- **Contribution $\varphi(\Sigma)$ of order s^3 to $\mathcal{B}_4^{(2)}$ is the KZ invariant**

$$\varphi(\Omega) = \int_{\Sigma^2} \omega_I(x)\overline{\omega}^J(x)\, \mathcal{G}(x,y|\Sigma)\, \omega_J(y)\overline{\omega}^I(y)$$

  - ⋆ introduced as a spectral invariant (Kawazumi 2008; Zhang 2008)
  - ⋆ satisfies an eigenvalue equation (ED, Green, Pioline, Russo 2014; also Kawazumi 2008)

$$\Delta\varphi = 5\,\varphi$$

  - ⋆ Δ is the Laplace-Beltrami operator on the genus-two moduli space $\mathcal{M}_2$

- $\mathcal{B}_4^{(2)}(s_{ij}|\Sigma)$ **generates an infinite family of modular invariants**

  (ED, Green 2013; ED, Green, Pioline 2017)

Modular graph tensors for arbitrary genus

- **Consider the integral for** $\varphi(\Omega)$ *without contracting indices*

$$\mathcal{A}_{IJ}{}^{KL}(\Omega) = \int_{\Sigma^2} \omega_I(x)\overline{\omega}^K(x)\, \mathcal{G}(x,y|\Omega)\, \omega_J(y)\overline{\omega}^L(y)$$

  ⋆ transforms as a modular tensor under $Sp(2h,\mathbb{Z})$

$$\tilde{\mathcal{A}}_{IJ}{}^{KL}(\tilde{\Omega}) = R^{-1}(\Omega)^K{}_P\, R^{-1}(\Omega)^L{}_Q\, \mathcal{A}_{MN}{}^{PQ}(\Omega)\, R(\Omega)^M{}_I\, R(\Omega)^N{}_J$$

  ⋆ using modular invariance of $\mathcal{G}$ and the $Sp(2h,\mathbb{Z})$ transforms

$$\tilde{\Omega} = (A\Omega + B)R(\Omega) \qquad \tilde{\omega} = \omega R(\Omega) \qquad R(\Omega) = (C\Omega + D)^{-1}$$

- **Generalization to** N **points** (ED, Schlotterer 2022)

$$\mathcal{B}_{I_1\cdots I_N}{}^{J_1\cdots J_N}(s_{ij}|\Omega) = \int_{\Sigma^N} \prod_{i=1}^N \omega_{I_i}(z_i)\overline{\omega}^{J_i}(z_i) \exp\left\{ \sum_{1\leq i<j\leq N} s_{ij}\, \mathcal{G}(z_i,z_j|\Omega) \right\}$$

  ⋆ Taylor expansion in s_{ij} gives modular graph tensors for arbitrary genus

POLYLOGARITHMS

on Riemann surfaces of arbitrary genus

Polylogarithms on $\mathbb{C}$ or the Riemann sphere

- **Rational functions $\mathbb{C}(z)$ close under differentiation but not under integration**
  - ⋆ Obstruction consists of functions with simple poles

$$\int_0^z \frac{dt}{t-1} = \ln(1-z)$$

  - ⋆ e.g. one iteration produces Euler's dilogarithm.

- **Polylogarithms form a ring that closes under differentiation and integration**

$$G(a_1, \cdots, a_n; z) = \int_0^z \frac{dt}{t-a_1} G(a_2, \cdots, a_n; t) \qquad\qquad G(\emptyset; z) = 1$$

  - ⋆ Only depend on homotopy class of path from 0 to z in $\mathbb{C} \setminus \{a_1, \cdots, a_n\}$
  - ⋆ Integration by parts in t and "Fay identities" that generalize

$$\frac{1}{(t-a_1)(t-a_2)} = \frac{1}{a_1-a_2}\left(\frac{1}{t-a_1} - \frac{1}{t-a_2}\right)$$

    allow one to reduce all integrations to combinations of G
  - ⋆ Applications to Feynman integrals and tree-level string amplitudes

Polylogarithms on the torus $\Sigma = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$

- **Elliptic functions on the torus play the role of rational functions on $\mathbb{C}$**
  - ⋆ Every elliptic function is a rational function in $\wp(z)$ and $\wp'(z)$ for $z \in \Sigma$
  - ⋆ $\mathbb{C}(\wp(z), \wp'(z))$ closes under differentiation but not under integration
  - ⋆ e.g. $\wp(z) = -\zeta'(z)$ but the Weierstrass ζ-function is multiple-valued on Σ

- **Meromorphic multiple-valued kernels $g^{(r)}(z)$** (Levin, Racinet 2007; Calaque, Enriquez, Etinghof 2009)

$$\frac{\vartheta_1'(0)\vartheta_1(z+b)}{\vartheta_1(z)\vartheta_1(b)} = \frac{1}{b} + \sum_{r=1}^{\infty} b^{r-1}\, g^{(r)}(z)$$

- **Real-analytic single-valued integration kernels $f^{(r)}(z)$** (Brown, Levin 2011)

$$e^{2\pi i b (\mathrm{Im}\, z)/\tau_2}\, \frac{\vartheta_1'(0)\vartheta_1(z+b)}{\vartheta_1(z)\vartheta_1(b)} = \frac{1}{b} + \sum_{r=1}^{\infty} b^{r-1}\, f^{(r)}(z)$$

  - ⋆ in terms of iterated integrals of the Green function $f^{(1)}(z) = -\partial_z g(z|\tau)$

$$f^{(r+1)}(z) = -\int_\Sigma \frac{d^2t}{\tau_2}\, f^{(1)}(z-t) f^{(r)}(t)$$

  - ⋆ related to Zagier's single-valued elliptic polylogarithms
  - ⋆ applications to string amplitudes e.g. (Mafra, Schlotterer 2018)

Polylogarithms on Σ of arbitrary genus

- **Real-analytic single-valued modular tensors** $f^{I_1\cdots I_r}{}_J(x,y)$
  - ⋆ constructed in terms of a flat connection $\mathcal{J}_{\mathsf{DHS}}(x,y)$ (ED, Hidding, Schlotterer 2023)

$$d_x \mathcal{J}_{\mathsf{DHS}}(x,y) - \mathcal{J}_{\mathsf{DHS}}(x,y) \wedge \mathcal{J}_{\mathsf{DHS}}(x,y) = 0$$

  - ⋆ valued in a Lie algebra freely generated by $\{a^1, \cdots, a^h, b_1, \cdots, b_h\}$
  - ⋆ path-ordered exponential depends only on the homotopy class of γ

$$\mathsf{P}\exp \int_\gamma \mathcal{J}_{\mathsf{DHS}}(x,y)$$

  - ⋆ Expansion in Lie algebra generators gives integration kernels

$$\mathcal{J}_{\mathsf{DHS}}(x,y) = -\pi \overline{\omega}^I(x) b_I + \sum_{r=1}^\infty f^{I_1\cdots I_r}{}_J(x,y)[b_{I_1}, [\cdots b_{I_r}, a^J]\cdots]$$

  - ⋆ $f^{I_1\cdots I_r}{}_J(x,y)$ are modular tensors under $Sp(2h,\mathbb{Z})$
  - ⋆ satisfy Fay-like identities (ED, Schlotterer 2024; Baune, Broedel, Im, Lisitsyn, Moeckli 2024)
  - ⋆ close under integration !

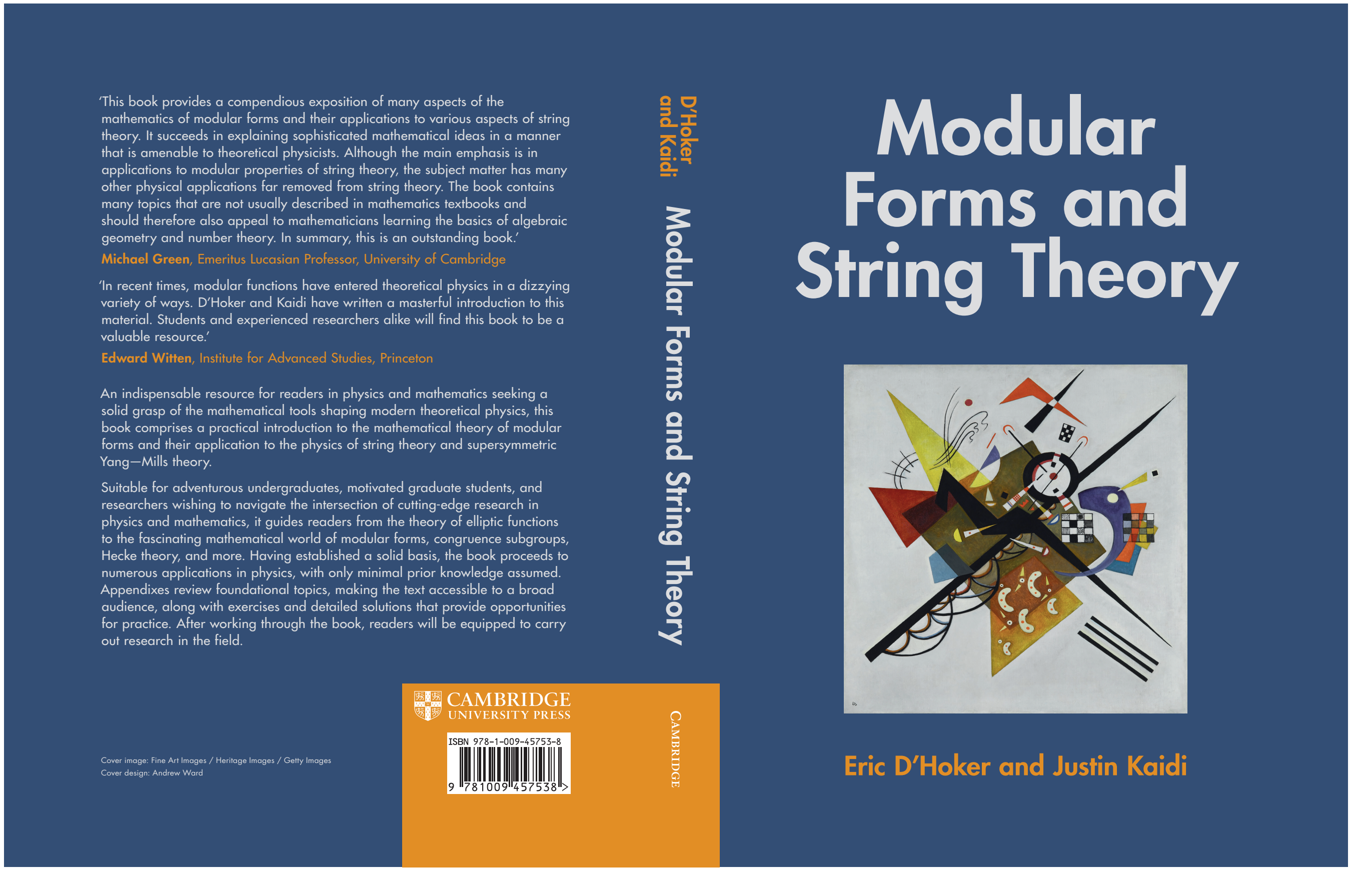
- **Meromorphic multiple-valued integration kernels** $g^{I_1\cdots I_r}{}_J(x,y)$
  - ⋆ defined formally through their monodromies (Enriquez 2011)
  - ⋆ related to $f^{I_1\cdots I_r}{}_J(x,y)$ but not modular tensors (ED, Enriquez, Schlotterer, Zerbini 2025)
  - ⋆ represented as iterated integrals on homology cycles
    involving Abelian differentials of first and third kind (ED, Schlotterer 2025)

Outlook

- **Modular graph forms for genus one**
  - ⋆ arithmetic significance ? (e.g. Hecke operators)
- **Modular graph functions and tensors on higher genus Riemann surfaces**
  - ⋆ Differential identities ?
  - ⋆ Theta-lifts ? cfr (Pioline 2015) for KZ invariant
- **Polylogarithms on surfaces of arbitrary genus**
  - ⋆ decomposing products of Szegö kernels into spin structure dependent modular tensors and spin structure independent polylogarithms

    (ED, Hidding, Schlotterer 2023) & this visit
- **Applications to field theory and string theory amplitudes**
  - ⋆ organization of Feynman integrals e.g. (Marzucca, McLoed, Page, Pögel, Weinzierl 2023)
  - ⋆ computing all N-graviton amplitudes for $h=2$
  - ⋆ bootstrapping string amplitudes for $h \geq 3$ e.g. (Geyer, Monteiro, Stark-Muchao 2021)

THANK YOU



‘This book provides a compendious exposition of many aspects of the mathematics of modular forms and their applications to various aspects of string theory. It succeeds in explaining sophisticated mathematical ideas in a manner that is amenable to theoretical physicists. Although the main emphasis is in applications to modular properties of string theory, the subject matter has many other physical applications far removed from string theory. The book contains many topics that are not usually described in mathematics textbooks and should therefore also appeal to mathematicians learning the basics of algebraic geometry and number theory. In summary, this is an outstanding book.’

Michael Green, Emeritus Lucasian Professor, University of Cambridge

‘In recent times, modular functions have entered theoretical physics in a dizzying variety of ways. D’Hoker and Kaidi have written a masterful introduction to this material. Students and experienced researchers alike will find this book to be a valuable resource.’

Edward Witten, Institute for Advanced Studies, Princeton

An indispensable resource for readers in physics and mathematics seeking a solid grasp of the mathematical tools shaping modern theoretical physics, this book comprises a practical introduction to the mathematical theory of modular forms and their application to the physics of string theory and supersymmetric Yang—Mills theory.

Suitable for adventurous undergraduates, motivated graduate students, and researchers wishing to navigate the intersection of cutting-edge research in physics and mathematics, it guides readers from the theory of elliptic functions to the fascinating mathematical world of modular forms, congruence subgroups, Hecke theory, and more. Having established a solid basis, the book proceeds to numerous applications in physics, with only minimal prior knowledge assumed. Appendixes review foundational topics, making the text accessible to a broad audience, along with exercises and detailed solutions that provide opportunities for practice. After working through the book, readers will be equipped to carry out research in the field.

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