Towards higher multiplicity string amplitudes

– GSO projection and modular tensors –

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String Amplitudes

Program to understand structure of perturbative string amplitudes
  ★ kinematic structure and relation to QFT amplitudes
    – e.g. KLT, double copy, BCJ, monodromy relations
  ★ modular structure of integrands
  ★ matching low energy expansion with susy and S-duality predictions

A subject dear to Paolo

1. 1969 – Lorentz expansion for the Veneziano amplitude, with S. Ferrara
2. 1972 – General properties of the dual resonance model, “DDF”, with E. Del Giudice, S. Fubini
4. 1976 – A Locally Supersymmetric … Action for the Spinning String, with L. Brink, J. Scherk
5. 1979 – Chiral Estimate of the Electric Dipole Moment of the neutron …, with R.J. Crewther, G. Veneziano, E. Witten
6. 1987 – N String Vertex and Loop Calculation in the Bosonic String, with M. Frau, A. Lerda, S. Sciuto
7. 1988 – N String, g Loop Vertex for the Fermionic String, with M. Frau, K. Hornfeck, A. Lerda, S. Sciuto
8. 2016 – Soft Theorems from String Theory, with R. Marotta, M. Mojaza
9. 2020 – Universality of ultra-relativistic gravitational scattering, with C. Heissenberg, R. Russo, G. Veneziano
The RNS formulation of superstrings

- The RNS formulation is based on two sectors
  - Ramond – space-time spinors and fermions [Ramond 1971]
  - Neveu-Schwarz – space-time bosons [Neveu, Schwarz 1971]

- globally supersymmetric worldsheet action [Gervais, Sakita 1971]

- Decoupling of negative norm states requires local symmetries
  - Diffeomorphism invariance on the worldsheet [Nambu 1970; Goto 1971]
  - Local supersymmetry on the worldsheet [Brink, Di Vecchia, Howe 1976; Deser, Zumino 1976]

\[
\mathcal{L} = -\frac{1}{2}g^{mn}\partial_m x^\mu \partial_n x_\mu - \frac{i}{2} \bar{\psi}^\mu \gamma^m \partial_m \psi_\mu \\
+ \frac{i}{2} \bar{\chi} m \gamma^n \gamma^m \psi^\mu \partial_n x_\mu + \frac{1}{8} (\bar{\chi} m \gamma^n \gamma^m \psi^\mu)(\bar{\chi} n \psi_\mu)
\]

- Weyl invariance in the critical dimension \(d = 10\) [Polyakov 1981]
Gauge fixing local worldsheet symmetries leaves equivalence classes

- **metric** \( g_{\mu\nu} \rightarrow \text{moduli} \) [Alessandrini, Amati 1970; Mandelstam 1974; ED, Phong 1985]
- **gravitino** \( \chi_{\mu} \rightarrow \text{super-moduli} \) [Friedan 1985; Moore, Nelson, Polchinski 1986; ED, Phong 1986]
- **BRST formulation** [Friedan, Martinec, Shenker 1985]; Alberto Lerda’s talk

Functional integrals over \( g_{\mu\nu} \) and \( \chi_{\mu} \) reduce to supermoduli space \( \mathcal{M}_h \)

\[
\dim \mathcal{M}_h = \begin{cases} 
(0|0) & h = 0 \\
(1|0) \text{ or } (1|1) & h = 1 \text{ even or odd spin structure} \\
(3h - 3|2h - 2) & h \geq 2 
\end{cases}
\]

Supermoduli enter non-trivially starting at genus 2
The Gliozzi-Scherk-Olive projection

• The GSO projection selects the superstrings [Gliozzi, Scherk, Olive 1977]
  ★ worldsheet spinors $\psi, \chi$ require specifying a spin structure $\kappa$
  ★ GSO projection $\equiv$ summation over spin structures $\kappa$
  – consistently with modular invariance of the amplitudes

• inequivalent summations project to different string theories
  ★ Type IIA versus Type IIB [Green, Schwarz 1982]
  ★ Heterotic $E_8 \times E_8$ versus $\text{Spin}(32)/\mathbb{Z}_2$ [Gross, Harvey, Martinec, Rohm 1985]

• independently for left and right movers
  ★ cancellation of the holomorphic anomaly [Belavin, Kniznik 1986]
  ★ via loop momenta and chiral splitting [Verlinde, Verlinde 1988; ED, Phong 1988]
Status of superstring amplitudes

- **Tree-level:** general amplitudes are known eg [Mafra, Schlotterer 2022]

- **One-loop:** spin structure summations using Riemann relations
  eg Snowmass White paper [Berkovits, ED, Green, Johansson, Schlotterer, 2022]

- **Two loops:** some low multiplicity amplitudes are known
  - ★ four massless string amplitudes [ED, Phong 2005]; pure spinors [Berkovits 2005]
  - ★ five massless string amplitudes [ED, Mafra, Pioline, Schlotterer 2020; ED, Schlotterer 2021]

- **Three loops:** no first principles construction realized so far
  - ★ both RNS and pure spinor approaches present obstacles
  - ★ coefficient of $D^6R^4$ low energy effective interaction [Gomez, Mafra 2014]
  - ★ conjectured RNS measure [Cacciatori, Dalla Plazza, van Geemen 2008]
  - ★ conjectured four string amplitude [Geyer, Monteiro, Stark-Muchao 2021]
This talk: GSO projection and modular tensors

- **Spin structure summations for higher multiplicity and higher genus**
  - Spin structure summation implements space-time supersymmetry via GSO
  - Space-time supersymmetry greatly simplifies amplitudes
  - What is the mathematical structure of spin structure summations?

- **For genus 2**
  - **Explicit spin structure summation** [ED, Hidding, Schlotterer 2022]
  - **Full six massless NS amplitude** [ED, Hidding, Schlotterer] in progress

- **For arbitrary genus**
  - **Reduce to summation over modular tensors** [ED, Hidding, Schlotterer] in progress
Compact Riemann surfaces of genus $g$

- **Homology** $H_1(\Sigma, \mathbb{Z}) \approx \mathbb{Z}^{2g}$ with intersection pairing $\mathcal{J}(\cdot, \cdot) \to \mathbb{Z}$
  - Canonical basis $\mathcal{J}(\mathcal{A}_I, \mathcal{A}_J) = \mathcal{J}(\mathcal{B}_I, \mathcal{B}_J) = 0$, $\mathcal{J}(\mathcal{A}_I, \mathcal{B}_J) = \delta_{IJ}$ for $I, J = 1, \cdots, g$

  ![Image of a Riemann surface with homology groups]

- **Modular group** $Sp(2g, \mathbb{Z})$ acts on $H_1(\Sigma, \mathbb{Z})$ leaving $\mathcal{J}(\cdot, \cdot)$ invariant

  $$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad M^t \mathcal{J} M = \mathcal{J}$$

- **Canonical basis of holomorphic one-forms** $\omega_I$ in $H^{(1,0)}(\Sigma)$

  $$\oint_{\mathcal{A}_I} \omega_J = \delta_{IJ} \quad \oint_{\mathcal{B}_I} \omega_J = \Omega_{IJ}$$

  - Period matrix $\Omega$ obeys Riemann relations $\Omega^t = \Omega$, $\text{Im} (\Omega) > 0$
  - Moduli space $\mathcal{M}_2 = \{ \Omega^t = \Omega, \text{Im} (\Omega) > 0 \}/Sp(4, \mathbb{Z})$ (minus the diagonal)
    $$\mathcal{M}_3 = \{ \Omega^t = \Omega, \text{Im} (\Omega) > 0 \}/Sp(6, \mathbb{Z})$$ (modulo hyperelliptic)
  - $\mathcal{M}_g$, $g \geq 4$ Schottky problem
Modular tensors

- Modular tensors transform under $Sp(2g, \mathbb{Z})$
  - tensor $\mathcal{T}$ of rank $r = (r, 0)$ and weight $(w, 0)$
  - transforms under \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbb{Z}) \) by
    \[
    \mathcal{T}^{I_1 \cdots I_r} \rightarrow (\det R)^w R^{I_1 J_1} \cdots R^{I_r J_r} \mathcal{T}^{J_1 \cdots J_r} \quad R = C\Omega + D
    \]
  - $\mathcal{T}$ is a section of a holomorphic vector bundle over Torelli space
  - May be generalized to tensors of rank $(r, \bar{r})$ and weight $(w, \bar{w})$

- Siegel modular forms are holomorphic with rank $r = 0$ and weight $w$
  - $Sp(4, \mathbb{Z})$: polynomial ring generated by $\Psi_4, \Psi_6, \Psi_{10}, \Psi_{12}, \Psi_{35}$ [Igusa 1960's]
  - $Sp(6, \mathbb{Z})$: polynomial ring with 19 generators [Tsuyumine 1986; Lercier, Ritzenthaler 2019]

- Modular tensors in mathematics and physics
  - holomorphic [van der Geer 2015]; non-holomorphic [Kawazumi 2016]
  - modular graph tensors [ED, Schlotterer 2020]
    - generalizing genus 1 modular graph forms [ED, Green, 2016]
    - generalizing higher genus modular graph functions [ED, Green, Pioline 2018]
      (both of which arise in the $\alpha'$ expansion of string amplitudes)
Two-loop amplitudes for four massless strings

• **Type II in the RNS formulation** (similarly Heterotic) [ED, Phong 2001-2005]

\[ A_4^{(2)} = g_s^2 t_8 \tilde{t}_8 \int_{\mathcal{M}_2} \frac{|d\Omega|^2}{(\det \text{Im } \Omega)^5} \int_{\Sigma^4} \mathcal{Y} \wedge \bar{\mathcal{Y}} \exp \left\{ \sum_{i<j} s_{ij} G(z_i, z_j|\Omega) \right\} \]

★ Kinematics \( s_{ij} = -\frac{\alpha'}{4}(k_i + k_j)^2 \) with factorized \( \varepsilon_i^\mu \tilde{\varepsilon}_i^\nu \) with \( f_{\mu\nu} = \varepsilon_i^\mu k_i^\nu - \varepsilon_i^\nu k_i^\mu \)

★ \( t_8(f_1, f_2, f_3, f_4)\tilde{t}_8(\tilde{f}_1, \tilde{f}_2, \tilde{f}_3, \tilde{f}_4) \sim R^4 \) where \( R \) stands for the Riemann tensor

★ Measure on \( \Sigma^4 \) interlaces kinematic and worldsheet data

\[
\mathcal{Y} = t \Delta(z_1, z_2) \Delta(z_3, z_4) - s \Delta(z_1, z_4) \Delta(z_2, z_3)
\]

\[
\Delta(z, w) = \omega_1(z)\omega_2(w) - \omega_2(z)\omega_1(w)
\]

\[
G(z, w|\Omega) = -\ln |E(z, w|\Omega)|^2 + 2\pi (\text{Im } \Omega)_{IJ}^{-1} \text{Im} \int_w^z \omega_I \text{Im} \int_w^z \omega_J
\]

★ The prime form \( E(z, w|\Omega) \) generalizes \( \vartheta_1(z - w|\tau) \) to higher genus

• **Type II in the pure spinor formulation** [Berkovits 2005; Berkovits, Mafra 2005]

★ generalized to include full supergravity multiplet of external strings
Two-loop amplitude for five massless strings

- **Amplitude for massless NS strings (conserving parity)** [ED, Schlotterer 2021]

\[
\mathcal{A}_5^{(2)} = g_s^2 \int dp^I \int_{\mathcal{M}_2} |d\Omega|^2 \int_{\Sigma^5} \tilde{F}_5 F_5
\]

★ the chiral amplitude is given by

\[
\mathcal{F}_5 = \mathcal{I}_5 \sum_i \left\{ \mathcal{P}^I(z_i) \cdot (\varepsilon_i t_i \mathcal{Y}_I + k_i \chi_i) - \sum_{j \neq i} \mathcal{Y}_I t_{ij} g_{i,j}^I \right\}
\]

★ in terms of the universal chiral Koba-Nielsen factor

\[
\mathcal{I}_5 = \exp \left\{ i \pi \Omega_{IJ} p^I p^J + 2\pi i p^I \sum_i k_i \int_{z_0}^{z_i} \omega_I + \sum_{i<j} s_{ij} \ln E(z_i, z_j) \right\}
\]

★ and universal meromorphic combinations

\[
\mathcal{P}^I(z_i) = 2\pi i p^I + \sum_{j \neq i} g_{i,j}^I k_j
\]

\[
g_{i,j}^I = \partial^I \ln \vartheta[\nu](z_j - z_i|\Omega)
\]

★ kinematic factors adapted to the five-point amplitude

\[
t_1 = t_8(f_2, f_3, f_4, f_5) \quad t_{12} = t_8([f_1, f_2], f_3, f_4, f_5) \quad & \text{cyclic}
\]

★ and holomorphic forms generalizing those of the four-point amplitude

\[
\mathcal{Y}_I = 4s_{12} \omega_I(4) \Delta(5, 1) \Delta(2, 3) + \text{cycl}(1, 2, 3, 4, 5)
\]

\[
\mathcal{I}_{1I} = (t_{12} - 2t_1 \varepsilon_1 \cdot k_2) \left\{ \omega_I(3) \Delta(1, 5) \Delta(2, 4) + \text{cycl}(3, 4, 5) \right\} + \text{cycl}(2, 3, 4, 5)
\]
Spin structure sums for higher multiplicity

- Major efforts go into carrying out the spin structure sums
  - for the four and five genus two string amplitudes using
    - the Riemann identities
    - the Fay trisecant identity (cfr bosonization)
    - and every other trick we could think of
  - for higher multiplicity these methods alone do not appear promising
    - the problem was also considered in [Tsuchiya 2012; 2017; 2022]

- Fermion correlator for spin structure $\delta$ is given by the Szegő kernel
  - Restrict to even spin structures and NS external states
    \[-\langle \psi(z)\psi(w) \rangle = S_\delta(z, w) = \frac{\vartheta[\delta](\int_{w}^{z} \omega|\Omega)}{\vartheta[\delta](0|\Omega)E(z, w)}\]
    - the Riemann $\vartheta$-function for spin structure $\delta = [\delta'|\delta''] \in \{0, \frac{1}{2}\}^{2g}$ is defined by
      \[\vartheta[\delta](\zeta|\Omega) = \sum_{n \in \mathbb{Z}^{2}} \exp\left\{ i\pi(n + \delta')^{t}\Omega(n + \delta') + 2\pi i(n + \delta')^{t}(\zeta + \delta'') \right\}\]
  - String amplitude integrands involve cyclic products of Szegő kernels
    \[C_\delta(z_1, \cdots, z_n) = S_\delta(z_1, z_2)S_\delta(z_2, z_3)\cdots S_\delta(z_{n-1}, z_n)S_\delta(z_n, z_1)\]
    - they also involve other products that may be treated similarly
Spin structure sums for genus 2

- **Theorem 1** [ED, Hidding, Schlotterer 2022]

  The spin structure sum of $C_\delta(z_1, \cdots, z_n)$ for genus 2 and arbitrary $n$ reduces to the spin structure sums for the cases $n = 0, 2, 3, 4$

  - The proof is constructive and formulated in the hyper-elliptic formulation
  - The result will be translated into the $\vartheta$-function formulation
  - The spin structure sums for $n = 0, 1, 2, 3, 4$ are well-known

- **Every genus two surface $\Sigma$ is hyper-elliptic**

  - namely a double cover of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$
  - ramified over 6 branch points $u_1, \cdots, u_6$
  - points $z \in \Sigma$ parametrized by $z = (x, s)$ where $s^2 = (x - u_1) \cdots (x - u_6)$
  - Moduli space $M_2$ isomorphic to $\{u_1, \cdots, u_6\}/\text{SL}(2, \mathbb{C}) \times S_6$
Sketch of proof of the Theorem

★ An even spin structure $\delta$ is isomorphic to a $3 + 3$ partition of branch points

$$\{u_1, \ldots, u_6\} = A \cup B \quad A \cap B = \emptyset \quad |A| = |B| = 3$$

(an odd spin structure is isomorphic to a 1+5 partition)

★ The Szegö kernel is given in terms of this partition by

$$S_\delta(z_1, z_2) = \frac{s_A(x_1)s_B(x_2) + s_B(x_1)s_A(x_2)}{2(x_1 - x_2)} \left[ \frac{dx_1 \, dx_2}{s(x_1) \, s(x_2)} \right]^{\frac{1}{2}}$$

where $s_A(x)s_B(x) = s(x)$ and $s_A(x)^2$ and $s_B(x)^2$ are polynomials given by

$$s_A(x)^2 = \prod_{r \in A} (x - u_r) \quad s_B(x)^2 = \prod_{r \in B} (x - u_r)$$

★ The cyclic product of Szegö kernels is thus given by (using $x_{n+1} = x_1$)

$$C_\delta(z_1, \ldots, z_n) = \frac{\prod_{i=1}^n (s_A(x_i)s_B(x_{i+1}) + s_B(x_i)s_A(x_{i+1}))}{2^n \, x_1 x_2 x_3 \cdots x_n} \frac{dx_1 \cdots dx_n}{s(x_1) \cdots s(x_n)}$$

★ All spin structure dependence is contained in polynomials with $2m \leq n$

$$Q_\delta(i_1, \ldots, i_m | j_1, \ldots, j_m) = \prod_{\alpha=1}^m s_A(x_{i\alpha})^2 s_B(x_{j\alpha})^2 + (A \leftrightarrow B)$$
Sketch of proof of the Theorem (cont’d)

• **Lemma 1**

  All spin structure dependence of $Q_\delta$ is polynomial in $\ell_\delta^{11}, \ell_\delta^{12} = \ell_\delta^{21}, \ell_\delta^{22}$

  \[
  \begin{align*}
  \ell_\delta^{11} &= \frac{1}{4} \alpha_2 \beta_2 - \frac{3}{20} \mu_4 \\
  \ell_\delta^{12} &= \frac{1}{4} (\alpha_1 \beta_2 + \alpha_2 \beta_1) - \frac{9}{40} \mu_3 \\
  \ell_\delta^{22} &= \frac{1}{4} \alpha_1 \beta_1 - \frac{3}{20} \mu_2
  \end{align*}
  \]

  \[
  \begin{align*}
  s_A(x)^2 &= x^3 - \alpha_1 x^2 + \alpha_2 x - \alpha_3 \\
  s_B(x)^2 &= x^3 - \beta_1 x^2 + \beta_2 x - \beta_3 \\
  s(x)^2 &= x^6 - \mu_1 x^5 + \cdots - \mu_5 x + \mu_6
  \end{align*}
  \]

• **Lemma 2: The trilinear relations**

  Every trilinear $\ell_\delta^{a_1 a_2} \ell_\delta^{a_3 a_4} \ell_\delta^{a_5 a_6}$ may be expressed as a polynomial of total degree two in the combinations $\ell_\delta^{11}, \ell_\delta^{12}$ and $\ell_\delta^{22}$ whose coefficients are polynomials in $\mu_1, \cdots, \mu_6$.

• **Combining Lemmas 1 and 2 implies that all spin structure dependence of $Q_\delta$ and $C_\delta$ is given by a quadratic polynomial in $\ell_\delta^{11}, \ell_\delta^{12}, \ell_\delta^{22}$ with coefficients that depend only on $\mu_i$.**

• **The spin structure sums of the linears $\ell_\delta^{a_1 a_2}$ and of the bilinears $\ell_\delta^{a_1 a_2} \ell_\delta^{a_3 a_4}$ are determined by $n$-point functions with $n \leq 4$, which concludes the proof of the Theorem.**
SL(2, C) tensorial structure of the trilinear relations

• Component form of the trilinear relations e.g.

\[
(\ell^{11}_\delta)^3 = \frac{\mu_4(\ell^{11}_\delta)^2}{20} - \frac{\mu_5\ell^{11}_\delta\ell^{12}_\delta}{4} + \frac{\mu_6\ell^{11}_\delta\ell^{22}_\delta}{4} - \frac{\mu_4\ell^{12}_\delta}{50} - \frac{9\mu_3\mu_5\ell^{11}_\delta}{160} + \frac{3\mu_2\mu_6\ell^{11}_\delta}{20} + \frac{\mu_4\mu_5\ell^{12}_\delta}{40}
\]

\[
- \frac{9\mu_3\mu_6\ell^{12}_\delta}{80} + \frac{\mu_2\ell^{12}_\delta}{16} + \frac{3\mu_4\mu_6\ell^{22}_\delta}{20} - \frac{3\mu_4^3}{2000} + \frac{9\mu_3\mu_4\mu_5}{1600} - \frac{3\mu_2\mu_5^2}{320} + \frac{81\mu_3^2\mu_6}{6400} + \frac{9\mu_2\mu_4\mu_6}{400}
\]

• The \( \ell^{ab}_\delta \) transform under the 3-dimensional irrep of SL(2, C) by

\[
\ell^{ab}_\delta \rightarrow J g^a_c g^b_d \ell^{cd}_\delta \quad g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SL}(2, \mathbb{C}) \quad J = \prod_{j=1}^{6} (\gamma u_j + \delta)^{-1}
\]

• The trilinear relations in SL(2, C) tensorial form

\[
7 \quad \ell^{(a_1a_2}\ell^{a_3a_4}\ell^{a_5a_6)}_\delta = M_1^{b_1b_2(a_1...a_4} \ell^{a_5a_6)}_\delta \ell^{c_1c_2}_\delta \varepsilon_{b_1c_1} \varepsilon_{b_2c_2} + \cdots
\]

\[
3 \quad (\det \ell^a_d) \ell^{a_1a_2}_\delta = \frac{3}{2} M_1^{a_1a_2b_1...b_4} \ell^{c_1c_2}_\delta \ell^{c_3c_4}_\delta \varepsilon_{b_1c_1} \cdots \varepsilon_{b_4c_4} + \cdots
\]

* where \( M_1 \) is the symmetric rank 6 tensor under SL(2, C) with components

\[
M_1^{111111} = \mu_6 \quad M_1^{111112} = \frac{\mu_5}{6} \quad M_1^{111122} = \frac{\mu_4}{15} \cdots
\]
**Sp(4, ℤ) tensorial structure of the trilinear relations**

- **Correspondence between hyper-elliptic and \( \vartheta \)-function formulations**
  - via standard Thomae formulas and holomorphic 1-forms \( \omega_I \)
    \[
    \varpi_1 = \frac{dx}{s(x)} \quad \varpi_2 = -\frac{x \, dx}{s(x)} \quad \omega_I(z) = \varpi_a(z) \, \sigma^a_I
    \]
  - we obtain the modular tensors \( \mathcal{L}_\delta \) and \( \mathcal{M}_1 \) (\( \delta \) transforms)
    \[
    \mathcal{L}_{\delta}^{ab} = \sigma^a_I \, \sigma^b_J \, \mathcal{L}_{\delta}^{IJ} \quad \mathcal{M}_{1}^{a_1 \cdots a_6} = (\det \sigma)^{-2} \, \sigma_1^{a_1} \cdots \sigma_6^{a_6} \, \mathcal{M}_{1}^{I_1 \cdots I_6}
    \]
    \[
    \mathcal{L}_{\delta}^{IJ} = \frac{\pi}{5i} \partial^{IJ} \ln \left\{ \vartheta[\delta](0)^{20} \right\} \quad \mathcal{M}_{1}^{I_1 \cdots I_6} = \Psi_{10}^{-\frac{1}{2}} \partial^{I_1} \vartheta[\nu_1](0) \cdots \partial^{I_6} \vartheta[\nu_6](0)
    \]
  - where \( \nu_1, \cdots, \nu_6 \) are the six (distinct) odd spin structures

- **Trilinear relations are between \( Sp(4, ℤ) \) modular tensors**
  \[
  \mathcal{L}_{\delta}^{I_1 I_2} \mathcal{L}_{\delta}^{I_3 I_4} \mathcal{L}_{\delta}^{I_5 I_6} = \mathcal{M}_1^{J_1 J_2 (I_1 \cdots I_4} \mathcal{L}_{\delta}^{I_5 I_6) \mathcal{L}_{\delta}^{K_1 K_2} \varepsilon_{J_1 K_1} \varepsilon_{J_2 K_2} + \cdots
  \]
  \[
  (\det \mathcal{L}_{\delta}) \mathcal{L}_{\delta}^{I_1 I_2} = \frac{3}{2} \, \mathcal{M}_1^{I_1 I_2 J_1 \cdots J_4} \, \mathcal{L}_{\delta}^{K_1 K_2} \mathcal{L}_{\delta}^{K_3 K_4} \varepsilon_{J_1 K_1} \cdots \varepsilon_{J_4 K_4} + \cdots
  \]
Spin structure sums for genus $g \geq 2$

- **Generic surfaces for genus** $g \geq 3$ are no longer hyper-elliptic

- **Theorem 2** [ED, Hidding, Schlotterer] in progress

  The spin structure sum of $C_\delta(z_1, \cdots, z_n)$ for arbitrary genus and arbitrary $n$

  reduces to the spin structure sums of $z_i$-independent modular tensors

- **Proof:** is constructive by a descent method
Summary and outlook

• Two-loop superstring amplitudes
  ★ explicit summation over even spin structures
  ★ paves the way to higher multiplicity amplitudes
  ★ relate kinematics to QFT amplitudes?

• Higher loops
  ★ even spin structure dependence reduced to modular tensors
  ★ what is the dimension and structure of modular tensor spaces?
  ★ which subspace is needed for string amplitudes?
  ★ can one build an efficient library?
Happy Birthday Paolo