Cascade flow from $\mathcal{N} = 2$ to adjoint QCD

Eric D'Hoker
Mani L. Bhaumik Institute for Theoretical Physics, UCLA

Professor Norisuke Sakai Memorial Symposium
Supersymmetry, Solitons, and Resurgence

5 August 2023

PHYSICAL REVIEW D VOLUME 54, NUMBER 12 15 DECEMBER 1996

Gauge symmetry breaking through soft masses in supersymmetric gauge theories

Eric D'Hoker*
Department of Physics and Astronomy, University of California at Los Angeles, Los Angeles, California 90024

Yukihiro Mimura† and Norisuke Sakai‡
Department of Physics, Tokyo Institute of Technology, Oh-okayama, Meguro, Tokyo 152, Japan

(Received 1 April 1996)

We analyze the effects of soft supersymmetry-breaking terms on $N=1$ supersymmetric QCD with N_f flavors and color gauge group $SU(N_c)$. The mass squared of some squarks may be negative, as long as vacuum stability is ensured by a simple mass inequality. For $N_f<N_c$, we include the dynamics of the nonperturbative superpotential and use the original (s)quark and gauge fields while, for $N_f>N_c+1$, we formulate the dynamics in terms of dual (s)quarks and a dual gauge group $SU(N_f-N_c)$. The presence of negative squark mass-squared terms leads to spontaneous breakdown of flavor and color symmetry. We determine this breaking pattern, derive the spectrum, and argue that the masses vary smoothly as one crosses from the Higgs phase into the confining phase. [S0556-2821(96)03824-6]

Motivation

- **Seiberg-Witten solution for $\mathcal{N} = 2$ super Yang-Mills**
  - ⋆ provides exact low energy effective action and BPS spectrum
  - ⋆ dyons become massless at singular points on Coulomb branch

- **Softly breaking $\mathcal{N} = 2$ supersymmetry**
  - ⋆ **exploit the enhanced control provided by the SW solution**

- **Earlier investigations into softly breaking $\mathcal{N} = 2$**
  - ⋆ confinement via magnetic monopole condensation [Seiberg, Witten 1994]
  - ⋆ also [Alvarez-Gaumé, Distler, Kounnas, Marino, 1996; Luty, Ratazzi 1999; Edelstein, Fuertes, Mas, Guilarte, 2000]
  - ⋆ embedding $SU(2)$ adjoint QCD into $\mathcal{N} = 2$ [Cordova, Dumitrescu 2018]

This talk

- $\mathcal{N}=2$ **super-Yang-Mills with gauge group** $SU(N)$
  - ⋆ gauge multiplet $(\phi, \lambda^1, \lambda^2, v_\mu)$ in adjoint rep of $SU(N)$
  - ⋆ no $SU(N)$ hypermultiplets
- **Add mass term** $M^2\, tr(\phi^\dagger \phi)$ **for gauge scalars** ϕ
  - ⋆ softly breaks all supersymmetries
  - ⋆ but preserves all other symmetries and 't Hooft anomalies
  - ⋆ ϕ decouples as $M \to \infty$ to adjoint QCD $(\lambda^1, \lambda^2, v_\mu)$ with two flavors
- **Phase structure along the flow** $0 < M < \infty$ **?**
- **Proposal: a magnetic dual Abelian Higgs model**

$\Rightarrow$ **Cascade of phase transitions through partial Coulomb/Higgs phases**

with Thomas Dumitrescu, Efrat Gerchkovitz and Emily Nardoni

2012.11843; 2208.11502; 23XX.XXXXX

$\mathcal{N}=2$ super Yang-Mills with gauge group $SU(N)$

- **Coulomb branch vacua** $[\phi^\dagger, \phi]=0$ **have unbroken** $\mathcal{N}=2$
  - ⋆ at generic point on the Coulomb branch

$$SU(N) \longrightarrow U(1)^{N-1}$$

  - ⋆ $U(1)_R \to \mathbb{Z}_{4N}$ by ABJ anomaly

- **Low energy** $\mathcal{N}=2$ **gauge** $U(1)_k$ **multiplets for** $k=1,\cdots,N-1$
  - ⋆ in terms of $\mathcal{N}=1$ chiral A_k and gauge V_k superfields

$$\mathcal{L}_{\mathsf{SW}} = \frac{\mathrm{Im}}{4\pi}\sum_{k=1}^{N-1}\int d^4\theta\, A_{Dk}\bar{A}_k + \frac{\mathrm{Im}}{8\pi}\sum_{k,\ell=1}^{n-1}\int d^2\theta\, \tau_{k\ell}\, W_k W_\ell$$

  - ⋆ specified by a locally holomorphic pre-potential $\mathcal{F}(A_1,\cdots,A_{N-1})$

$$A_{Dk} = \frac{\partial \mathcal{F}}{\partial A_k} \qquad \tau_{k\ell} = \frac{\partial^2\mathcal{F}}{\partial A_k\,\partial A_\ell} \qquad W_k^\alpha = -\tfrac{1}{4}\bar{D}\bar{D}D^\alpha V_k$$

  - ⋆ subject to $\mathrm{Im}\,\tau > 0$ for positive kinetic term of $U(1)_k$ gauge fields

The Seiberg-Witten solution

- **The Seiberg-Witten solution determines the pre-potential**
  - ⋆ $\mathcal{F}$ as a function of the vevs $\langle A_k \rangle = a_k$ and $\langle A_{Dk} \rangle = a_{Dk}$
- **SW curves** for $SU(N)$ [Klemm, Lerche, Yankielowicz, Theisen; Argyres, Faraggi 1994]
  - ⋆ parametrized by gauge-invariant moduli $u_n = \mathrm{tr}(\phi^n)$ for $n = 2, \cdots N$

$$y^2 = C(x)^2 - \Lambda^{2N} \qquad\qquad C(x) = x^N - \sum_{n=2}^{N} u_n x^{N-n}$$

  - ⋆ hyper-elliptic of genus $N-1$

- **Solution given by periods of SW differential**
  - ⋆ Canonical homology basis $\mathfrak{A}_k, \mathfrak{B}_k$ for $k = 1, \cdots, N-1$

$$a_k = \oint_{\mathfrak{A}_k} \frac{xC'dx}{2\pi i\, y} \qquad\qquad a_{Dk} = \oint_{\mathfrak{B}_k} \frac{xC'dx}{2\pi i\, y}$$

  - ⋆ Mass of BPS dyon with electric/magnetic charges $\mathbf{q}, \mathbf{m} \in \mathbb{Z}^{N-1}$

$$M_{\mathsf{BPS}}(\mathbf{q}; \mathbf{m}) = \sqrt{2}|Z| \qquad\qquad Z = \sum_{k=1}^{N-1} \left(q_k a_k + m_k a_{Dk} \right)$$

Special points on the Coulomb branch

- **Special points: enhanced symmetry and/or vanishing masses**

$\star$ a single $\mathbb{Z}_{2N}$ symmetric point
– with curve $y^2 = x^{2N} - \Lambda^{2N}$
– no massless BPS states
$\Longrightarrow$ a convergent Taylor series exists at the $\mathbb{Z}_{2N}$ point [ED, Dumitrescu, Nardoni 2022]

$\star$ two different $\mathbb{Z}_N$ symmetric "Argyres-Douglas points" for $N \geq 3$
– with curves $y^2 = x^N(x^N \pm 2\Lambda^N)$
– $\frac{1}{2}N(N-1)$ massless dyons that are *mutually non-local*

$\star$ N different $\mathbb{Z}_2$ symmetric "multi-monopole points"
– mapped into one another by $\mathbb{Z}_N$
– with curves $y^2 = \Lambda^{2N} \sinh^2 \big(N \operatorname{arccos}(x/2N)\big)$ (Chebyshev polynomials)
– $N-1$ massless magnetic monopoles that are *mutually local*

Near a multi-monopole point

- **Approaching a multi-monopole point** [Douglas, Shenker 1995]
  - ⋆ magnetic periods $a_{Dk} \to 0$ so that $M_{\mathrm{BPS}}(\mathbf{0}, \mathbf{m}) \to 0$
  - ⋆ electric periods $a_k \not\to 0$ so that $M_{\mathrm{BPS}}(\mathbf{q}, \mathbf{0})$ remains finite

- **Massless states produce singularities in τ**

$$\tau_{k\ell} = -\frac{i\,\delta_{k\ell}}{2\pi} \ln \frac{a_{Dk}}{\Lambda} + \mathcal{O}(a_D^0)$$

  - ⋆ the SW low energy effective Lagrangian breaks down because it integrated out light magnetic monopoles

- **Viable low energy effective theory requires keeping massless states**

Effective Abelian Higgs model

- **Introduce $N-1$ magnetic monopole fields**
  - $\star$ hyper-multiplets $\mathcal{H}_k$ for gauge group $U(1)_k$ for $k=1,\cdots,N-1$
  - $\star$ in terms of $\mathcal{N}=1$ superfields with auxiliary fields $F_k^{\pm}$

$$\mathcal{H}_k^+ = (h_{1k}, \psi_{+k}, F_k^+) \qquad\qquad \mathcal{H}_k^- = (\bar{h}_k^2, \psi_{-k}, F_k^-)$$

- **Effective Lagrangian including magnetic monopole fields**
  - $\star$ dictated by $\mathcal{N}=2$ supersymmetry

$$\begin{aligned}\mathcal{L}_{\mathsf{SW}}^{\mathrm{eff}} &= \sum_{k=1}^{N-1}\Big[\int d^4\theta\Big(\bar{\mathcal{H}}_k^+ e^{-2V_k}\mathcal{H}_k^+ + \bar{\mathcal{H}}_k^- e^{+2V_k}\mathcal{H}_k^- + \frac{\mathrm{Im}}{2\pi}\bar{A}_k A_{Dk}\Big)\\ &\qquad +2\mathrm{Re}\int d^2\theta\, A_{Dk}\mathcal{H}_k^+\mathcal{H}_k^-\Big] + \sum_{k,\ell=1}^{N-1}\frac{\mathrm{Im}}{4\pi}\int d^2\theta\,\tau_{Dk\ell}^{\mathrm{eff}} W_k W_\ell\end{aligned}$$

  - $\star$ V_k and $W_k^\alpha = -\frac{1}{4}\bar{D}\bar{D}D^\alpha V_k$ is the $\mathcal{N}=1$ gauge field of $U(1)_k$

Effective periods and gauge couplings

- **Retaining magnetic monopole fields changes the RG flow**
  - ⋆ near a multi monopole point we still have $a_{Dk} \to 0$
  - ⋆ introducing the magnetic monopole fields below RG scale μ removes their contribution from the β-function

- **Gauge couplings $\tau_{k\ell}$ are now free of singularities as $a_{Dk} \to 0$**
  - ⋆ precise normalizations [ED, Dumitrescu, Gerchkovitz, Nardoni 2020]

$$\tau_{k\ell}^{\text{eff}} = \frac{i}{2\pi} \left(\delta_{k\ell} \ln \frac{\Lambda}{\mu} + \ln L_{k\ell} \right) + \mathcal{O}(a_D)$$

  - ⋆ where L is given in terms of $c_k = \cos(k\pi/N)$ and $s_k = \sin(k\pi/N)$

$$L_{kk} = 16 N s_k^3 \qquad\qquad L_{k\ell} = \frac{1 - c_{k+\ell}}{1 - c_{k-\ell}} \qquad k \neq \ell$$

  - ⋆ order $\mathcal{O}(a_D)$ corrections to τ are known as well [ED, Phong 1997]

Soft supersymmetry breaking operator

- **Flow from $\mathcal{N}=2$ to adjoint QCD is controlled by operator $\mathcal{T}$**

$$\mathcal{T} = 2g^{-2}\,\mathrm{tr}(\phi^\dagger \phi)$$

  - $\star$ $\mathcal{L}_{\mathsf{SU(N)}} \to \mathcal{L}_{\mathsf{SU(N)}} - M^2\mathcal{T}$ breaks susy completely

- **IR behavior of $\mathcal{T}$ in $\mathcal{N}=2$ theory governed by SW theory**
  - $\star$ $\mathcal{T}$ is the lowest component of the $\mathcal{N}=2$ stress tensor multiplet
    – as such its dimension is protected
  - $\star$ in SW theory, flows towards the Kähler potential is given by

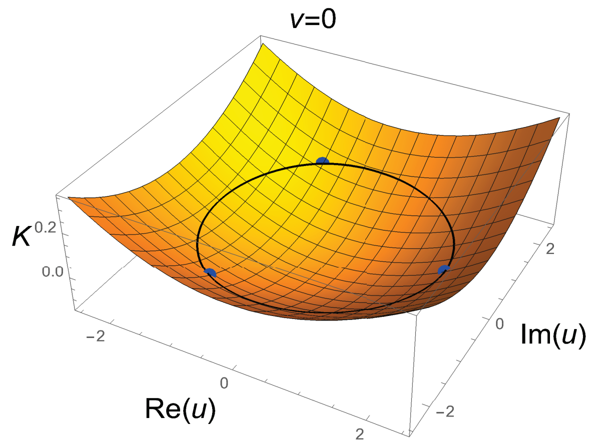
$$\mathcal{T} \to \frac{i}{4\pi}\sum_{k=1}^{N-1}\Big(\bar a_{Dk} a_k - \bar a_k a_{Dk}\Big)$$

  - $\star$ with magnetic monopole hyper-multiplet fields integrated in

$$\mathcal{T} \to \frac{i}{4\pi}\sum_{k=1}^{N-1}\Big(\bar a^{\,\mathrm{eff}}_{Dk}\, a^{\mathrm{eff}}_k - \bar a^{\,\mathrm{eff}}_k\, a^{\mathrm{eff}}_{Dk}\Big) - \tfrac{1}{2}\sum_k \bar h_k h_k$$

The Kähler potential on the Coulomb branch

- $M^2 \mathcal{T}$ **will drive vacuum towards minimum of** $\mathcal{T}$
  - ⋆ at the $\mathbb{Z}_{2N}$ symmetric point [ED, Dumitrescu, Nardoni 2022]
  - ⋆ for example for $SU(3)$ with $u = u_2$ and $v = u_3$



- **Quadratic approximation for** $\mathcal{T}$

$$\mathcal{T} \approx \sum_{k=1}^{N-1} \Big(\frac{N\Lambda}{\pi^2} s_k \, \mathrm{Im}\,(a_{Dk}) - \tfrac{1}{2} \bar{h}_k h_k \Big) + \sum_{k,\ell=1}^{N-1} t_{k\ell} \, \bar{a}_{Dk} \, a_{D\ell}$$

  - ⋆ matrix $t \sim \mathrm{Im}\, \tau^{\rm eff}$ is positive definite for $\mu/\Lambda \ll 1$

The Abelian Higgs model with susy breaking

- **Assembling all the contributions into the Abelian Higgs model**
  - ⋆ incorporate susy breaking operator $M^2\mathcal{T}$ in the quadratic approximation
  - ⋆ retain only the effective potential of the model (to study vacua)

$$\mathcal{V} = \sum_{k=1}^{N-1} \left(\frac{M^2 N \Lambda}{\pi^2} s_k \mathrm{Im}\,(a_{Dk}) + \left[2|a_{Dk}|^2 - \tfrac{1}{2} M^2 \right] \bar{h}_k h_k \right)$$

$$+ \sum_{k,\ell=1}^{N-1} \left(M^2 t_{k\ell} a_{Dk} \bar{a}_{D\ell} + (t^{-1})_{k\ell} \left[(\bar{h}_k h_\ell)(\bar{h}_\ell h_k) - \tfrac{1}{2} (\bar{h}_k h_k)(\bar{h}_\ell h_\ell) \right] \right)$$

- **Proposal:** Abelian Higgs model is dual to flow from $\mathcal{N}=2$ to adjoint QCD
  - ⋆ for small M back-reaction of $\mathcal{T}$ on flow can be ignored
  - ⋆ for larger M we present evidence in favor of a coherent picture

Vacuum alignment

- **Minima of $\mathcal{V}$ occur at $\mathrm{Re}\,(a_{Dk})=0$**

- **The Higgs fields h_k align perfectly as $SU(2)_R$ doublets**
  - ⋆ minimize $\mathcal{V}$ for given values of $h_k^\dagger h_k$;

$$\mathcal{V}\Big|_{\bar h_k h_k} = \sum_{k,\ell}(t^{-1})_{k\ell}\,\mathbf{v}_k\cdot\mathbf{v}_\ell \qquad \mathbf{v}_k = h_k^\dagger \boldsymbol{\sigma} h_k \qquad h_k = \begin{pmatrix} h_k^1 \\ h_k^2 \end{pmatrix}$$

  - ⋆ ground state is ferromagnetic under the mild assumption

$$(t^{-1})_{k\ell} < 0 \qquad k \neq \ell$$

  - ⋆ holds for sufficiently small μ/Λ

- **$SU(2)_R$ is spontaneously broken as soon as any $h_k \neq 0$**
  - ⋆ producing two Goldstone bosons $SU(2)_R \to U(1)_R$ i.e. CP^1-phase
  - ⋆ matches expected chiral symmetry breaking in adjoint QCD $\langle \lambda^i \lambda^j \rangle \neq 0$

Simplified Abelian Higgs model

- **Vacuum alignment greatly simplifies the analysis of the potential**

$$\begin{cases} a_{Dk} = -iM\, x_k \\ h^i_k = M \delta^i_1\, h_k \end{cases} \qquad \kappa = \frac{N\Lambda}{\pi^2 M}$$

  ⋆ in terms of these dimensionless real variables

$$\frac{\mathcal{V}}{M^4} = \sum_{k=1}^{N-1} \Big(\tfrac{1}{2}(4x_k^2 - 1) h_k^2 - \kappa s_k x_k \Big) + \sum_{k,\ell=1}^{N-1} \Big(t_{k\ell} x_k x_\ell + \tfrac{1}{2} (t^{-1})_{k\ell} h_k^2 h_\ell^2 \Big)$$

- **Reduced field equations for space-time independent VEVs** x_k, h_k

$$\begin{aligned} \kappa s_k &= 2h_k^2 x_k + \sum_{\ell=1}^{N-1} t_{k\ell}\, x_\ell \\ 0 &= h_k \Big(4x_k^2 - 1 + 2 \sum_{\ell=1}^{N-1} (t^{-1})_{k\ell} h_\ell^2 \Big) \end{aligned} \tag{1}$$

Organization of semi-classical analysis

- **Steps in semi-classical analysis**
  - ⋆ Existence of solutions for given N, κ
  - ⋆ Local stability of solutions: positive Hessian on the solution
  - ⋆ Global stability of solutions: global minimum of $\mathcal{V}$ for given N, κ

- **Solutions to Higgs eqs organized by partitions $A|B$**

$$
\begin{array}{ll}
k \in A & h_k = 0 \\
k \in B & 4x_k^2 - 1 + 2\sum_{\ell=1}^{N-1} (t^{-1})_{k\ell} h_\ell^2 = 0
\end{array}
$$

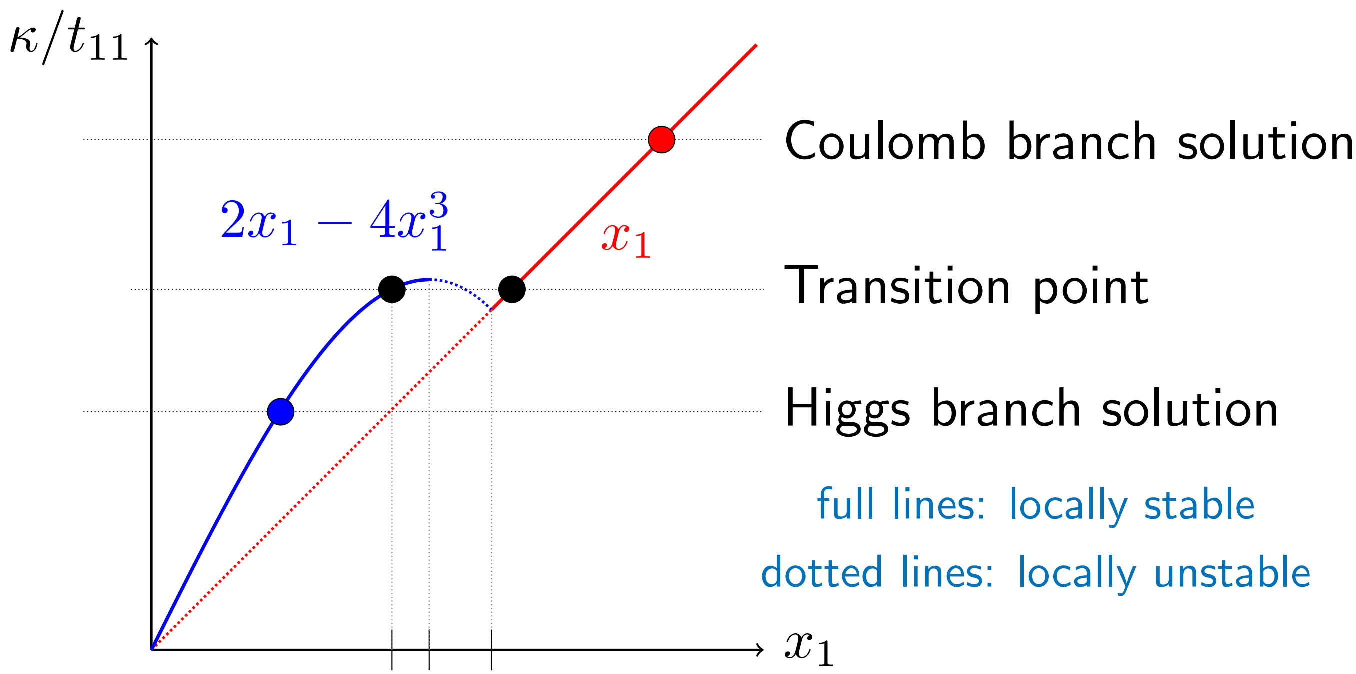
  - ⋆ Clearly $A \cup B = \{1, \cdots, N-1\}$ and $A \cap B = \emptyset$

- **In a given partition $A|B$ solve for h_k with $k \in B$ in terms of x**
  - ⋆ resulting in $|B|$ coupled cubics in x_k for $k \in B$

Gauge group $SU(2)$

- **Two possible branches** [Cordova, Dumitrescu 2018]
  - ⋆ Coulomb $h_1 = 0$ implies $x_1 = \kappa/t_{11}$; potential $\mathcal{V}_{\rm red} = \mathcal{V}_0$
  - ⋆ Higgs $h_1 \neq 0$ implies $2x_1 - 4x_1^3 = \kappa/t_{11}$

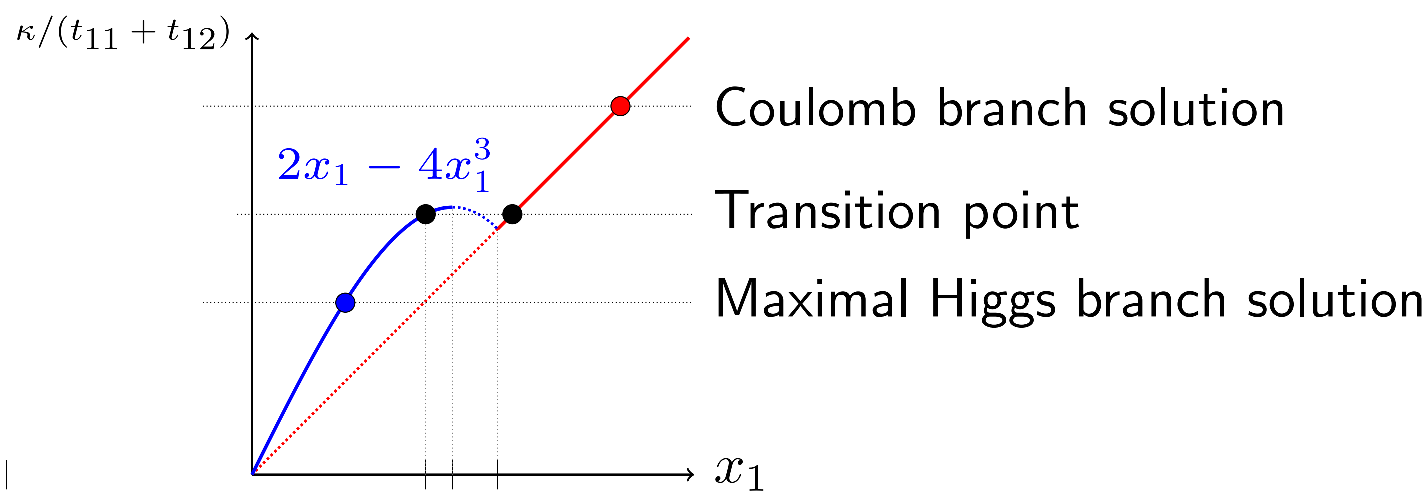
$$\mathcal{V}_{\rm red} = \mathcal{V}_0 - \tfrac{1}{8} t_{11}(1-8x_1^2)(1-4x_1^2)^2$$



Gauge group $SU(3)$

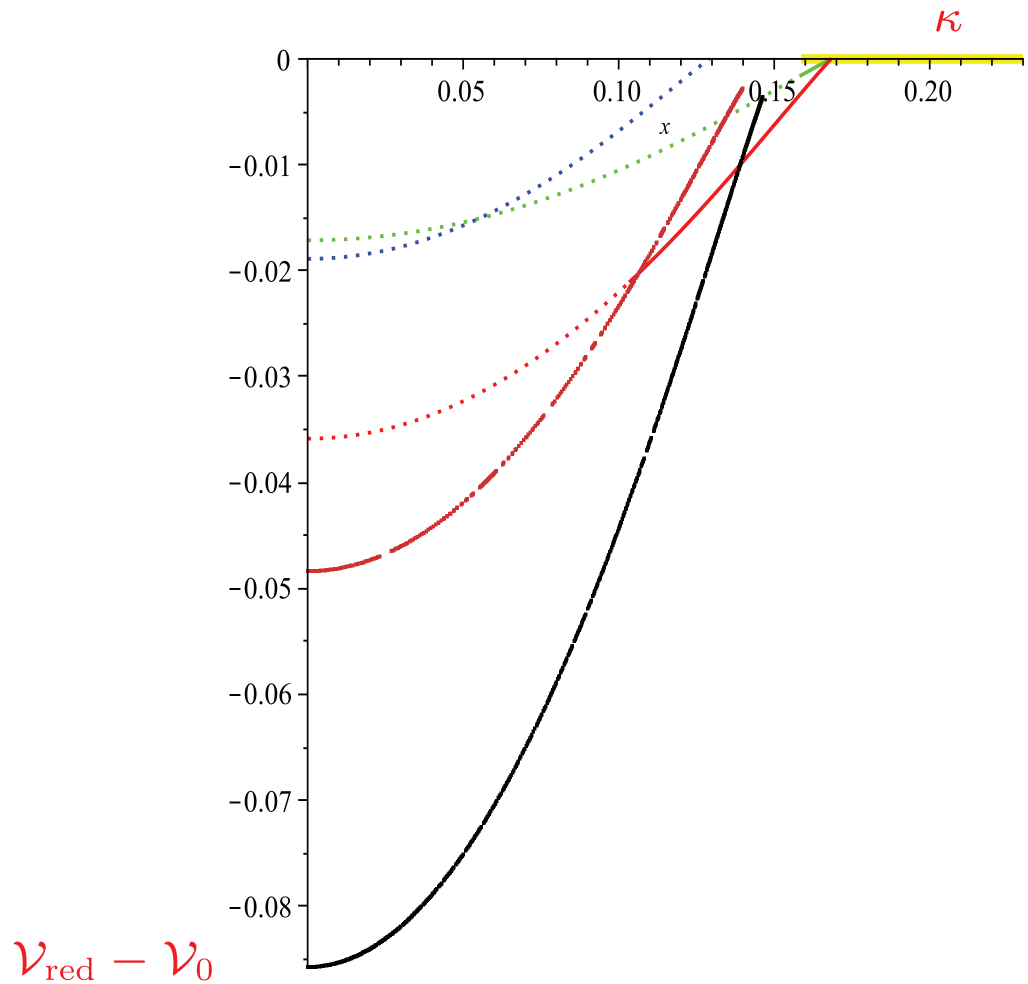
- **Three possible branches**
  - ⋆ Coulomb branch $h_k = 0$ implies $x_k = \kappa(t^{-1})_k$ for $k = 1, 2$
  - ⋆ Maximal Higgs branch $h_k \neq 0$ for $k = 1, 2$
    - local stability requires $h_1 = h_2$ when it exists
  - ⋆ Mixed branch $h_1 \neq 0, h_2 = 0$ (or $h_2 \neq 0, h_1 = 0$)
    - has higher potential than $h_1 = h_2$

- **phase diagram of $SU(2)$ with $h_1 = h_2$ and $x_1 = x_2$**



Gauge group $SU(4)$

- **Possible C-inequivalent branches**
  - ⋆ Coulomb $h_k = 0$ for $k = 1, 2, 3$
  - ⋆ Single Higgs $h_1 \neq 0, h_2, h_3 = 0$
  - ⋆ Single Higgs $h_2 \neq 0, h_1, h_3 = 0$
  - ⋆ Double Higgs $h_1, h_2 \neq 0, \ h_3 = 0$
  - ⋆ Double Higgs $h_1, h_3 \neq 0, \ h_2 = 0$
  - ⋆ Maximal Higgs $h_1, h_2, h_3 \neq 0$



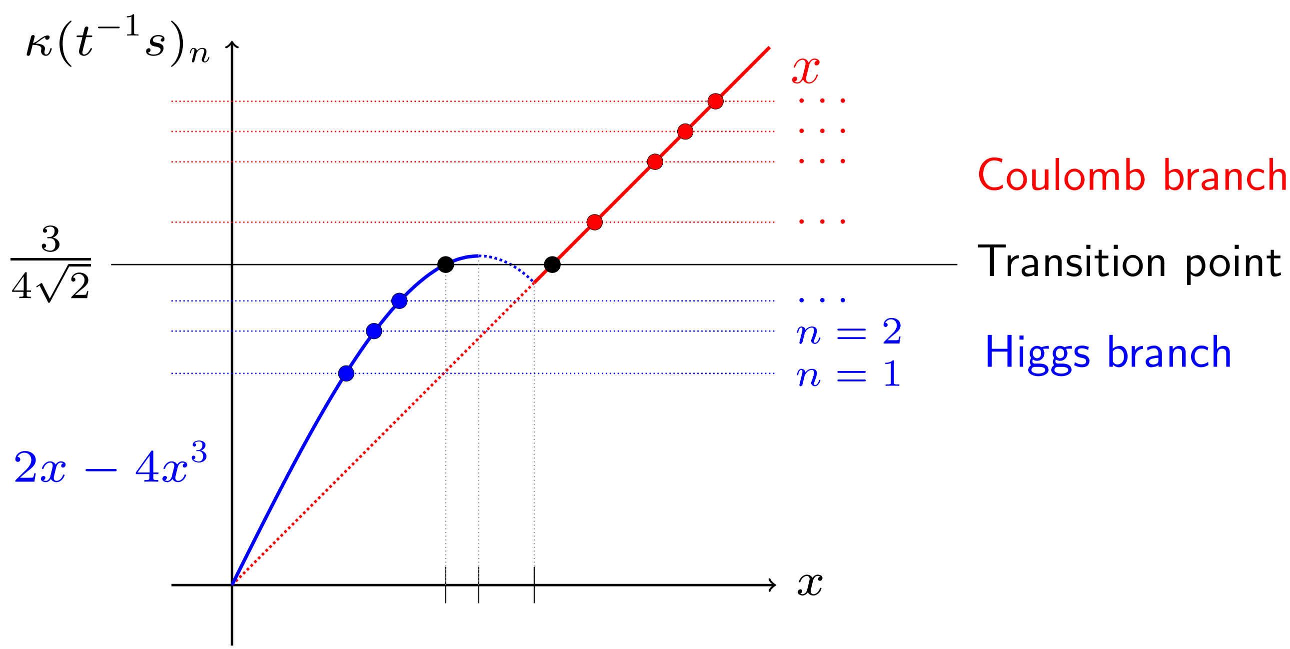
⋆ All solutions in double Higgs branch have $h_1 = h_3 \neq 0, \ h_2 = 0$

Cascade flow for arbitrary gauge group $SU(N)$

- **For larger values of N more mixed phases are stable**

- **Expansion in $(\ln \mu/\Lambda)^{-1}$ makes diagonal of matrix t dominate**
  - ⋆ Pairs $h_n = h_{N-n}$ decouple from one another
  - ⋆ reduce to $N-1$ copies of $SU(2)$ case + perturbations

- **Confirmed by numerical studies of coupled cubics**



Summary

- **Magnetic Abelian Higgs model dual for flow from $\mathcal{N} = 2$ to adjoint QCD**
  - ⋆ Magnetic monopole fields have been "integrated in"
  - ⋆ Soft susy breaking operator flows to Kähler potential of SW theory

- **Semi-classical Abelian Higgs model matches $SU(N)$ theory**
  - ⋆ For small susy breaking mass M stay in Coulomb branch
  - ⋆ For increasing M, spontaneous breaking of $SU(2)_R$
    - – matches chiral symmetry breaking in adjoint QCD
  - ⋆ Vacuum alignment guarantees only two Goldstone bosons CP^1 phase
  - ⋆ For large M, Abelian Higgs model predicts monopole condensation
    - – matches confinement in adjoint QCD

- **Semi-classical analysis of Abelian Higgs model predicts**
  - ⋆ Intermediate phases between Coulomb and Maximal Higgs phases
  - ⋆ Approximations predict a cascade of phases
    - – in which more Higgs successively acquire VEVs

Dedicated to the memory of friend and collaborator Professor Norisuke Sakai

Mount Elbert (4401 m) and Mount Massive (4398 m), Colorado