Towards higher multiplicity string amplitudes

– GSO projection and modular tensors –

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String Amplitudes

Program to understand structure of perturbative string amplitudes

⋆ kinematic structure and relation to QFT amplitudes
  – e.g. KLT, double copy, BCJ, monodromy relations

⋆ modular structure of integrands

⋆ matching low energy expansion with susy and S-duality predictions

A subject dear to Paolo

- 1969 – *Lorentz expansion for the Veneziano amplitude*, with S. Ferrara
- 1972 – *General properties of the dual resonance model, "DDF"*, with E. Del Giudice, S. Fubini
- 1975 – *Soft Dilations and Scale Renormalization in Dual Theories*, with M. Ademollo, et al.
- 1976 – *A Locally Supersymmetric ... Action for the Spinning String*, with L. Brink, J. Scherk
- 1979 – *Chiral Estimate of the Electric Dipole Moment of the neutron ...* , with R.J. Crewther, G. Veneziano, E. Witten
- 1987 – *N String Vertex and Loop Calculation in the Bosonic String*, with M. Frau, A. Lerda, S. Sciuto
- 1988 – *N String, g Loop Vertex for the Fermionic String*, with M. Frau, K. Hornfeck, A. Lerda, S. Sciuto
- 2016 – *Soft Theorems from String Theory*, with R. Marotta, M. Mojaza
- 2020 – *Universality of ultra-relativistic gravitational scattering*, with C. Heissenberg, R. Russo, G. Veneziano

The RNS formulation of superstrings

- **The RNS formulation is based on two sectors**
  - ⋆ Ramond – space-time spinors and fermions [Ramond 1971]
  - ⋆ Neveu-Schwarz – space-time bosons [Neveu, Schwarz 1971]

  – globally supersymmetric worldsheet action [Gervais, Sakita 1971]

- **Decoupling of negative norm states requires local symmetries**
  - ⋆ Diffeomorphism invariance on the worldsheet [Nambu 1970; Goto 1971]
  - ⋆ Local supersymmetry on the worldsheet [Brink, Di Vecchia, Howe 1976; Deser, Zumino 1976]

$$\begin{aligned}\mathcal{L} \quad &= \quad -\tfrac{1}{2} g^{mn} \partial_m x^\mu \partial_n x_\mu - \tfrac{i}{2} \bar\psi^\mu \gamma^m \partial_m \psi_\mu \\ &\quad\quad + \tfrac{i}{2} \bar\chi_m \gamma^n \gamma^m \psi^\mu \partial_n x_\mu + \tfrac{1}{8} (\bar\chi_m \gamma^n \gamma^m \psi^\mu)(\bar\chi_n \psi_\mu)\end{aligned}$$

  - ⋆ Weyl invariance in the critical dimension $d = 10$ [Polyakov 1981]

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A LOCALLY SUPERSYMMETRIC AND REPARAMETRIZATION INVARIANT ACTION FOR THE SPINNING STRING

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We construct an action for the spinning string which is locally supersymmetric and reparametrization invariant using the techniques of supergravity. In a special gauge it is shown that the equations of motion and the constraints are those of the Neveu-Schwarz-Ramond model.

- **Gauge fixing local worldsheet symmetries leaves equivalence classes**
  - ⋆ metric $g_{\mu\nu} \Longrightarrow$ moduli [Alessandrini, Amati 1970; Mandelstam 1974; ED, Phong 1985]
  - ⋆ gravitino $\chi_\mu \Longrightarrow$ super-moduli [Friedan 1985; Moore, Nelson, Polchinski 1986; ED, Phong 1986]
  - ⋆ BRST formulation [Friedan, Martinec, Shenker 1985]; Alberto Lerda's talk
- **Functional integrals over $g_{\mu\nu}$ and χ_μ reduce to supermoduli space $\mathfrak{M}_h$**

$$\dim \mathfrak{M}_h = \begin{cases} (0|0) & h=0 \\ (1|0) \text{ or } (1|1) & h=1 \text{ even or odd spin structure} \\ (3h-3|2h-2) & h \geq 2 \end{cases}$$

- **Supermoduli enter non-trivially starting at genus 2**

The Gliozzi-Scherk-Olive projection

- **The GSO projection selects the superstrings** [Gliozzi, Scherk, Olive 1977]
  - ⋆ worldsheet spinors ψ, χ require specifying a spin structure κ
  - ⋆ GSO projection $=$ summation over spin structures κ
    - – consistently with modular invariance of the amplitudes

- **inequivalent summations project to different string theories**
  - ⋆ Type IIA versus Type IIB [Green, Schwarz 1982]
  - ⋆ Heterotic $E_8 \times E_8$ versus Spin$(32)/\mathbb{Z}_2$ [Gross, Harvey, Martinec, Rohm 1985]

- **independently for left and right movers**
  - ⋆ cancellation of the holomorphic anomaly [Belavin, Kniznik 1986]
  - ⋆ via loop momenta and chiral splitting [Verlinde, Verlinde 1988; ED, Phong 1988]

Status of superstring amplitudes

- **Tree-level: general amplitudes are known** eg [Mafra, Schlotterer 2022]
- **One-loop: spin structure summations using Riemann relations**
  eg Snowmass White paper [Berkovits, ED, Green, Johansson, Schlotterer, 2022]
- **Two loops: some low multiplicity amplitudes are known**
  - ⋆ four massless string amplitudes [ED, Phong 2005]; pure spinors [Berkovits 2005]
  - ⋆ five massless string amplitudes [ED, Mafra, Pioline, Schlotterer 2020; ED, Schlotterer 2021]
- **Three loops: no first principles construction realized so far**
  - ⋆ both RNS and pure spinor approaches present obstacles
  - ⋆ coefficient of $D^6\mathcal{R}^4$ low energy effective interaction [Gomez, Mafra 2014]
  - ⋆ conjectured RNS measure [Cacciatori, Dalla Plazza, van Geemen 2008]
  - ⋆ conjectured four string amplitude [Geyer, Monteiro, Stark-Muchao 2021]

This talk: GSO projection and modular tensors

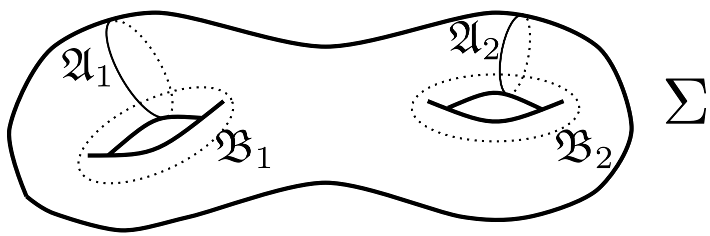
- **Spin structure summations for higher multiplicity and higher genus**
  - ⋆ spin structure summation implements space-time supersymmetry via GSO
  - ⋆ space-time supersymmetry greatly simplifies amplitudes
  - ⋆ what is the mathematical structure of spin structure summations ?

- **For genus 2**
  - ⋆ Explicit spin structure summation [ED, Hidding, Schlotterer 2022]
  - ⋆ Full six massless NS amplitude [ED, Hidding, Schlotterer] in progress

- **For arbitrary genus**
  - ⋆ Reduce to summation over **modular tensors** [ED, Hidding, Schlotterer] in progress

Compact Riemann surfaces of genus g

- **Homology $H_1(\Sigma,\mathbb{Z}) \approx \mathbb{Z}^{2g}$ with intersection pairing $\mathfrak{J}(\cdot,\cdot) \to \mathbb{Z}$**
  - ⋆ Canonical basis $\mathfrak{J}(\mathfrak{A}_I,\mathfrak{A}_J) = \mathfrak{J}(\mathfrak{B}_I,\mathfrak{B}_J) = 0$, $\mathfrak{J}(\mathfrak{A}_I,\mathfrak{B}_J) = \delta_{IJ}$ for $I,J = 1,\cdots,g$



- **Modular group $Sp(2g,\mathbb{Z})$ acts on $H_1(\Sigma,\mathbb{Z})$ leaving $\mathfrak{J}(\cdot,\cdot)$ invariant**

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad M^t \mathfrak{J} M = \mathfrak{J} \qquad \begin{pmatrix} \mathfrak{B} \\ \mathfrak{A} \end{pmatrix} \to M \begin{pmatrix} \mathfrak{B} \\ \mathfrak{A} \end{pmatrix}$$

- **Canonical basis of holomorphic one-forms ω_I in $H^{(1,0)}(\Sigma)$**

$$\oint_{\mathfrak{A}_I} \omega_J = \delta_{IJ} \qquad \oint_{\mathfrak{B}_I} \omega_J = \Omega_{IJ}$$

  - ⋆ Period matrix Ω obeys Riemann relations $\Omega^t = \Omega,\ \mathrm{Im}\,(\Omega) > 0$
  - ⋆ Moduli space $\mathcal{M}_2 = \{\Omega^t = \Omega,\ \mathrm{Im}\,(\Omega) > 0\}/Sp(4,\mathbb{Z})$ (minus the diagonal)
    $\mathcal{M}_3 = \{\Omega^t = \Omega,\ \mathrm{Im}\,(\Omega) > 0\}/Sp(6,\mathbb{Z})$ (modulo hyperelliptic)
    $\mathcal{M}_g,\ g \geq 4$ Schottky problem

Modular tensors

- **Modular tensors transform under $Sp(2g,\mathbb{Z})$**
  - ⋆ tensor $\mathfrak{T}$ of rank $r=(r,0)$ and weight $(w,0)$
  - ⋆ transforms under $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp(2g,\mathbb{Z})$ by

$$\mathfrak{T}^{I_1\cdots I_r} \to (\det R)^w \, R^{I_1}{}_{J_1} \cdots R^{I_r}{}_{J_r} \, \mathfrak{T}^{J_1\cdots J_r} \qquad\qquad R = C\Omega + D$$

  - ⋆ $\mathfrak{T}$ is a section of a holomorphic vector bundle over Torelli space
  - ⋆ May be generalized to tensors of rank $(r,\bar r)$ and weight $(w,\bar w)$

- **Siegel modular forms are holomorphic with rank $r=0$ and weight w**
  - ⋆ $Sp(4,\mathbb{Z})$: polynomial ring generated by $\Psi_4,\Psi_6,\Psi_{10},\Psi_{12},\Psi_{35}$ [Igusa 1960's]
  - ⋆ $Sp(6,\mathbb{Z})$: polynomial ring with 19 generators [Tsuyumine 1986; Lercier, Ritzenthaler 2019]

- **Modular tensors** in mathematics and physics
  - ⋆ holomorphic [van der Geer 2015]; non-holomorphic [Kawazumi 2016]
  - ⋆ modular graph tensors [ED, Schlotterer 2020]
    - – generalizing genus 1 modular graph forms [ED, Green, 2016]
    - – generalizing higher genus modular graph functions ED, Green, Pioline 2018

      (both of which arise in the α' expansion of string amplitudes)

Two-loop amplitudes for four massless strings

- **Type II in the RNS formulation** (similarly Heterotic) [ED, Phong 2001-2005]

$$\mathcal{A}_4^{(2)} = g_s^2\, t_8 \tilde t_8 \int_{\mathcal{M}_2} \frac{|d\Omega^3|^2}{(\det \mathrm{Im}\, \Omega)^5} \int_{\Sigma^4} \mathcal{Y} \wedge \bar{\mathcal{Y}} \exp \Big\{ \sum_{i<j} s_{ij}\, G(z_i, z_j|\Omega) \Big\}$$

  - ⋆ Kinematics $s_{ij} = -\frac{\alpha'}{4}(k_i + k_j)^2$ with factorized $\varepsilon_i^\mu \tilde\varepsilon_i^\nu$ with $f_i^{\mu\nu} = \varepsilon_i^\mu k_i^\nu - \varepsilon_i^\nu k_i^\mu$
  - ⋆ $t_8(f_1, f_2, f_3, f_4)\tilde t_8(\tilde f_1, \tilde f_2, \tilde f_3, \tilde f_4) \sim \mathcal{R}^4$ where $\mathcal{R}$ stands for the Riemann tensor
  - ⋆ Measure on Σ^4 interlaces kinematic and worldsheet data

$$\begin{aligned} \mathcal{Y} &= t\Delta(z_1, z_2)\Delta(z_3, z_4) - s\Delta(z_1, z_4)\Delta(z_2, z_3) \\ \Delta(z, w) &= \omega_1(z)\omega_2(w) - \omega_2(z)\omega_1(w) \\ G(z, w|\Omega) &= -\ln |E(z, w|\Omega)|^2 + 2\pi(\mathrm{Im}\,\Omega)^{-1}_{IJ}\, \mathrm{Im} \int_w^z \omega_I\, \mathrm{Im} \int_w^z \omega_J \end{aligned}$$

  - ⋆ The prime form $E(z, w|\Omega)$ generalizes $\vartheta_1(z - w|\tau)$ to higher genus

- **Type II in the pure spinor formulation** [Berkovits 2005; Berkovits, Mafra 2005]
  - ⋆ generalized to include full supergravity multiplet of external strings

Two-loop amplitude for five massless strings

- **Amplitude for massless NS strings (conserving parity)** [ED, Schlotterer 2021]

$$\mathcal{A}_5^{(2)} = g_s^2 \int dp^I \int_{\mathcal{M}_2} |d\Omega^3|^2 \int_{\Sigma^5} \tilde{\mathcal{F}}_5 \, \mathcal{F}_5$$

⋆ the chiral amplitude is given by

$$\mathcal{F}_5 = \mathcal{I}_5 \sum_i \Big\{ \mathfrak{P}^I(z_i) \cdot \big(\varepsilon_i \, \mathrm{t}_i \, \mathcal{Y}_I + k_i \, \mathfrak{T}_{iI} \big) - \sum_{j \neq i} \mathcal{Y}_I \, \mathrm{t}_{ij} \, g^I_{i,j} \Big\}$$

⋆ in terms of the universal chiral Koba-Nielsen factor

$$\mathcal{I}_5 = \exp \Big\{ i\pi \Omega_{IJ} p^I p^J + 2\pi i p^I \sum_i k_i \int_{z_0}^{z_i} \omega_I + \sum_{i<j} s_{ij} \ln E(z_i, z_j) \Big\}$$

⋆ and universal meromorphic combinations

$$\mathfrak{P}^I(z_i) = 2\pi i p^I + \sum_{j \neq i} g^I_{i,j} k_j \qquad\qquad g^I_{i,j} = \partial^I \ln \vartheta[\nu](z_j - z_i | \Omega)$$

⋆ kinematic factors adapted to the five-point amplitude

$$\mathrm{t}_1 = t_8(f_2, f_3, f_4, f_5) \qquad\qquad \mathrm{t}_{12} = t_8([f_1, f_2], f_3, f_4, f_5) \ \ \& \ \ \text{cyclic}$$

⋆ and holomorphic forms generalizing those of the four-point amplitude

$$\mathcal{Y}_I = 4 s_{12} \, \omega_I(4) \Delta(5,1) \Delta(2,3) + \ \text{cycl}(1,2,3,4,5)$$

$$\mathfrak{T}_{1I} = (\mathrm{t}_{12} - 2\mathrm{t}_1 \varepsilon_1 \cdot k_2) \Big\{ \omega_I(3) \Delta(1,5) \Delta(2,4) + \ \text{cycl}(3,4,5) \Big\} + \ \text{cycl}(2,3,4,5)$$

Spin structure sums for higher multiplicity

- **Major efforts go into carrying out the spin structure sums**
  - ⋆ for the four and five genus two string amplitudes using
    - – the Riemann identities
    - – the Fay trisecant identity (cfr bosonization)
    - – and every other trick we could think of
  - ⋆ for higher multiplicity these methods alone do not appear promising
    - – the problem was also considered in [Tsuchiya 2012; 2017; 2022]

- **Fermion correlator for spin structure δ is given by the Szegö kernel**
  - ⋆ Restrict to even spin structures and NS external states

$$-\langle \psi(z)\psi(w)\rangle = S_\delta(z,w) = \frac{\vartheta[\delta](\int_w^z \omega|\Omega)}{\vartheta[\delta](0|\Omega)E(z,w)}$$

  - – the Riemann ϑ-function for spin structure $\delta = [\delta'|\delta''] \in \{0, \frac{1}{2}\}^{2g}$ is defined by

$$\vartheta[\delta](\zeta|\Omega) = \sum_{n\in\mathbb{Z}^2} \exp\Big\{ i\pi(n+\delta')^t\Omega(n+\delta') + 2\pi i(n+\delta')^t(\zeta+\delta'')\Big\}$$

  - ⋆ String amplitude integrands involve cyclic products of Szegö kernels

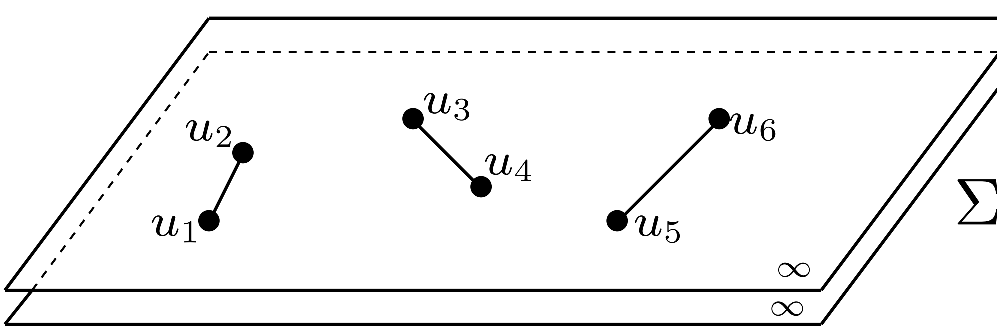
$$C_\delta(z_1,\cdots,z_n) = S_\delta(z_1,z_2)S_\delta(z_2,z_3)\cdots S_\delta(z_{n-1},z_n)S_\delta(z_n,z_1)$$

  - – they also involve other products that may be treated similarly

Spin structure sums for genus 2

- **Theorem 1** [ED, Hidding, Schlotterer 2022]
  The spin structure sum of $C_\delta(z_1, \cdots, z_n)$ for genus 2 and arbitrary n reduces to the spin structure sums for the cases $n = 0, 2, 3, 4$
  - The proof is constructive and formulated in the hyper-elliptic formulation
  - The result will be translated into the ϑ-function formulation
  - The spin structure sums for $n = 0, 1, 2, 3, 4$ are well-known

- **Every genus two surface Σ is hyper-elliptic**
  - ⋆ namely a double cover of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$
  - ⋆ ramified over 6 branch points $u_1, \cdots, u_6$
  - ⋆ points $z \in \Sigma$ parametrized by $z = (x, s)$ where $s^2 = (x - u_1) \cdots (x - u_6)$
  - ⋆ Moduli space $\mathcal{M}_2$ isomorphic to $\{u_1, \cdots, u_6\}/\mathrm{SL}(2, \mathbb{C}) \times \mathfrak{S}_6$



Sketch of proof of the Theorem

⋆ An even spin structure δ is isomorphic to a $3+3$ partition of branch points

$$\{u_1, \cdots, u_6\} = A \cup B \qquad A \cap B = \emptyset \qquad |A| = |B| = 3$$

(an odd spin structure is isomorphic to a 1+5 partition)

⋆ The Szegö kernel is given in terms of this partition by

$$S_\delta(z_1, z_2) = \frac{s_A(x_1)s_B(x_2) + s_B(x_1)s_A(x_2)}{2(x_1 - x_2)} \left[\frac{dx_1 \, dx_2}{s(x_1)\, s(x_2)} \right]^{\frac{1}{2}}$$

where $s_A(x)s_B(x) = s(x)$ and $s_A(x)^2$ and $s_B(x)^2$ are polynomials given by

$$s_A(x)^2 = \prod_{r \in A} (x - u_r) \qquad s_B(x)^2 = \prod_{r \in B} (x - u_r)$$

⋆ The cyclic product of Szegö kernels is thus given by (using $x_{n+1} = x_1$)

$$C_\delta(z_1, \cdots z_n) = \frac{\prod_{i=1}^n (s_A(x_i)s_B(x_{i+1}) + s_B(x_i)s_A(x_{i+1})}{2^n \, x_{12}x_{23} \cdots x_{n1}} \frac{dx_1 \cdots dx_n}{s(x_1) \cdots s(x_n)}$$

⋆ All spin structure dependence is contained in polynomials with $2m \leq n$

$$Q_\delta(i_1, \cdots, i_m | j_1, \cdots, j_m) = \prod_{\alpha=1}^m s_A(x_{i_\alpha})^2 s_B(x_{j_\alpha})^2 + (A \leftrightarrow B)$$

Sketch of proof of the Theorem (cont'd)

- **Lemma 1**

  All spin structure dependence of Q_δ *is polynomial in* $\ell_\delta^{11}, \ell_\delta^{12} = \ell_\delta^{21}, \ell_\delta^{22}$

$$\ell_\delta^{11} = \tfrac{1}{4}\alpha_2\beta_2 - \tfrac{3}{20}\mu_4 \qquad s_A(x)^2 = x^3 - \alpha_1 x^2 + \alpha_2 x - \alpha_3$$

$$\ell_\delta^{12} = \tfrac{1}{4}(\alpha_1\beta_2 + \alpha_2\beta_1) - \tfrac{9}{40}\mu_3 \qquad s_B(x)^2 = x^3 - \beta_1 x^2 + \beta_2 x - \beta_3$$

$$\ell_\delta^{22} = \tfrac{1}{4}\alpha_1\beta_1 - \tfrac{3}{20}\mu_2 \qquad s(x)^2 = x^6 - \mu_1 x^5 + \cdots - \mu_5 x + \mu_6$$

- **Lemma 2: The trilinear relations**

  Every trilinear $\ell_\delta^{a_1a_2}\,\ell_\delta^{a_3a_4}\,\ell_\delta^{a_5a_6}$ *may be expressed as a polynomial of total degree two in the combinations* $\ell_\delta^{11}, \ell_\delta^{12}$ *and* ℓ_δ^{22} *whose coefficients are polynomials in* $\mu_1, \cdots, \mu_6$

- *Combining Lemmas 1 and 2 implies that all spin structure dependence* of Q_δ and C_δ is given by a quadratic polynomial in $\ell_\delta^{11}, \ell_\delta^{12}, \ell_\delta^{22}$ with coefficients that depend only on μ_i.

- *The spin structure sums of the linears* $\ell_\delta^{a_1a_2}$ *and of the bilinears* $\ell_\delta^{a_1a_2}\ell_\delta^{a_3a_4}$ *are determined by* n*-point functions with* $n \leq 4$, *which concludes the proof of the Theorem.*

$\mathrm{SL}(2,\mathbb{C})$ tensorial structure of the trilinear relations

- **Component form of the trilinear relations** e.g.

$$(\boldsymbol{\ell}_\delta^{11})^3 = \frac{\mu_4(\boldsymbol{\ell}_\delta^{11})^2}{20} - \frac{\mu_5\boldsymbol{\ell}_\delta^{11}\boldsymbol{\ell}_\delta^{12}}{4} + \mu_6\boldsymbol{\ell}_\delta^{11}\boldsymbol{\ell}_\delta^{22} - \frac{\mu_6(\boldsymbol{\ell}_\delta^{12})^2}{4} + \frac{\mu_4^2\boldsymbol{\ell}_\delta^{11}}{50} - \frac{9\mu_3\mu_5\boldsymbol{\ell}_\delta^{11}}{160} + \frac{3\mu_2\mu_6\boldsymbol{\ell}_\delta^{11}}{20} + \frac{\mu_4\mu_5\boldsymbol{\ell}_\delta^{12}}{40}$$
$$- \frac{9\mu_3\mu_6\boldsymbol{\ell}_\delta^{12}}{80} - \frac{\mu_5^2\boldsymbol{\ell}_\delta^{22}}{16} + \frac{3\mu_4\mu_6\boldsymbol{\ell}_\delta^{22}}{20} - \frac{3\mu_4^3}{2000} + \frac{9\mu_3\mu_4\mu_5}{1600} - \frac{3\mu_2\mu_5^2}{320} - \frac{81\mu_3^2\mu_6}{6400} + \frac{9\mu_2\mu_4\mu_6}{400}$$

- **The ℓ_δ^{ab} transform under the 3-dimensional irrep of $\mathrm{SL}(2,\mathbb{C})$ by**

$$\boldsymbol{\ell}_\delta^{ab} \to J\, g^a_c\, g^b_d\, \boldsymbol{\ell}_\delta^{cd} \qquad g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}) \qquad J = \prod_{j=1}^{6} (\gamma u_j + \delta)^{-1}$$

- **The trilinear relations in $\mathrm{SL}(2,\mathbb{C})$ tensorial form**

$$\mathbf{7} \qquad \boldsymbol{\ell}_\delta^{(a_1a_2}\, \boldsymbol{\ell}_\delta^{a_3a_4}\, \boldsymbol{\ell}_\delta^{a_5a_6)} = \mathbf{M}_1^{b_1b_2(a_1\cdots a_4}\, \boldsymbol{\ell}_\delta^{a_5a_6)}\, \boldsymbol{\ell}_\delta^{c_1c_2}\, \varepsilon_{b_1c_1}\, \varepsilon_{b_2c_2} + \cdots$$

$$\mathbf{3} \qquad (\det \boldsymbol{\ell}_\delta)\, \boldsymbol{\ell}_\delta^{a_1a_2} = \tfrac{3}{2}\, \mathbf{M}_1^{a_1a_2b_1\cdots b_4}\, \boldsymbol{\ell}_\delta^{c_1c_2} \boldsymbol{\ell}_\delta^{c_3c_4} \varepsilon_{b_1c_1} \cdots \varepsilon_{b_4c_4} + \cdots$$

$\star$ where $\mathbf{M}_1$ is the symmetric rank 6 tensor under $\mathrm{SL}(2,\mathbb{C})$ with components

$$\mathbf{M}_1^{111111} = \mu_6 \qquad \mathbf{M}_1^{111112} = \tfrac{\mu_5}{6} \qquad \mathbf{M}_1^{111122} = \tfrac{\mu_4}{15} \cdots$$

$\mathrm{Sp}(4,\mathbb{Z})$ tensorial structure of the trilinear relations

- **Correspondence between hyper-elliptic and ϑ-function formulations**
  - ⋆ via standard Thomae formulas and holomorphic 1-forms ω_I

$$\varpi_1 = \frac{dx}{s(x)} \qquad \varpi_2 = -\frac{x\,dx}{s(x)} \qquad \omega_I(z) = \varpi_a(z)\,\sigma^a{}_I$$

  - ⋆ we obtain the *modular tensors* $\mathfrak{L}_\delta$ and $\mathfrak{M}_1$ (δ transforms)

$$\boldsymbol{\ell}_\delta^{ab} = \sigma^a{}_I\,\sigma^b{}_J\,\mathfrak{L}_\delta^{IJ} \qquad \mathrm{M}_1^{a_1\cdots a_6} = (\det\sigma)^{-2}\,\sigma^{a_1}{}_{I_1}\cdots\sigma^{a_6}{}_{I_6}\,\mathfrak{M}_1^{I_1\cdots I_6}$$

$$\mathfrak{L}_\delta^{IJ} = \frac{\pi}{5i}\partial^{IJ}\ln\left\{\frac{\vartheta[\delta](0)^{20}}{\Psi_{10}}\right\} \qquad \mathfrak{M}_1^{I_1\cdots I_6} = \Psi_{10}^{-\frac{1}{2}}\partial^{(I_1}\vartheta[\nu_1](0)\cdots\partial^{I_6)}\vartheta[\nu_6](0)$$

  - ⋆ where $\nu_1,\cdots,\nu_6$ are the six (distinct) odd spin structures

- **Trilinear relations are between $Sp(4,\mathbb{Z})$ modular tensors**

$$\begin{aligned}
\mathfrak{L}_\delta^{(I_1I_2}\,\mathfrak{L}_\delta^{I_3I_4}\,\mathfrak{L}_\delta^{I_5I_6)} &= \mathfrak{M}_1^{J_1J_2(I_1\cdots I_4}\,\mathfrak{L}_\delta^{I_5I_6)}\,\mathfrak{L}_\delta^{K_1K_2}\,\varepsilon_{J_1K_1}\,\varepsilon_{J_2K_2} + \cdots \\
(\det\mathfrak{L}_\delta)\,\mathfrak{L}_\delta^{I_1I_2} &= \tfrac{3}{2}\,\mathfrak{M}_1^{I_1I_2J_1\cdots J_4}\,\mathfrak{L}_\delta^{K_1K_2}\,\mathfrak{L}_\delta^{K_3K_4}\,\varepsilon_{J_1K_1}\cdots\varepsilon_{J_4K_4} + \cdots
\end{aligned}$$

Spin structure sums for genus $g \geq 2$

- **Generic surfaces for genus $g \geq 3$ are no longer hyper-elliptic**

- **Theorem 2** [ED, Hidding, Schlotterer] in progress
  The spin structure sum of $C_\delta(z_1, \cdots, z_n)$ for arbitrary genus and arbitrary n reduces to the spin structure sums of z_i-independent modular tensors

⋆ *Proof:* is constructive by a descent method

Summary and outlook

- **Two-loop superstring amplitudes**
  - ⋆ explicit summation over even spin structures
  - ⋆ paves the way to higher multiplicity amplitudes
  - ⋆ relate kinematics to QFT amplitudes ?

- **Higher loops**
  - ⋆ even spin structure dependence reduced to modular tensors
  - ⋆ what is the dimension and structure of modular tensor spaces ?
  - ⋆ which subspace is needed for string amplitudes ?
  - ⋆ can one build an efficient library ?

Happy Birthday Paolo