

Flat connections, Fay identities and spin structure sums

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Congratulations to the Ma \int cAmp team

Ruth, Francis, Axel and Oliver

Structure of Amplitudes

- Schematic structure

$$\text{Amplitude} = \sum_{\# \text{ loops}} \int_{\text{loop momenta}} \sum_{\Gamma} \int_{\mathcal{M}} \mathcal{I}$$

- Quantum field theory

Γ = different Feynman diagrams

\mathcal{M} = Feynman parameters

\mathcal{I} = sums of products of propagators and vertices

- String theory

Γ = spin structures (RNS: defines the string theory & implements supersymmetry)

\mathcal{M} = moduli (or supermoduli) of compact Riemann surfaces

\mathcal{I} = CFT \times $\overline{\text{CFT}}$ at fixed loop momenta by chiral splitting

This talk: mathematical structure of the integrand

- Identify a space of functions that is closed under

- ★ adding, multiplying, differentiating and taking primitives

- ⇒ Polylogarithms via flat connections on arbitrary Riemann surfaces

- ⇒ Flatness of connection is equivalent to Fay identities

- ⇒ Disentangle dependence on spin structures from vertex points

- cfr. Low energy expansion

- ★ organized via Modular Graph Functions & Forms

[ED, Green, Vanhove 2015; Zerbini 2015; ED, Green, Gürdogan, Vanhove 2015; Basu 2015-16; Berg, Buchberger, Schlotterer 2016;

ED, Kaidi 2016; Zerbini 2017; Kleinschmidt, Verschinin 2017; Brown 2017; Gerken, Kaidi 2018; Broedel, Schlotterer, Zerbini 2018; ...]

Polylogarithms from flat connections

- Flat connection $\mathcal{J}(x; c)$ in principal \mathfrak{g} -bundle over Σ
 - ★ Σ = compact Riemann surface of arbitrary genus h ;
 - ★ \mathfrak{g} = Lie algebra freely generated by $c = \{c_1, \dots, c_n\}$;

$$d_x \mathcal{J}(x; c) - \mathcal{J}(x; c) \wedge \mathcal{J}(x; c) = 0$$

- ★ Flatness guarantees integrability of

$$d_x \Gamma(x, y; c) = \mathcal{J}(x; c) \Gamma(x, y; c)$$

- ★ whose solution depends only on the homotopy class of the path from y to x

- Polylogarithms $\Gamma(\mathfrak{w}; x, y)$ obtained by expanding in words \mathfrak{w}

$$\Gamma(x, y; c) = \sum_{\mathfrak{w} \in \mathcal{W}(c)} \mathfrak{w} \Gamma(\mathfrak{w}; x, y)$$

- ★ where $\mathcal{W}(c)$ is the set of words formed from the letters c_1, \dots, c_n .

Flat connections at low genus

- Meromorphic connections in a single variable on S^2 [Knizhnik, Zamolodchikov 1984]

$$\mathcal{J}(x; c) = \sum_{i=1}^n \frac{c_i dx}{x - p_i}$$

- ★ Closure using integration by parts in x and the “Fay identity”

$$\frac{1}{(x - p_i)(x - p_j)} = \frac{1}{p_i - p_j} \left(\frac{1}{x - p_i} - \frac{1}{x - p_j} \right)$$

- Flat connections on the torus $z \in \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$

- ★ Lie algebra \mathfrak{g} freely generated by a, b (use $Bc = [b, c]$ for all $c \in \mathfrak{g}$)

- ★ meromorphic but not single-valued [Levin, Racinet 2007; Calaque, Enriquez, Etinghof 2009]

$$\mathcal{J}_{\text{LRCEE}}(z; a, b) = dz F(z; B) B a \qquad F(x; \alpha) = \frac{\vartheta_1'(0) \vartheta_1(z + \alpha)}{\vartheta_1(z) \vartheta_1(\alpha)}$$

- ★ single-valued but non-meromorphic [Brown, Levin 2011]

$$\mathcal{J}_{\text{BL}}(z; a, b) = 2\pi i \frac{dz - d\bar{z}}{\tau - \bar{\tau}} b + dz e^{2\pi i(z - \bar{z})/(\tau - \bar{\tau})B} F(z; B) B a$$

- ★ “Fay identity” = Fay trisecant identity [Fay 1973] for the torus

$$F(s - p; \alpha) F(s - q; \beta) = F(p - q; \beta) F(s - p; \alpha + \beta) + F(q - p; \alpha) F(s - q; \alpha + \beta)$$

Flat connections in n variables for arbitrary genus h

- **Compact Riemann surface Σ of arbitrary genus h**

- ★ canonical homology basis $\mathfrak{A}^I, \mathfrak{B}_J$ cycles for $I, J = 1, \dots, h$
- ★ holomorphic $(1, 0)$ forms ω_I normalized $\oint_{\mathfrak{A}^I} \omega_J = \delta_J^I, \oint_{\mathfrak{B}^I} \omega_J = \Omega_{IJ}$

- **The Lie algebra \mathfrak{g} is no longer freely generated for $n \geq 2$**

- ★ $\mathfrak{g} = \hat{\mathfrak{t}}_{h,n}$ is (the degree completion of the algebra) generated by

$$a_i^I \quad b_{iI} \quad t_{ij} \quad I = 1, \dots, h, \quad i, j = 1, \dots, n$$

- ★ and is presented by the defining relations (for $h \geq 2$)

$$[a_i^I, a_j^J] = [b_{iI}, b_{jJ}] = 0 \quad [b_{iI}, a_j^J] = \delta_I^J t_{ij} \quad \sum_I [b_{iI}, a_i^I] + \sum_{j \neq i} t_{ij} = 0$$

- **Connections in n variables $\mathbf{x} = (x_1, \dots, x_n) \in \Sigma^n$ valued in $\hat{\mathfrak{t}}_{h,n}$**

- ★ meromorphic on Σ^n but multiple-valued [Enriquez 2011]
- ★ single-valued on Σ^n but non-meromorphic [ED, Schlotterer 2025]

(the corresponding DHS connection for $n = 1$ [ED, Hidding, Schlotterer 2023])

Require flatness on configuration space $\text{Cf}_n(\Sigma) = \Sigma^n \setminus \text{diagonals}$

The Enriquez connection

- **The Enriquez connection in n variables** $\mathbf{x} = (x_1, \dots, x_n) \in \Sigma^n$

★ is a $(1, 0)$ form on Σ^n valued in $\hat{\mathfrak{t}}_{h,n}$,

$$\mathcal{K}_E(\mathbf{x}; a, b, t) = \sum_{i=1}^n K_i(\mathbf{x}; a, b, t) dx_i$$

★ whose individual components are linear in a_i^I and t_{ij} ,

$$K_i(\mathbf{x}; a, b, t) = \sum_{r=0}^{\infty} B_{iI_1} \cdots B_{iI_r} \left(\varpi^{I_1 \cdots I_r} \varpi^J(x_i) a_i^J + \sum_{j \neq i} \chi^{I_1 \cdots I_r}(x_i, x_j) t_{ij} \right)$$

★ has vanishing \mathfrak{A} -monodromy while \mathfrak{B} -monodromy is given by,

$$K_i(\mathfrak{B}_{jI} \cdot \mathbf{x}; a, b, t) = e^{-2\pi i b_{jI}} K_i(\mathbf{x}; a, b, t) e^{2\pi i b_{jI}}$$

★ meromorphic in \mathbf{x} with simple pole in x_i at x_j with residue t_{ij}

- **Theorem 1: The connection \mathcal{K}_E is flat on $\text{Cf}_n(\Sigma)$ [Enriquez 2011]**

★ alternative proof by direct methods [ED, Schlotterer 2025]

The DS connection

- **The DS connection in n variables** $\mathbf{x} = (x_1, \dots, x_n) \in \Sigma^n$
 - ★ is a single-valued $(1, 0) \oplus (0, 1)$ form on Σ^n valued in $\hat{\mathfrak{t}}_{h,n}$,

$$\mathcal{J}_{\text{DS}}(\mathbf{x}; a, b, t) = \sum_{i=1}^n J_i(\mathbf{x}; a, b, t) dx_i - \pi \sum_{i=1}^n \bar{\omega}^I(x_i) b_{iI} d\bar{x}_i$$

- ★ The individual components of J_i are linear in a_i^J and t_{ij} ,

$$J_i(\mathbf{x}; a, b, t) = \sum_{r=0}^{\infty} B_{iI_1} \cdots B_{iI_r} \left(\partial_i \Phi^{I_1 \cdots I_r}{}_J(x_i) a_i^J + \sum_{j \neq i} \partial_i \mathcal{G}^{I_1 \cdots I_r}(x_i, x_j) t_{ij} \right)$$

- ★ $\partial \Phi^\emptyset{}_J(x) = \omega_J(x)$ and $\mathcal{G}^\emptyset(x, y) = \mathcal{G}(x, y)$ is the Arakelov Green function,

$$\Phi^{I_1 \cdots I_r}{}_J(x) = \int_{\Sigma} d^2 z \mathcal{G}(x, z) \bar{\omega}^{I_1}(z) \partial_z \Phi^{I_2 \cdots I_r}{}_J(z)$$

$$\mathcal{G}^{I_1 \cdots I_r}(x, y) = \int_{\Sigma} d^2 z \mathcal{G}(x, z) \bar{\omega}^{I_1}(z) \partial_z \mathcal{G}^{I_2 \cdots I_r}(z, y)$$

- **Theorem 2: The connection \mathcal{J}_{DS} is flat on $\text{Cf}_n(\Sigma)$ and invariant under the modular group $Sp(2h, \mathbb{Z})$ [ED, Schlotterer 2025]**

Relating the Enriquez and DS connections

- **Theorem 3: The connections \mathcal{K}_E and \mathcal{J}_{DS} are related**

- ★ by the composition of a gauge transformation $\mathcal{U}(\mathbf{x}, \mathbf{y})$

$$d - \mathcal{K}_E(\mathbf{x}; a, b, t) = \mathcal{U}(\mathbf{x}, \mathbf{y})^{-1} (d - \mathcal{J}_{DS}(\mathbf{x}; \hat{a}, \hat{b}, \hat{t})) \mathcal{U}(\mathbf{x}, \mathbf{y})$$

- ★ and a Lie algebra automorphism $\hat{\mathfrak{t}}_{h,n} \rightarrow \hat{\mathfrak{t}}_{h,n} : (\hat{a}, \hat{b}, \hat{t}) \rightarrow (a, b, t)$

- ★ the gauge transformation is constructed out of \mathcal{J}_{DS}

$$\mathcal{U}(\mathbf{x}, \mathbf{y}) = \text{P exp} \int_{\mathbf{y}}^{\mathbf{x}} \mathcal{J}_{DS}$$

Single-variable case [ED, Enriquez, Schlotterer, Zerbini 2025]

Multi-variable case [ED, Schlotterer 2025]

- **The relation allows one to convert**

- ★ between connections

(meromorphic, multiple-valued) \Leftrightarrow (single-valued, modular invariant)

- ★ and associated polylogarithms

Fay identities \iff integrability

- Integration kernels for Enriquez and DS connections satisfy identities

★ Interchange Lemma (IL) on two points $x, y \in \Sigma$ e.g. for DS [ED, Schlotterer 2020]

$$\omega_J(x)\partial_y\mathcal{G}(y,x) + \omega_J(y)\partial_x\mathcal{G}(x,y) - \omega_I(x)\partial_y\Phi^I_J(y) - \omega_I(y)\partial_x\Phi^I_J(x) = 0$$

★ Fay identities on three points $x, y, z \in \Sigma$ e.g. for DS

$$\begin{aligned} \partial_x\mathcal{G}(x,z)\partial_y\mathcal{G}(y,z) &= \partial_x\mathcal{G}(x,y)\partial_y\mathcal{G}(y,z) + \partial_y\mathcal{G}(y,x)\partial_x\mathcal{G}(x,z) \\ &\quad - \omega_I(x)\partial_y\mathcal{G}^I(y,z) - \omega_I(y)\partial_x\mathcal{G}^I(x,z) + \partial_x\partial_y\mathcal{G}_2(x,y) \end{aligned}$$

Both infinite families were constructed for Enriquez and DS in [ED, Schlotterer 2024]

Shown to be exhaustive for Enriquez kernels [Baune, Broedel, Im, Lisitsyn, Moeckli 2024]

- **Theorem 4: (Interchange Lemma & Fay) \iff flatness of \mathcal{K}_E and \mathcal{J}_{DS}**

★ The IL and Fay identities arise from the most difficult components

$$0 = [K_i(\mathbf{x}; a, b, t), K_j(\mathbf{x}; a, b, t)] = \text{linear combo of all IL \& Fay for Enriquez}$$

$$0 = [J_i(\mathbf{x}; a, b, t), J_j(\mathbf{x}; a, b, t)] = \text{linear combo of all IL \& Fay for DS}$$

[ED, Schlotterer 2025]

Decomposition of fermion correlators

- The RNS formulation of string amplitudes appeals to worldsheet fermions
 - ★ For even spin structure δ , generic moduli, the fermion propagator satisfies

$$\partial_{\bar{x}} S_{\delta}(x, y) = \pi \delta(x, y)$$

- ★ Fermion correlators include cyclic products, with $\mathbf{x} = (x_1, \dots, x_n)$

$$C_{\delta}(\mathbf{x}) = S_{\delta}(x_1, x_2) S_{\delta}(x_2, x_3) \cdots S_{\delta}(x_{n-1}, x_n) S_{\delta}(x_n, x_1)$$

- **Theorem 5: Cyclic products satisfy the decomposition** [ED, Schlotterer 2025]

$$C_{\delta}(\mathbf{x}) = \sum_{r=0}^n \mathcal{V}_{I_1 \dots I_r}(\mathbf{x}) C_{\delta}^{I_1 \dots I_r}$$

- ★ $\mathcal{V}_{I_1, \dots, I_r}$ are independent of δ , single-valued on Σ^n , meromorphic in \mathbf{x} and entirely expressible in terms of DHS kernels
- ★ $C_{\delta}^{I_1 \dots I_r}$ are independent of \mathbf{x} and contain all δ -dependence
- ★ both are tensors under the congruence subgroup $\Gamma_h(2) \subset Sp(2h, \mathbb{Z})$

There is a corresponding result for Enriquez kernels, but the elements are no longer modular tensors

Outlook

- **Summary**

- ★ **Developed complementary flat connections on arbitrary Riemann surfaces**
- ★ **Related those connections to one another**
- ★ **Exhibited their intimate relation with the Fay identities**
- ★ **Decomposed products of Szegő kernels onto modular tensors**

- **Ongoing and immediate future projects**

- ★ **Individual results now need to be implemented for string amplitudes**
 - Summation over spin structures for genus 2 known [ED, Hidding, Schlotterer 2023]
 - Moduli variations \implies differential eqs in moduli [ED, Enriquez, Schlotterer, Zerbini 202x]
 - Structure of modular tensors $C_\delta^{I_1 \dots I_r}$ at higher genus

- **Let's talk**