Cascade flow from $\mathcal{N} = 2$ to adjoint QCD

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Sertifikat
Hermed bekrefter vi at
Leo Dhoker
i dag passerte polarsirkelen.
Vi håper at dette besøk i "midnattsolens land" blir et lyst minne for Dem
for hele livet.
GLOBETROTTER
Polarsirkelen, den 5. Juli 1968
GLOBE-TROTTER
Reiselederen

Motivation

- **Seiberg-Witten solution for $\mathcal{N}=2$ super Yang-Mills** [Seiberg, Witten 1994]
  - ⋆ on the Coulomb branch via Abelian gauge theory
  - ⋆ provides the exact low energy effective action and BPS spectrum
  - ⋆ dyons become massless at singular points on Coulomb branch
- **Softly breaking $\mathcal{N}=2$ supersymmetry**
  - ⋆ **exploit the enhanced control provided by the SW solution**
- **Earlier investigations into softly breaking $\mathcal{N}=2$**
  - ⋆ e.g. $\rightarrow \mathcal{N}=1$ exhibits confinement [Seiberg, Witten 1994]
    via magnetic monopole condensation
  - ⋆ e.g. other proposals
    dilaton spurion [Alvarez-Gaumé, Distler, Kounnas, Marino, 1996]
    explicit breaking [Luty, Ratazzi 1999; Edelstein, Fuertes, Mas, Guilarte, 2000]
  - ⋆ embedding adjoint QCD in $\mathcal{N}=2$ [Cordova, Dumitrescu 2018] (more soon)

This talk

- **Consider $\mathcal{N} = 2$ super-Yang-Mills with gauge group $SU(N)$**
  - ⋆ no $SU(N)$ hypermultiplets
  - ⋆ gauge multiplet $(\phi, \lambda^1, \lambda^2, v_\mu)$ in adjoint rep of $SU(N)$
  - ⋆ Coulomb branch = VEVs for ϕ with zero potential and unbroken $\mathcal{N} = 2$

- **Add mass term $M^2\, tr(\phi^\dagger \phi)$ for gauge scalars ϕ to potential**
  - ⋆ softly breaks all supersymmetries
  - ⋆ but preserves all other symmetries and 't Hooft anomalies
  - ⋆ ϕ decouples as $M \to \infty$ to adjoint QCD $(\lambda^1, \lambda^2, v_\mu)$ with two flavors

- **What is the phase structure across the flow $0 < M < \infty$?**
  - ⋆ We will propose and (approximately) solve semi-classically

magnetic dual Abelian Higgs model

⇒ Cascade of phase transitions through partial Coulomb/Higgs phases

$\mathcal{N}=2$ super Yang-Mills with gauge group $SU(N)$

- $\mathcal{N}=2$ **gauge multiplet** $(\phi, \lambda^1, \lambda^2, v_\mu)$

$$\mathcal{L}_{\mathsf{SU(N)}} = -\frac{1}{2g^2}\,\mathrm{tr}(v_{\mu\nu}v^{\mu\nu}) - \frac{2}{g^2}\,\mathrm{tr}(D^\mu\phi^\dagger D_\mu\phi) - \frac{1}{g^2}\mathrm{tr}[\phi^\dagger,\phi]^2 + \text{ fermions}$$

⋆ $SU(2)_R$ symmetry of the $\mathcal{N}=2$ super Poincaré algebra
⋆ $U(1)_R \to \mathbb{Z}_{4N}$ by anomaly and instanton induced 't Hooft interaction

- **Coulomb branch vacua** $[\phi^\dagger,\phi]=0$ **have unbroken** $\mathcal{N}=2$

⋆ at generic point on the Coulomb branch $SU(N) \longrightarrow U(1)^{N-1}$
⋆ low energy : $\mathcal{N}=2$ gauge $U(1)_k$ multiplets for $k=1,\cdots,N-1$
decomposed into $\mathcal{N}=1$ chiral A_k and gauge V_k multiplets

$$\mathcal{L}_{\mathsf{SW}} = \frac{\mathrm{Im}}{4\pi}\sum_{k=1}^{N-1}\int d^4\theta\, A_{Dk}\bar{A}_k + \frac{\mathrm{Im}}{8\pi}\sum_{k,\ell=1}^{n-1}\int d^2\theta\,\tau_{k\ell}\,W_kW_\ell$$

⋆ fully specified by a locally holomorphic pre-potential $\mathcal{F}(A_1,\cdots A_{N-1})$

$$A_{Dk} = \frac{\partial\mathcal{F}}{\partial A_k} \qquad \tau_{k\ell} = \frac{\partial^2\mathcal{F}}{\partial A_k\,\partial A_\ell} \qquad W_k^\alpha = -\tfrac{1}{4}\bar{D}\bar{D}D^\alpha V_k$$

⋆ subject to $\mathrm{Im}\,\tau > 0$ for positive kinetic term

The Seiberg-Witten solution

- **The Seiberg-Witten solution determines the pre-potential**
  - ⋆ $\mathcal{F}$ only depends on the fields A_k and not on their derivatives
  - ⋆ It suffices to evaluate $\mathcal{F}$ on the vevs $\langle A_k \rangle = a_k$ and $\langle A_{Dk} \rangle = a_{Dk}$

- **Family of SW curves** for $SU(N)$ [Klemm, Lerche, Yankielowicz, Theisen; Argyres, Faraggi 1994]
  - ⋆ parametrized by gauge-invariant moduli $u_n = \mathrm{tr}(\phi^n)$

$$y^2 = C(x)^2 - \Lambda^{2N} \qquad C(x) = x^N - \sum_{n=2}^{N} u_n x^{N-n}$$

  - ⋆ Hyper-elliptic of genus $N-1$: Canonical basis $\mathfrak{A}_k, \mathfrak{B}_k$ for $k = 1, \cdots, N-1$ namely $\#(\mathfrak{A}_k, \mathfrak{A}_\ell) = \#(\mathfrak{B}_k, \mathfrak{B}_\ell) = 0$ and $\#(\mathfrak{A}_k, \mathfrak{B}_\ell) = \delta_{k\ell}$

- **Solution given by periods of SW differential** λ

$$a_k = \oint_{\mathfrak{A}_k} \lambda \qquad a_{Dk} = \frac{\partial \mathcal{F}}{\partial a_k} = \oint_{\mathfrak{B}_k} \lambda \qquad \lambda = \frac{x C' dx}{2\pi i \, y}$$

  - ⋆ Riemann bilinear relations guarantee $\mathrm{Im}\, \tau > 0$
  - ⋆ Mass of BPS dyon with electric charges q_k and magnetic charges m_k

$$M_{\mathsf{BPS}}(\mathbf{q}; \mathbf{m}) = \sqrt{2} |Z| \qquad Z = \sum_{k=1}^{N-1} \left(q_k a_k + m_k a_{Dk} \right) \qquad \mathbf{q}, \mathbf{m} \in \mathbb{Z}^{N-1}$$

Special points on the Coulomb branch

- **Special points: enhanced symmetry or vanishing masses or both**

⋆ a single $\mathbb{Z}_{2N}$ symmetric point
  – with curve $y^2 = x^{2N} - \Lambda^{2N}$
  – no massless BPS states

$\Longrightarrow$ a convergent Taylor series exists at the $\mathbb{Z}_{2N}$ point [ED, Dumitrescu, Nardoni 2022]

⋆ two different $\mathbb{Z}_N$ symmetric "Argyres-Douglas points"
  – with curves $y^2 = x^N(x^N \pm 2\Lambda^N)$
  – $\frac{1}{2}N(N-1)$ massless dyons that are mutually non-local for $N \geq 3$

⋆ N different $\mathbb{Z}_2$ symmetric "multi-monopole points"
  – mapped into one another by $\mathbb{Z}_N$
  – with curves $y^2 = \Lambda^{2N} \sinh^2 \big(N \operatorname{arccos}(x/2N)\big)$ (Chebyshev polynomials)
  – $N-1$ massless magnetic monopoles that are mutually local

Near a multi-monopole point

- **Approaching a multi-monopole point** [Douglas, Shenker 1995]
  - ⋆ magnetic periods $a_{Dk} \to 0$ so that $M_{\text{BPS}}(\mathbf{0},\mathbf{m}) \to 0$
  - ⋆ electric periods $a_k \not\to 0$ so that $M_{\text{BPS}}(\mathbf{q},\mathbf{0})$ remains finite

$$a_k = -\frac{2N\Lambda}{\pi} \sin\frac{k\pi}{N} - \frac{i}{2\pi} a_{Dk} \ln\frac{a_{Dk}}{\Lambda} + \mathcal{O}(a_D)$$

- **Massless states produce singularities in τ**
  - ⋆ expected running of the $U(1)$ gauge couplings
    - – one-loop exact RG β function

$$\tau_{k\ell} = -\frac{i\,\delta_{k\ell}}{2\pi} \ln\frac{a_{Dk}}{\Lambda} + \mathcal{O}(a_D^0)$$

  - ⋆ the Seiberg-Witten low energy effective Lagrangian breaks down
    - – because it integrated out the light/massless magnetic monopole states

$\Longrightarrow$ viable low energy effective theory obtained by keeping massless states

Effective Abelian Higgs model

- **Introduce $N-1$ magnetic monopole fields $\mathcal{H}_k$,** $\quad k=1,\cdots,N-1$
  - ⋆ hyper-multiplets of $\mathcal{N}=2$ for gauge group $U(1)^{N-1}$
  - ⋆ with charge vector $(\mathbf{0};\mathbf{m}_k)$ where $(\mathbf{m}_k)^\ell=\delta_k^\ell$

$$\mathcal{H}_k=(h_{ik},\psi_{+k},\bar\psi_{-k}) \qquad\qquad \bar{\mathcal{H}}_k=(\bar h_k^i,\bar\psi_{+k},\psi_{-k})$$

  - ⋆ the index $i=1,2$ labels the doublet representation of $SU(2)_R$
  - ⋆ in terms of $\mathcal{N}=1$ superfields with auxiliary fields $F_k^\pm$

$$\mathcal{H}_k^+=(h_{1k},\psi_{+k},F_k^+) \qquad\qquad \mathcal{H}_k^-=(\bar h_k^2,\psi_{-k},F_k^-)$$

- **Effective Lagrangian including magnetic monopole hyper-multiplets**
  - ⋆ dictated by $\mathcal{N}=2$ supersymmetry

$$\begin{aligned}\mathcal{L}_{\textsf{SW}}^{\text{eff}} \;=\; & \sum_{k=1}^{N-1}\Big[\int d^4\theta\Big(\bar{\mathcal{H}}_k^+e^{-2V_k}\mathcal{H}_k^+ + \bar{\mathcal{H}}_k^-e^{+2V_k}\mathcal{H}_k^- + \frac{\text{Im}}{2\pi}\bar A_k A_{Dk}\Big)\\ & +2\text{Re}\int d^2\theta\, A_{Dk}\mathcal{H}_k^+\mathcal{H}_k^-\Big] + \sum_{k,\ell=1}^{N-1}\frac{\text{Im}}{4\pi}\int d^2\theta\,\tau_{Dk\ell}^{\text{eff}}W_kW_\ell\end{aligned}$$

  - ⋆ where $W_k^\alpha=-\frac{1}{4}\bar D\bar D D^\alpha V_k$ and V_k are the $\mathcal{N}=1$ gauge multiplets of $U(1)^{N-1}$

Effective periods and gauge couplings

- **Retaining magnetic monopole fields changes the periods**
  - ⋆ near a multi monopole point we still have $a_{Dk} \to 0$
  - ⋆ including the magnetic monopole fields below renormalization scale μ
    - – removes their contribution from the one-loop exact β-function

$$a_k^{\rm eff} = -\frac{2N\Lambda}{\pi} \sin \frac{k\pi}{N} - \frac{i}{2\pi} a_{Dk} \ln \frac{\mu}{\Lambda} + \mathcal{O}(a_D)$$

- **Gauge couplings $\tau_{k\ell}$ are now free of singularities as $a_{Dk} \to 0$**
  - ⋆ we shall need the precise normalizations [ED, Dumitrescu, Gerchkovitz, Nardoni 2020]

$$\tau_{k\ell}^{\rm eff} = \frac{i}{2\pi} \left(\delta_{k\ell} \ln \frac{\Lambda}{\mu} + \ln L_{k\ell} \right) + \mathcal{O}(a_D)$$

  - ⋆ where L is given in terms of $c_k = \cos(k\pi/N)$ and $s_k = \sin(k\pi/N)$

$$L_{kk} = 16 N s_k^3 \qquad\qquad L_{k\ell} = \frac{1 - c_{k+\ell}}{1 - c_{k-\ell}} \qquad k \neq \ell$$

  - ⋆ order $\mathcal{O}(a_D)$ corrections to τ are known as well [ED, Phong 1997]

Soft supersymmetry breaking operator

- **Flow from $\mathcal{N}=2$ to adjoint QCD is controlled by the operator $\mathcal{T}$**

$$\mathcal{T} = 2g^{-2}\,\mathrm{tr}(\phi^\dagger \phi)$$

  ⋆ $\mathcal{L}_{\mathsf{SU(N)}} \to \mathcal{L}_{\mathsf{SU(N)}} - M^2 \mathcal{T}$ breaks susy completely
  ⋆ but preserves all other symmetries and ‘t Hooft anomalies
  ⋆ $M \to \infty$ decouples ϕ leaving adjoint QCD with two Weyl fermions

- **IR behavior of $\mathcal{T}$ in $\mathcal{N}=2$ theory governed by SW theory**
  ⋆ $\mathcal{T}$ is the lowest component of the $\mathcal{N}=2$ stress tensor multiplet
    – as such its dimension is protected
  ⋆ flows towards the Kähler potential which, in SW theory, is given by

$$\mathcal{T} \to \frac{i}{4\pi} \sum_{k=1}^{N-1} \left(\bar{A}_{Dk} A_k - \bar{A}_k A_{Dk} \right)$$

  ⋆ In the effective theory near a multi-monopole point
    – with magnetic monopole hyper-multiplet fields integrated in
    – $\mathcal{T}$ evaluates on the corresponding periods

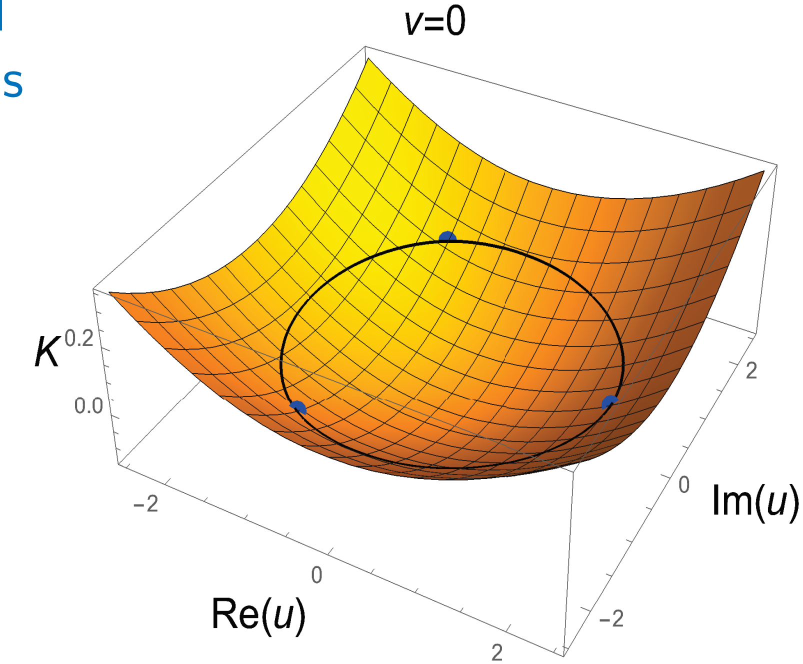
$$\mathcal{T} \to \frac{i}{4\pi} \sum_{k=1}^{N-1} \left(\bar{a}^{\mathrm{eff}}_{Dk}\, a^{\mathrm{eff}}_k - \bar{a}^{\mathrm{eff}}_k\, a^{\mathrm{eff}}_{Dk} \right) - \tfrac{1}{2} \sum_k \bar{h}_k h_k$$

The Kähler potential on the Coulomb branch

- $M^2\mathcal{T}$ **will drive vacuum towards minimum of** $\mathcal{T}$
  - ⋆ How does the Kähler potential $\mathcal{T}$ behave ?
  - ⋆ Clearly $\mathcal{T} = 0$ at multi-monopole points since $a_{Dk} = 0$ for all N

- **Hyper-elliptic periods are not standard special functions for** $N \geq 3$
  - ⋆ Multiple Taylor series expansion at $\mathbb{Z}_{2N}$ point, finite radius
  - ⋆ Numerical integration of Kähler potential
  - ⋆ Numerical integration of Picard-Fuchs eqs
  - ⋆ $N = 3$ curve $y^2 = (x^3 - ux - v)^2 - \Lambda^6$
    - $\Longrightarrow$ minimum of $\mathcal{T}$ at $\mathbb{Z}_6$ point
  - ⋆ $N > 3$ compelling evidence
    - $\Longrightarrow$ minimum of $\mathcal{T}$ at $\mathbb{Z}_{2N}$ point

[ED, Dumitrescu, Nardoni 2022]



The Abelian Higgs model with susy breaking

- **Quadratic approximation for $\mathcal{T}$**

$$\mathcal{T} \approx \sum_{k=1}^{N-1} \left(\frac{N\Lambda}{\pi^2} s_k \, \mathrm{Im}\,(a_{Dk}) - \tfrac{1}{2} \bar h_k h_k \right) + \sum_{k,\ell=1}^{N-1} t_{k\ell}\, \bar a_{Dk}\, a_{D\ell} \qquad\qquad t_{k\ell} = \frac{\mathrm{Im}\, \tau_{k\ell}^{\mathrm{eff}}}{2\pi}$$

  - ⋆ symmetric matrix t is positive definite with positive entries for $\mu/\Lambda \ll 1$
  - ⋆ leads to renormalizable effective theory

- **Assembling all the contributions to the Abelian Higgs model**
  - ⋆ incorporate susy breaking operator $M^2\mathcal{T}$ in the quadratic approximation
  - ⋆ retain only the effective potential of the model (to study vacua)

$$\mathcal{V} = \sum_{k=1}^{N-1} \left(\frac{M^2 N\Lambda}{\pi^2} s_k \mathrm{Im}\,(a_{Dk}) + \left[2|a_{Dk}|^2 - \tfrac{1}{2} M^2 \right] \bar h_k h_k \right)$$

$$+ \sum_{k,\ell=1}^{N-1} \left(M^2 t_{k\ell} a_{Dk} \bar a_{D\ell} + (t^{-1})_{k\ell} \left[(\bar h_k h_\ell)(\bar h_\ell h_k) - \tfrac{1}{2} (\bar h_k h_k)(\bar h_\ell h_\ell) \right] \right)$$

- **Proposal:** Abelian Higgs model is dual to flow from $\mathcal{N}=2$ to adjoint QCD
  - ⋆ for small M back-reaction of $\mathcal{T}$ on flow can be ignored
  - ⋆ for larger M we present evidence in favor of a coherent picture

Vacuum alignment

- **Minima of $\mathcal{V}$ occur at $\mathrm{Re}\,(a_{Dk})=0$**
  - ⋆ preserves a combination of charge conjugation C, time reversal T, and $\mathbb{Z}_{4N}$

$$a_{Dk}=-iMx_k \qquad\qquad k\in\mathbb{R}$$

- **The Higgs fields h_k align perfectly as $SU(2)_R$ doublets**
  - ⋆ minimize $\mathcal{V}$ for given values of $\bar h_k h_k$; orientation dependence:

$$\mathcal{V}\Big|_{\bar h_k h_k}=\sum_{k,\ell}(t^{-1})_{k\ell}\,\mathbf{v}_k\cdot\mathbf{v}_\ell \qquad\qquad \mathbf{v}_k=\bar h_k\boldsymbol{\sigma}h_k \qquad\qquad h_k=\begin{pmatrix}h_k^1\\ h_k^2\end{pmatrix}$$

  - ⋆ diagonal contribution $k=\ell$ is given by fixing $\bar h_k h_k=\mathbf{v}_k^2$
  - ⋆ ground state is ferromagnetic under the mild assumptions

$$(t^{-1})_{k\ell}<0 \qquad\qquad k\neq\ell$$

  - ⋆ found to hold for sufficiently small μ/Λ
  - ⋆ choose direction $h_k^1=Mh_k,\ h_k^2=0,\ h_k\in\mathbb{R}^+$

- **$SU(2)_R$ is spontaneously broken as soon as any $h_k\neq 0$**
  - ⋆ producing two Goldstone bosons $SU(2)_R\to U(1)_R$ i.e. CP^1-phase
  - ⋆ matches expected chiral symmetry breaking in adjoint QCD $\langle\lambda^i\lambda^j\rangle\neq 0$

Simplified Abelian Higgs model

- **Vacuum alignment greatly simplifies the analysis of the potential**

$$\begin{cases} a_{Dk} = -iM\, x_k \\ h_k^i = M\delta_1^i\, h_k \end{cases} \qquad \kappa = \frac{N\Lambda}{\pi^2 M}$$

⋆ in terms of these dimensionless real variables

$$\frac{\mathcal{V}}{M^4} = \sum_{k=1}^{N-1}\left(\tfrac{1}{2}(4x_k^2-1)h_k^2 - \kappa s_k x_k\right) + \sum_{k,\ell=1}^{N-1}\left(t_{k\ell}x_k x_\ell + \tfrac{1}{2}(t^{-1})_{k\ell}h_k^2 h_\ell^2\right)$$

- **Reduced field equations for space-time independent VEVs** x_k, h_k

$$2h_k^2 x_k + \sum_{\ell=1}^{N-1} t_{k\ell}\, x_\ell = \kappa s_k \qquad h_k\left(4x_k^2 - 1 + 2\sum_{\ell=1}^{N-1}(t^{-1})_{k\ell}h_\ell^2\right) = 0$$

- **Potential** $\mathcal{V}$ **and field eqs are invariant under charge conjugation** C

$$\begin{cases} s_k = s_{N-k} \\ t_{k,\ell} = t_{N-k,N-\ell} \end{cases} \qquad \begin{cases} x_k \to x_{N-k} \\ h_k \to h_{N-k} \end{cases}$$

Organization of semi-classical analysis

- **Steps in semi-classical analysis**
  - ⋆ Existence of solutions for given N, κ
  - ⋆ Local stability of solutions: positive Hessian on the solution
  - ⋆ Global stability of solutions: global minimum of $\mathcal{V}$ for given N, κ
- **Solutions to Higgs eqs organized by partitions $A|B$**

$$k \in A \qquad\qquad h_k = 0$$

$$k \in B \qquad\qquad 4x_k^2 - 1 + 2\sum_{\ell=1}^{N-1} (t^{-1})_{k\ell} h_\ell^2 = 0$$

  - ⋆ Clearly $A \cup B = \{1, \cdots, N-1\}$ and $A \cap B = \emptyset$
  - ⋆ Partitions are ordered: $A|B \neq B|A$
- **In a given partition $A|B$, solve for h_k, $k \in B$ in terms of x**
  - ⋆ define the submatrix u of dimension $\#(B)$ by

$$(t^{-1})_{k\ell} = (u^{-1})_{k\ell} \quad \text{for } k, \ell \in B$$

  - ⋆ The matrix u is positive, with positive definite entries, just as t is

$$2h_k^2 = \sum_{\ell \in B} u_{k\ell}(1 - 4x_\ell^2) \qquad\qquad k \in B$$

Reduced field equations and potential

- **Eliminate h_ℓ for $\ell \in B$; solve for x_k with $k \in A$ in terms of x_m, $m \in B$**
  - ⋆ First solve the equations for x_ℓ with $\ell \in B$

$$\ell \in B \qquad\qquad x_\ell + \sum_{m,n \in B} (u^{-1})_{\ell m} x_m u_{mn} (1 - 4x_n^2) = \kappa (t^{-1} s)_\ell$$

  - ⋆ Then solve for x_k with $k \in A$, using the matrix $\sigma = \left(t\big|_A \right)^{-1}$

$$k \in A \qquad\qquad x_k = \sum_{m \in A} \sigma_{km} \Big(\kappa s_m - \sum_{n \in B} t_{mn} x_n \Big)$$

- **The reduced potential may be expressed in terms of x_ℓ, $\ell \in B$**

$$\mathcal{V}_{\rm red} = \mathcal{V}_0 + \sum_{k,\ell \in B} u_{k\ell} \Big[(x_k - \kappa(t^{-1} s_k)(x_\ell - \kappa(t^{-1} s_\ell) - \tfrac{1}{8}(1 - 4x_k^2)(1 - 4x_k^2) \Big]$$

  - ⋆ where $\mathcal{V}_0$ is the potential for the partition $B = \emptyset$

$$\mathcal{V}_0 = -\kappa^2 \sum_{k,\ell=1}^{N-1} (t^{-1})_{k\ell} s_k s_\ell$$

  - ⋆ Partition $B = \emptyset$ and potential $\mathcal{V}_0$ correspond to the "Coulomb branch"

- **The Hessian may be similarly reduced to investigate local stability**

Large κ and small κ

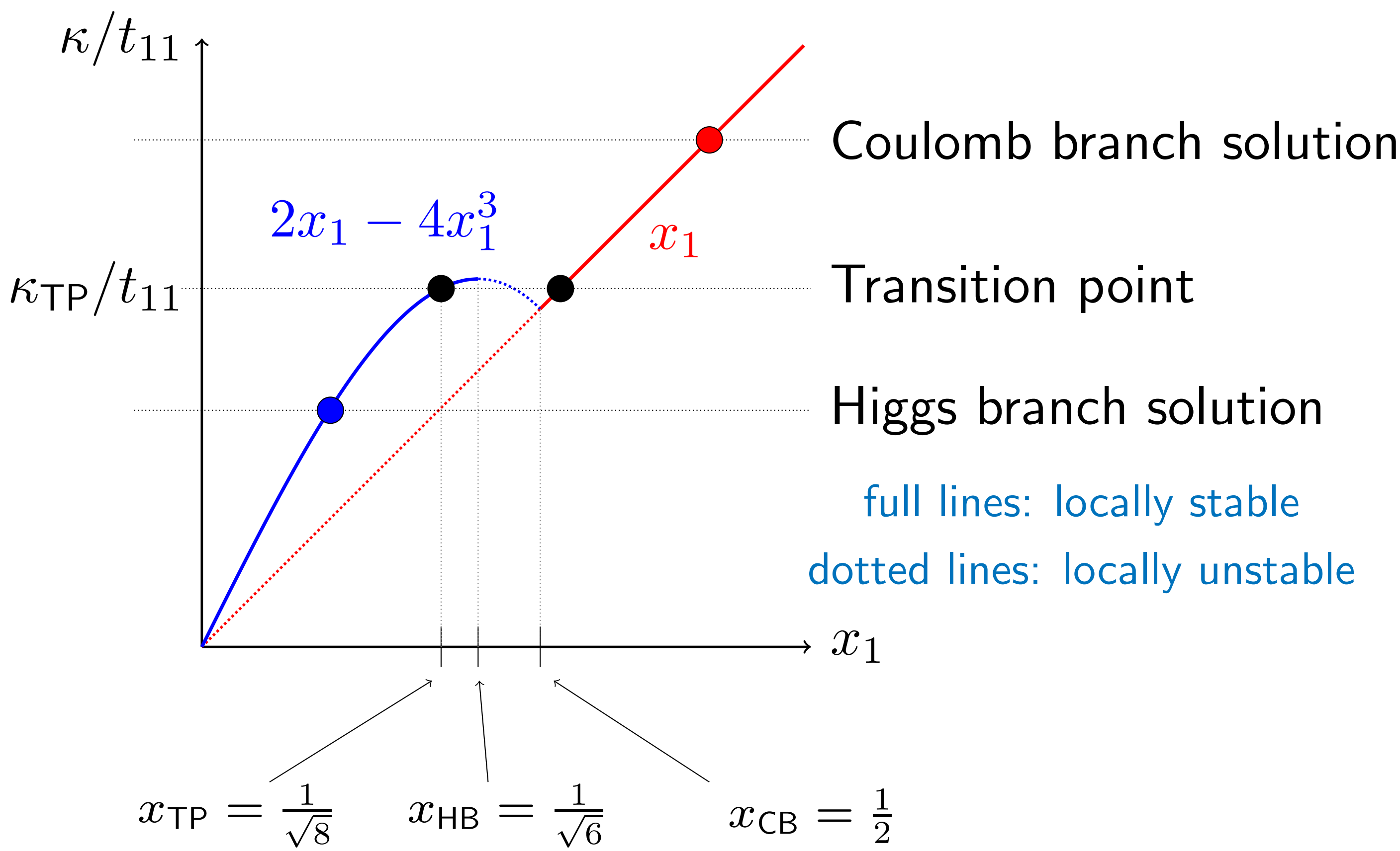
- **Coulomb branch for sufficiently large κ** (i.e. small M)
  - ⋆ Coulomb branch has $B=\emptyset$, namely all Higgs h_k vanishing
  - ⋆ locally stable when $2(t^{-1}s)_k < 1$ for all $k=\{1,\cdots,N-1\}$
  - ⋆ Charge conjugation invariant
  - √ Small perturbation of $\mathcal{N}=2$ theory stays in the Coulomb branch
- **Maximal Higgs branch for sufficiently small κ** (i.e. large M)
  - ⋆ Maximal Higgs branch has $A=\emptyset$, namely all Higgs h_k non-vanishing
  - ⋆ locally stable for sufficiently small κ
  - ⋆ Charge conjugation invariant
  - √ Magnetic monopole condensation implies confinement in adjoint QCD
  - √ Chiral $SU(2)_R$ symmetry of adjoint QCD is spontaneously broken to
- **Existence follows from $\ell \in B$-equations for arbitrary partition**

$$x_\ell + \sum_{m,n\in B} (u^{-1})_{\ell m} x_m u_{mn} (1-4x_n^2) = \kappa (t^{-1}s)_\ell$$

  - ⋆ $\kappa \to \infty$ requires $x_m \to \infty$ which would violate $h_\ell^2 > 0$
  - ⋆ $\kappa \to 0$ requires $x_m \to 0$ which forces $h_\ell^2 \neq 0$ for all $\ell \in \{1,\cdots,N-1\}$

Gauge group $SU(2)$

- **Only two possible branches** [Cordova, Dumitrescu 2018]
  - ⋆ Coulomb $h_1 = 0$ implies $x_1 = \kappa/t_{11}$; potential $\mathcal{V}_{\rm red} = \mathcal{V}_0$
  - ⋆ Higgs $h_1 \neq 0$ implies $2x_1 - 4x_1^3 = \kappa/t_{11}$
    - – potential $\mathcal{V}_{\rm red} = \mathcal{V}_0 - \frac{1}{8} t_{11}(1-8x_1^2)(1-4x_1^2)^2$

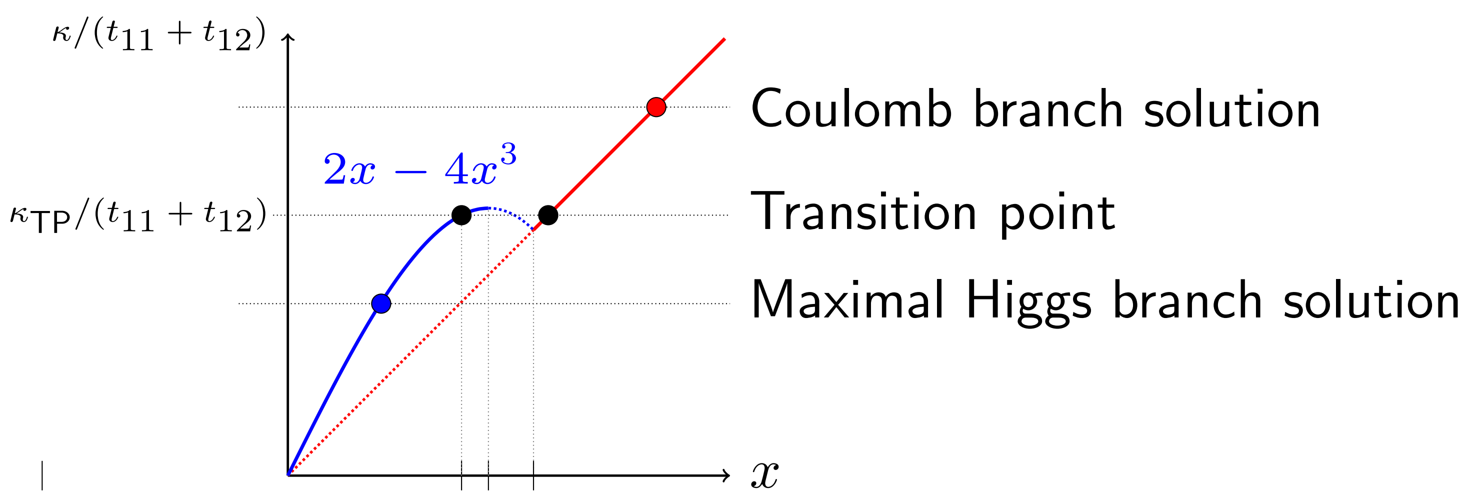


Gauge group $SU(3)$

- **Three possible C-inequivalent branches**
  - ⋆ Coulomb branch $h_k = 0$ implies $x_k = \kappa(t^{-1})_k$ for $k = 1, 2$
  - ⋆ Mixed branch $h_1 \neq 0, h_2 = 0$ (and its image under C: $h_2 \neq 0, h_1 = 0$)
  - ⋆ Maximal Higgs branch $h_k \neq 0$ for $k = 1, 2$

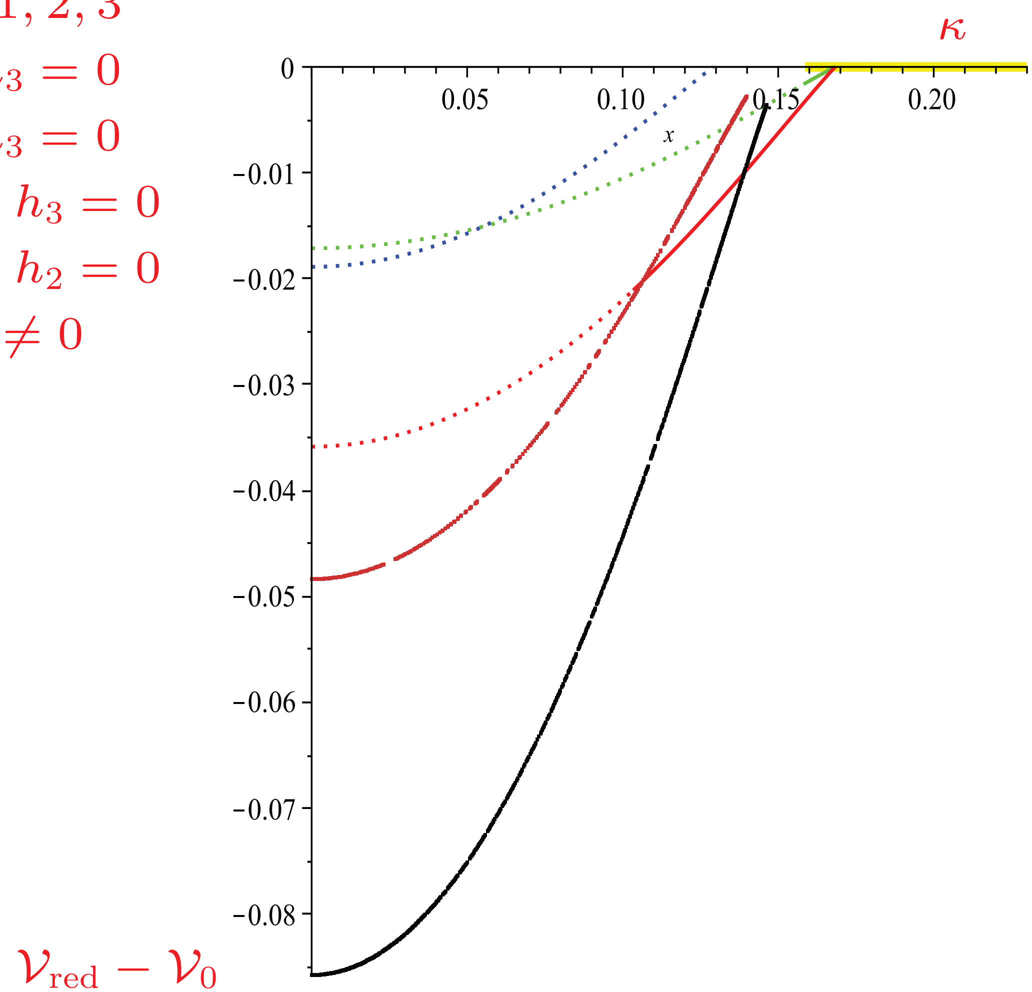
- **Immediate consequences of existence, local and global stability**
  - ⋆ Maximal Higgs subbranch $h_1, h_2 \neq 0, h_2 \neq h_1$ is locally unstable
  - ⋆ Maximal Higgs subbranch $h_1 = h_2 \neq 0$ is locally stable, when it exists
  - ⋆ Mixed branch $h_1 \neq 0, h_2 = 0$ has higher potential than $h_1 = h_2 \neq 0$

- **Phase diagram = $SU(2)$ with adapted parameters for $x = x_1 = x_2$**



Gauge group $SU(4)$

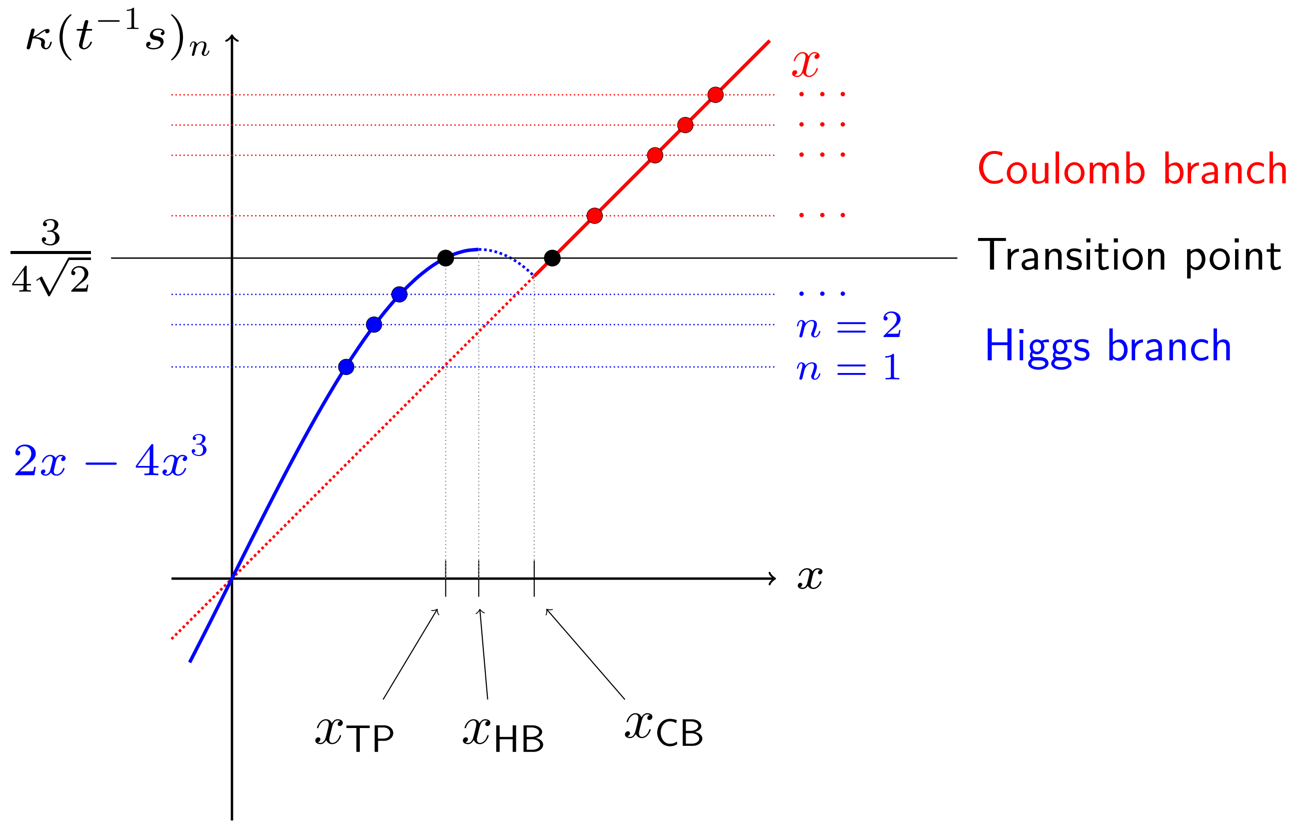
- **Possible C-inequivalent branches**
  - ⋆ Coulomb branch $h_k = 0$ for $k = 1, 2, 3$
  - ⋆ Single Higgs branch $h_1 \neq 0, h_2, h_3 = 0$
  - ⋆ Single Higgs branch $h_2 \neq 0, h_1, h_3 = 0$
  - ⋆ Double Higgs branch $h_1, h_2 \neq 0,\ h_3 = 0$
  - ⋆ Double Higgs branch $h_1, h_3 \neq 0,\ h_2 = 0$
  - ⋆ Maximal Higgs branch $h_1, h_2, h_3 \neq 0$



- ⋆ All solutions in double Higgs branch have $h_1 = h_3 \neq 0,\ h_2 = 0$

Cascade flow for arbitrary gauge group $SU(N)$

- **For larger values of N the numerics is underway, e.g. for $SU(7)$ etc**
  - ⋆ Evidence for phases intermediate between Coulomb and Maximal Higgs
  - ⋆ Make $\mu/\Lambda \ll 1$ so that diagonal of matrix t dominates
  - $\Longrightarrow$ Pairs $h_n = h_{N-n}$ decouple from one another: $N-1$ copies of $SU(2)$ case
    - – transition point governed by $\kappa(t^{-1}s)_n$ crossing $3/4\sqrt{2}$



Summary and Outlook

- **Proposal for Abelian Higgs model dual to flow from $\mathcal{N} = 2$ to adjoint QCD**
  - ⋆ Magnetic monopole fields have been "integrated in"
  - ⋆ Soft susy breaking operator flows to Kähler potential of SW theory
  - ⋆ Unique minimum of Kähler potential allows for quadratic approximation

- **Semi-classical Abelian Higgs model matches $SU(N)$ theory**
  - ⋆ For small susy breaking mass M, stay in Coulomb branch
  - ⋆ For increasing M, spontaneous breaking of $SU(2)_R$
    - – matches chiral symmetry breaking in adjoint QCD
  - ⋆ Vacuum alignment guarantees only two Goldstone bosons CP^1 phase
  - ⋆ For large M, Abelian Higgs model predicts monopole condensation
    - – matches confinement in adjoint QCD

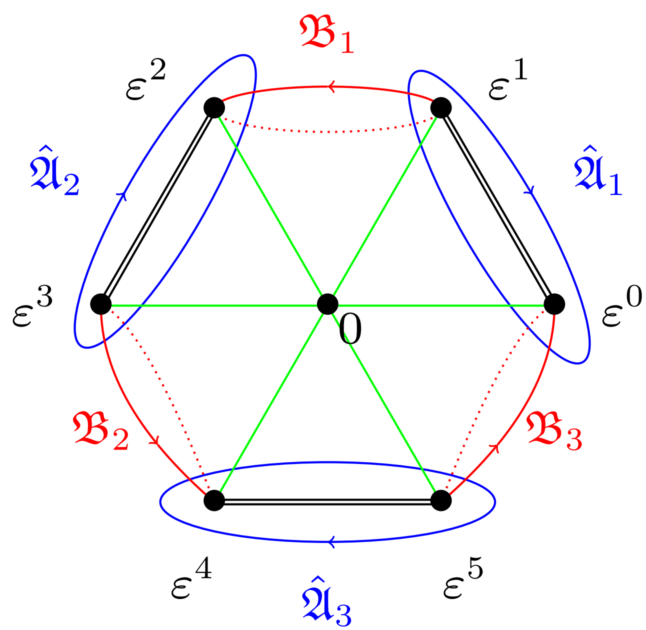
- **Semi-classical analysis of Abelian Higgs model predicts**
  - ⋆ Intermediate phases between Coulomb and Maximal Higgs phases
  - ⋆ Approximations predict a cascade of phases
    - – in which pairs of C-conjugate Higgs successively acquire VEVs

- **Solidify semi-classical calculations**
  - ⋆ consider additional tests on validity of dual picture

Thank you

Expansion around the $\mathbb{Z}_{2N}$ point

- **At the $\mathbb{Z}_{2N}$ point, the curve is $y^2 = x^{2N} - \Lambda^{2N}$**
  - ⋆ Branch points at $2N$-th roots of unity, generated by $\varepsilon = e^{2\pi i/2N}$



- **Recall the general curve**
  - ⋆ for arbitrary moduli u_n of the Coulomb branch

$$y^2 = C(x)^2 - \Lambda^{2N} \qquad\qquad C(x) = x^N - \sum_{n=2}^{N} u_n \, x^{N-2}$$

Expansion around the $\mathbb{Z}_{2N}$ point (cont'd)

⋆ The periods a_k, a_{Dk} are expressed as follows

$$a_I = \sum_{J=1}^{I} \{Q(\varepsilon^{2J-1}) - Q(\varepsilon^{2J-2})\} \qquad\qquad a_{DI} = Q(\varepsilon^{2I}) - Q(\varepsilon^{2I-1})$$

⋆ $Q(\xi)$ is a function on the $2N$-th roots of unity ξ (i.e. $\xi^{2N}=1$)

$$\pi i\, Q(\xi) = \int_0^\xi \lambda$$

⋆ The function $Q(\xi)$ has the following series expansion around $u_n = 0$

$$Q(\xi) = \sum_{\substack{\{\ell_n\}=0 \\ n=0,\dots,N-2}}^{\infty} V_{L,M}(\xi)\, \frac{u_0^{\ell_0} \cdots u_{N-2}^{\ell_{N-2}}}{\ell_0! \,\cdots\, \ell_{N-2}!} \qquad L = \sum_{j=0}^{N-2} j\ell_j \qquad M = \sum_{j=0}^{N-2} \ell_j$$

⋆ where the coefficients $V_{L,M}(\xi)$ are given by

$$V_{L,M}(\xi) = \frac{2^{M-(L+1)/N}}{2N}\, \xi^{NM+L+N+1} \frac{\Gamma(\frac{L+1}{N})}{\Gamma(\frac{2N+1+L-MN}{2N})^2}$$

⋆ Series is convergent in finite region around $\mathbb{Z}_{2N}$ point
$\Longrightarrow$ Reproduces Appell functions for $SU(3)$ [Klemm, Lerche, Theisen 1995]