Shortcuts to String Amplitudes

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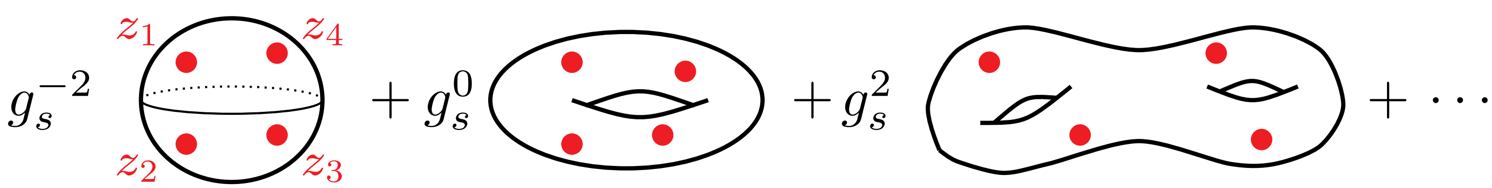
Perturbative string amplitudes

- **Sums over Riemann surfaces**
  - ⋆ expansion in the string coupling g_s
    $=$ expansion in the genus or equivalently number of loops



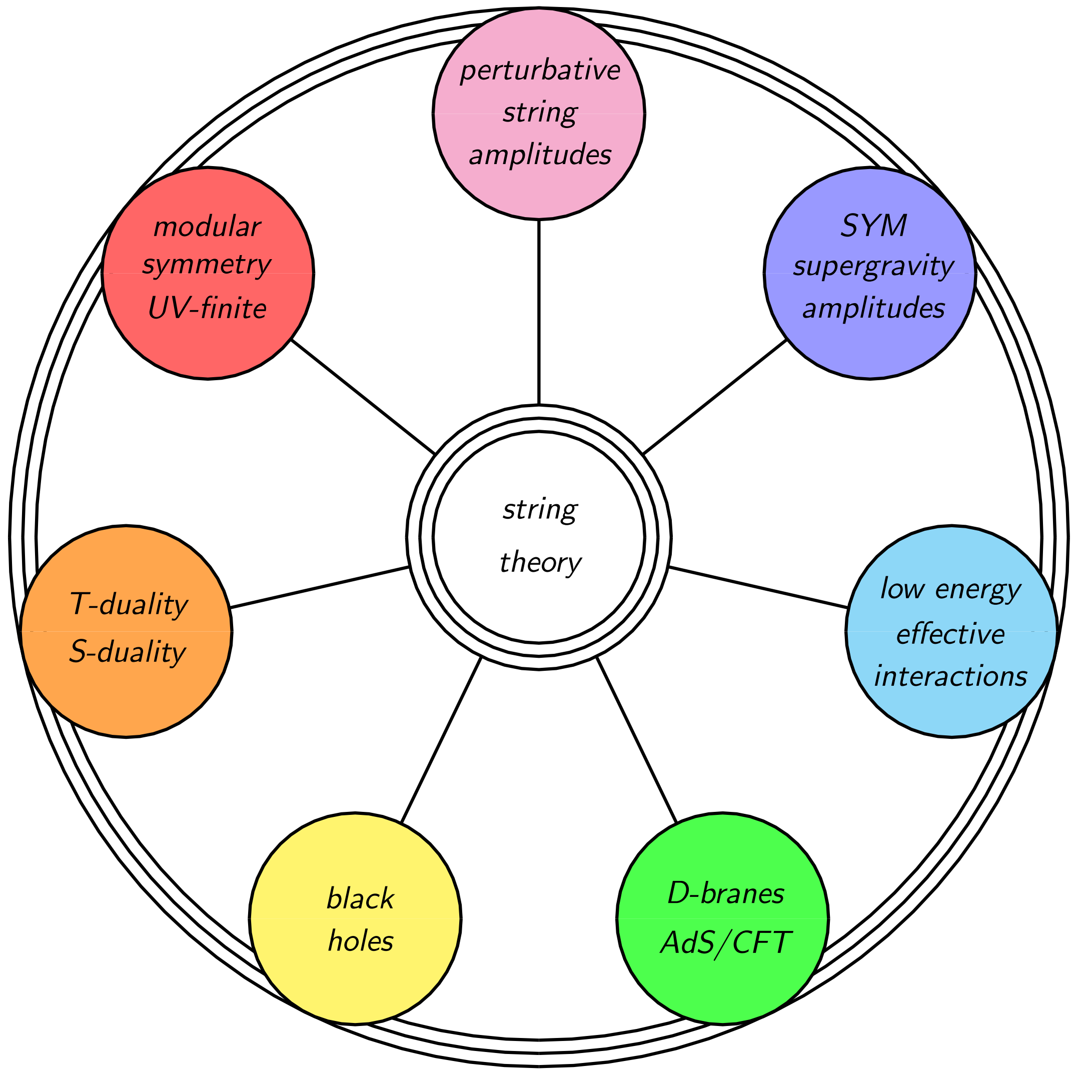
  - ⋆ each vertex point z_i represents an incoming or outgoing string
  - ⋆ for a given physical process, the number of vertex points N is fixed

- **For each genus h integrate**
  - ⋆ the N vertex points over the compact Riemann surface
  - ⋆ over the moduli space $\mathcal{M}_h$ of compact Riemann surfaces
  - ⋆ $N \leq 3$ amplitudes vanish by supersymmetry

- **Complexity increases rapidly with both h and N**

Motivation

- **High energy behavior of strings**
  - ⋆ UV-finiteness of quantum gravity
- **Effective interactions induced by strings on supergravity**
  - ⋆ Interplay with non-perturbative dynamics of string theory
  - ⋆ Matching S-duality and supersymmetry predictions in Type IIB
  - ⋆ Experimentally observable corrections ?
- **Interplay with QFT amplitudes**
  - ⋆ super-Yang-Mills
  - ⋆ supergravity
  - ⋆ gravitational waves
- **Mathematical aspects of string theory**
  - ⋆ modular forms
  - ⋆ super-moduli space



experiment – NATURE

What's new ?

- **Two-loop superstring amplitudes for 5 massless strings**
  - ⋆ Chiral splitting and pure spinors [ED, Mafra, Pioline, Schlotterer 2020]
  - ⋆ RNS first principles calculation [ED, Schlotterer 2021] in preparation

- **Known earlier**
  - ⋆ tree-level and one-loop 4-string amplitudes
    - – Type II [Green, Schwarz 1982]
    - – Heterotic [Gross, Harvey, Martinec, Rohm 1985]
  - ⋆ tree-level and one-loop amplitudes for more external states
    e.g. [Schlotterer, Stieberger 2012; Mafra, Schlotterer 2018]
  - ⋆ two-loop 4-string amplitudes
    - – Type II and Heterotic [D'Hoker, Phong 2001-2005]
    - – with external fermions [Berkovits 2005; Berkovits, Mafra 2005]

Tree-level amplitudes

- **The Riemann surface is a sphere** $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$
  - ⋆ Type II tree-level 4-string amplitude

$$\mathcal{A}_4^{(0)} = \frac{1}{g_s^2} \frac{t_8 \tilde{t}_8}{stu} \frac{\Gamma(1-s)\Gamma(1-t)\Gamma(1-u)}{\Gamma(1+s)\Gamma(1+t)\Gamma(1+u)}$$

  - ⋆ s, t, u are dimensionless Mandelstam $s = s_{12} = -\frac{\alpha'}{4}(k_1 + k_2)^2$ etc
  - ⋆ in a basis of polarization tensors $\varepsilon_i^\mu \tilde{\varepsilon}_i^\nu$ with $f_i^{\mu\nu} = \varepsilon_i^\mu k_i^\nu - \varepsilon_i^\nu k_i^\mu$

$$t_8 = \mathrm{tr}(f_1 f_2 f_3 f_4) - \tfrac{1}{4}\mathrm{tr}(f_1 f_2)\mathrm{tr}(f_3 f_4) + \mathrm{cycl}(2,3,4)$$

  - ⋆ in terms of the Riemann tensor $\mathcal{R}$ we have $t_8 \tilde{t}_8 \sim \mathcal{R}^4$

- **Low energy expansion**

$$\mathcal{A}_4^{(0)} = \frac{1}{g_s^2} \frac{t_8 \tilde{t}_8}{stu} \Big[\underbrace{\frac{1}{stu}}_{\mathcal{R}} + \underbrace{2\zeta(3)}_{\mathcal{R}^4} + \underbrace{\zeta(5)(s^2+t^2+u^2)}_{D^4\mathcal{R}^4} + \underbrace{2\zeta(3)^2 stu}_{D^6\mathcal{R}^4} + \cdots \Big]$$

  - ⋆ normalizes S-duality invariant effective interactions [Green, Gutperle 1997]
  - ⋆ exhibits uniform transcendentality

One-loop amplitudes

- **Type II one-loop four-graviton amplitude** (Green, Schwarz, 1982)

$$\mathcal{A}_4^{(1)} = t_8 \tilde t_8 \int_{\mathcal{M}_1} \frac{d^2\tau}{\tau_2^2} \prod_{k=1}^4 \int_\Sigma \frac{d^2 z_k}{\tau_2} \exp\Big\{ \sum_{1 \leq i<j \leq 4} s_{ij}\, g(z_i - z_j|\tau) \Big\}$$

  ⋆ Moduli space $\mathcal{M}_1 = \{\tau \in \mathbb{C},\, \tau_2 = \mathrm{Im}\, \tau > 0\}/SL(2,\mathbb{Z})$
  ⋆ Torus $\Sigma = \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$ with local complex coordinates $z, \bar z$
  ⋆ Scalar Green function on the torus $g(z|\tau)$

- **The integral cannot be evaluated explicitly**
  ⋆ is absolutely convergent only for $\mathrm{Re}\,(s_{ij}) = 0$
  ⋆ analytic continuation to $s_{ij} \in \mathbb{C}$ exists $\Longrightarrow$ branch cuts [ED, Phong 1994]

- **Low energy expansion** [Green, Russo Vanhove 2008]

$$\mathcal{A}_4^{(1)} = \frac{t_8\, \tilde t_8}{stu} \Big[\frac{0}{stu} + \frac{2\pi^2}{3} + 0 \cdot (s^2+t^2+u^2) + \frac{2\pi^2}{9} \zeta(3) stu + s^4 \ln(s) + \cdots \Big]$$

$\mathcal{R}$ $\quad$ $\mathcal{R}^4$ $\quad$ $D^4\mathcal{R}^4$ $\quad$ $D^6\mathcal{R}^4$ $\quad$ branch cuts

  ⋆ matches S-duality invariant effective interactions [Green, Gutperle, Vanhove 1997]
  ⋆ exhibits uniform transcendentality [ED, Green 2019], [ED, Geiser 2021] in preparation

Higher number of loops/genus

- **How to generalize the genus-one construction to higher genus ?**
  - ⋆ recall the genus-one generating function

$$\int_{\mathcal{M}_1} \frac{d^2\tau}{\tau_2^2} \prod_{i=1}^{4} \int_{\Sigma} \frac{d^2 z_i}{\tau_2} \, \exp\left\{ \sum_{1 \leq i < j \leq 4} s_{ij} \, g(z_i - z_j | \tau) \right\}$$

- **Compact Riemann surface Σ of genus $h \geq 2$ and N external states**
  - ⋆ we need a scalar Green function $G(z_i, z_j|\Sigma)$
  - ⋆ a measure $d\mu_N$ on Σ^N
  - ⋆ a measure $d\mu_\Sigma$ on moduli space $\mathcal{M}_h$ (for RNS on super-moduli space)

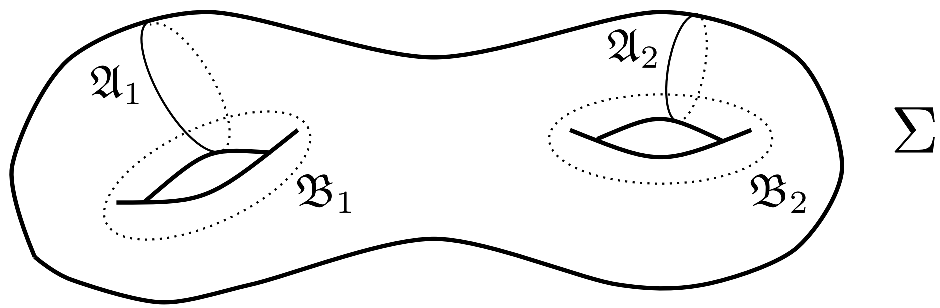
$$\int_{\mathcal{M}_2} d\mu_\Sigma \int_{\Sigma^N} d\mu_N \, \exp\left\{ \sum_{1 \leq i < j \leq N} s_{ij} \, G(z_i, z_j|\Sigma) \right\}$$

Compact Riemann surfaces Σ of genus h

- **Homology and cohomology**

  ⋆ One-cycles $H_1(\Sigma,\mathbb{Z}) \approx \mathbb{Z}^{2h}$ with intersection pairing $\mathfrak{J}(\cdot,\cdot) \to \mathbb{Z}$

  ⋆ Canonical basis $\mathfrak{J}(\mathfrak{A}_I,\mathfrak{A}_J) = \mathfrak{J}(\mathfrak{B}_I,\mathfrak{B}_J) = 0$, $\mathfrak{J}(\mathfrak{A}_I,\mathfrak{B}_J) = \delta_{IJ}$ for $1 \leq I, J \leq h$



- **Modular group** $Sp(2h,\mathbb{Z})$

  ⋆ map $H_1(\Sigma,\mathbb{Z}) \to H_1(\Sigma,\mathbb{Z})$ which leaves $\mathfrak{J}(\cdot,\cdot)$ invariant

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad M^t \mathfrak{J} M = \mathfrak{J} \qquad \begin{pmatrix} \mathfrak{B} \\ \mathfrak{A} \end{pmatrix} \to M \begin{pmatrix} \mathfrak{B} \\ \mathfrak{A} \end{pmatrix}$$

Compact Riemann surfaces Σ of genus h (cont'd)

- **Canonical dual basis of holomorphic one-forms ω_I in $H^{(1,0)}(\Sigma)$**

$$\oint_{\mathfrak{A}_I} \omega_J = \delta_{IJ} \qquad\qquad \oint_{\mathfrak{B}_I} \omega_J = \Omega_{IJ}$$

  ⋆ Period matrix Ω obeys Riemann relations $\Omega^t = \Omega, \ \mathrm{Im}\,(\Omega) > 0$

- **Siegel half space**

  ⋆ The space of all $h \times h$ positive symmetric matrices with complex entries (not necessarily arising as a period matrix of a Riemann surface)

$$\mathcal{H}_h = \{\Omega \in \mathbb{C}^{h^2}, \, \Omega^t = \Omega, \, \mathrm{Im}\,(\Omega) > 0\} = Sp(2h, \mathbb{R})/U(h)$$

  ⋆ possesses a unique $Sp(2h, \mathbb{R})$-invariant Kähler metric and measure $d\mu_\Omega$

- **Moduli spaces**

  ⋆ $\mathcal{M}_2 = \mathcal{H}_2/Sp(4, \mathbb{Z})$ (minus diagonal Ω) so that $d\mu_\Omega \to d\mu_\Sigma$

  ⋆ $\mathcal{M}_3 = \mathcal{H}_3/Sp(6, \mathbb{Z})$ (minus diagonal, orbifold hyper-elliptic) so that $d\mu_\Omega \to d\mu_\Sigma$

  ⋆ for $h \geq 4$ we have $\mathcal{M}_h \neq \mathcal{H}_h/Sp(2h, \mathbb{Z})$ = Schottky problem

Two-loop four-graviton Type II amplitude

- **Computed in RNS formulation** [ED & Phong 2001-2005]
  - ⋆ The final result is remarkably simple

$$\mathcal{A}_4^{(2)} = g_s^2\, t_8 \tilde t_8 \int_{\mathcal{M}_2} d\mu_\Sigma \int_{\Sigma^4} \frac{\mathcal{Y} \wedge \bar{\mathcal{Y}}}{(\det \mathrm{Im}\, \Omega)^2} \exp\Big\{ \sum_{1 \leq i < j \leq 4} s_{ij}\, G(z_i, z_j|\Omega) \Big\}$$

  - ⋆ Measure on Σ^4 interlaces kinematic and worldsheet data

$$\begin{aligned} \mathcal{Y} &= t\Delta(z_1, z_2)\Delta(z_3, z_4) - s\Delta(z_1, z_4)\Delta(z_2, z_3) \\ \Delta(z, w) &= \omega_1(z)\omega_2(w) - \omega_2(z)\omega_1(w) \\ G(z, w|\Omega) &= -\ln |E(z, w|\Omega)|^2 + 2\pi(\mathrm{Im}\,\Omega)^{-1}_{IJ}\, \mathrm{Im} \int_w^z \omega_I\, \mathrm{Im} \int_w^z \omega_J \end{aligned}$$

  - ⋆ prime form $E(z, w) \sim z - w$ is holomorphic [Fay 1973]

- **Low energy expansion**

$$\mathcal{A}_4^{(2)} = g_s^2 t_8 \tilde t_8 \Big[0 + \frac{2\pi^4}{135}(s^2 + t^2 + u^2) + \frac{2\pi^4}{15} stu + \cdots \Big]$$

  - ⋆ vanishing of $\mathcal{R}^4$ [ED, Phong 2005]
  - ⋆ coefficient of $D^4\mathcal{R}^4$ [ED, Gutperle, Phong 2005]
  - ⋆ coefficient of $D^6\mathcal{R}^4$ [ED, Green, Pioline, Russo 2014]

Beyond

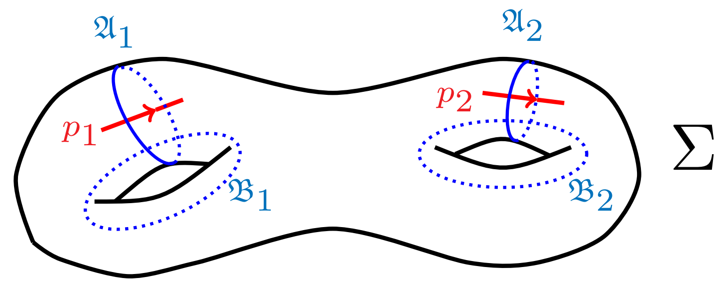
- **More external states**
  - ⋆ tree-level and one-loop: general construction available
    [Mafra, Stieberger, Schlotterer 2011; Mafra, Schlotterer 2018]
  - ⋆ two-loop 5-string amplitude: leading low energy effective interaction
    [Gomez, Mafra, Schlotterer 2015]
    - – limited by the inability to evaluate correlators in the pure spinor formulation
    - – full amplitude was not available

- **Higher number of loops**
  - ⋆ three-loop 4-string amplitude
    - – guessing the RNS measure [ED, Phong 2006], [Matone, Volpato 2008]
      [Cacciatori, Della Piazza, van Geemen [2008]
    - – attempt at "guessing" the amplitude failed [ED, Phong 2011] unpublished
    - – leading low energy effective interaction via pure spinors [Gomez, Mafra 2014]
      but divergences in zero-mode integrations in the pure spinor formulation
    - – attempt at "guessing" the amplitude partially successful [ED, Pioline 2019] unpublished
    - – very recently proposal [Geyer, Monteiro, Stark-Muchão] to appear

Chiral splitting and loop momenta

- **Loop momenta**
  - ⋆ Momentum flowing through a curve is the line integral of the $\partial_z x^\mu$
  - ⋆ Given a canonical homology basis there is a natural choice of loop momenta

$$p_I^\mu = \frac{1}{2\pi} \oint_{\mathfrak{A}_I} \partial_z x^\mu = -\frac{1}{2\pi} \oint_{\mathfrak{A}_I} \partial_{\bar z} x^\mu$$



- **Chiral splitting theorem** [ED, Phong 1988-89]
  - ⋆ The physical amplitude is given by pairing left and right chiral blocks

$$\mathcal{A}_N^{(2)} = \int_{\mathbb{R}^{20}} dp \int_{\mathcal{M}_2} d^6\Omega \int_{\Sigma^N} \mathcal{F}_N(\varepsilon, k, p|z, \Omega) \overline{\tilde{\mathcal{F}}_N(\tilde\varepsilon, k, p|z, \Omega)}$$

  - ⋆ chiral blocks $\mathcal{F}_N$ and $\tilde{\mathcal{F}}_N$ **meromorphic** in z_i and moduli Ω
    (actually on super-Riemann surface and supermoduli space in RNS)

  - ⋆ factorize $\mathcal{F}_N = \mathcal{K}_N \mathcal{I}_N$ into the "chiral Koba-Nielsen factor" $\mathcal{I}_N$

$$\mathcal{I}_N = \exp\left\{ i\pi \Omega_{IJ} p_I \cdot p_J + 2\pi i \sum_j k_j \cdot p_I \int_{z_0}^{z_j} \omega_I - \sum_{i<j} s_{ij} \ln E(z_i, z_j|\Omega) \right\}$$

Properties of chiral blocks

- **The exponential factor $\mathcal{I}_N$ is universal** (bosonic, Type II, Heterotic)
  - ⋆ holomorphic in z_i (away from $z_i = z_j$) and Ω (away from degeneration nodes)
  - ⋆ invariant under $N!$ permutations of z_i
  - ⋆ Green function G and measure are recovered upon integrating out p_I and combining the prime form $E(z_i, z_j)$ with the Abelian integrals

- **Invariance under homology shifts** [ED, Phong 1988-89]
  - ⋆ Moving a vertex point z_i around an $\mathfrak{A}_I$ or $\mathfrak{B}_I$ cycle

$$
\begin{aligned}
e^{-2\pi i k_j \cdot p^J} \mathcal{F}_N(\varepsilon, k_i, p | z_i + \delta_{ij} \mathfrak{A}_J, \Omega) &= \mathcal{F}_N(\varepsilon, k_i, p^I | z_i, \Omega) \\
\mathcal{F}_N(\varepsilon, k, p^I - \delta^I_J k_j | z_i + \delta_{ij} \mathfrak{B}_J, \Omega) &= \mathcal{F}_N(\varepsilon, k_i, p^I | z_i, \Omega)
\end{aligned}
$$

  - ⋆ Physical amplitude $\mathcal{A}_N^{(2)}$ is invariant, using translation invariance of $\int dp$

- **The prefactor $\mathcal{K}_N$ depends on the specific string theory**
  - ⋆ meromorphic $(1,0)$-form in each vertex z_i
  - ⋆ invariant under homology shifts acting on both z_i and p
  - ⋆ invariant under $N!$ permutations of z_i

Obtaining $\mathcal{K}_N$

- **Direct calculation in the RNS formulation**
  - ⋆ For 4-string amplitude $\mathcal{K}_4 = t_8 \mathcal{Y}$ [ED, Phong 2005]
  - ⋆ For 5-string amplitude $\mathcal{K}_5$ [ED, Schlotterer 2021] - in preparation

- **Could one have guessed/restricted the general form of $\mathcal{K}_4$?**
  - ⋆ From supersymmetry $\mathcal{K}_4 = t_8 \mathcal{Y}$ with $\mathcal{Y}$ independent of p
  - ⋆ No poles at $s_{ij} = 0$ realized for $\mathcal{Y}$ holomorphic
  - ⋆ Identity $\Delta(1,2)\Delta(3,4) + \text{cycl}(2,3,4) = 0$ implies $\mathcal{Y}$ must depend on s_{ij}
  - ⋆ Unique holomorphic, permutation-invariant, combination

$$\mathcal{Y} \sim (k_1 - k_2) \cdot (k_3 - k_4) \Delta(1,2)\Delta(3,4) + \text{cycl}(2,3,4)$$

- **One can similarly guess/restrict the general form of $\mathcal{K}_5^{\text{odd Parity}}$**
  - ⋆ Zero-mode counting in RNS $\mathcal{K}_5^{\text{odd P}} \sim \epsilon_{10}^{\mu}(\varepsilon_1, f_2, f_3, f_4, f_5)$
  - ⋆ Only possible contraction is with loop momenta and Abelian differential

$$\mathcal{K}_5^{\text{odd P}} \sim (k_4 \cdot k_5) p_I^{\mu} \epsilon_{10}^{\mu}(\varepsilon_1, f_2, f_3, f_4, f_5) \omega_I(1) \Delta(2,3)\Delta(4,5) + \text{perms}$$

  - ⋆ would vanish without $k_3 \cdot k_4$ factor due to identities
  - ⋆ borne out by explicit calculation – see the sequel

Ingredients of the pure spinor formulation

- **The Green-Schwarz formulation** [Green, Schwarz 1983]
  - ⋆ Space-time vector x^μ and spinor θ^α; manifestly supersymmetric
  - ⋆ Second class constraints; quantized only in light-cone gauge
- **The (non-minimal) pure spinor fields** [Berkovits 2000-2005]
  - ⋆ free-field theory of space-time vector x^μ and spinors θ^α or d_α etc.
  - ⋆ worldsheet $(0,0)$-forms: matter x^μ, θ^α; ghosts $\lambda^\alpha, \bar\lambda_\alpha, r_\alpha$
  - ⋆ worldsheet $(1,0)$-forms: matter d_α; ghosts $w_\alpha, \bar w^\alpha, s^\alpha$
  - ⋆ pure spinor relations $\lambda\gamma^\mu\lambda = \bar\lambda\gamma^\mu\bar\lambda = \bar\lambda\gamma^\mu r = 0$
    - "linearized" by decomposing $\mathbf{16}_{\rm SO(10)} = (\mathbf{1} \oplus \bar{\mathbf{5}} \oplus \mathbf{10})_{\rm SU(5)}$
- **BRST operator** $Q^2 = 0$

$$Q = \oint \Big(\lambda^\alpha d_\alpha + \bar w^\alpha r_\alpha \Big) \qquad \begin{cases} Q\theta^\alpha = \lambda^\alpha \\ Qx^\mu = \lambda\gamma^\mu\theta \end{cases} \qquad \begin{cases} Q\lambda^\alpha = 0 \\ Q^2 x^\mu = \lambda\gamma^\mu\lambda \end{cases}$$

  - ⋆ vertex operators

$$U = \partial_z \theta^\alpha A_\alpha(x,\theta) + d_\alpha W^\alpha(x,\theta) + \cdots$$

  - ⋆ BRST-invariance implies space-time supersymmetry

Ingredients of the pure spinor formulation (cont'd)

⋆ Composite b-ghost defined by $\{Q,b\}=T$
- Easy to solve for G^α defined by $\{Q,G^\alpha\}=\lambda^\alpha T$
- The field b is obtained using terminating descent [Berkovits 2005]

$$b=s^\alpha \partial \bar\lambda_\alpha + \frac{\bar\lambda_\alpha G^\alpha}{(\lambda\bar\lambda)} + \frac{(\bar\lambda\gamma^{\mu\nu\rho} r)}{192(\lambda\bar\lambda)^2}(d\gamma_{\mu\nu\rho}d) + \text{ three terms}$$

⋆ Difficulties: λ^α and $\bar\lambda_\alpha$ are independent holomorphic fields
- singularities due to vanishing $(\lambda\bar\lambda)$ and integration as $\lambda^\alpha,\bar\lambda_\alpha\to\infty$
- general Wick contractions exceedingly complicated

⋆ Zero modes come to the rescue for low genus and few external states
- $(0,0)$-forms $\theta^\alpha,\lambda^\alpha,\bar\lambda_\alpha,r_\alpha$ have 16 zero modes (modulo constraints)
- $(1,0)$-forms $d_\alpha,w_\alpha,\bar w^\alpha,s^\alpha$ have $16\times h$ zero modes (modulo constraints)
- The zero modes of λ^α and $\bar\lambda_\alpha$ may be identified as complex conjugates $\Longrightarrow$ denominators positive definite

⋆ Perform Wick contractions leaving only zero modes integrations
- governed by non-trivial BRST cohomology of ghost number 3

$$\langle(\lambda\gamma^\mu\theta)(\lambda\gamma^\nu\theta)(\lambda\gamma^\rho\theta)(\theta\gamma_{\mu\nu\rho}\theta)\rangle=1$$

Two-loop four-string amplitude (revisited)

- **The four-string amplitude** [Berkovits 2005]

$$\mathcal{K}_4 = T_{1,2|3,4} \Delta(1,3)\Delta(2,4) + \mathrm{cycl}(2,3,4)$$

⋆ BRST-invariant
⋆ Fully permutation invariant using properties of T and Δ
⋆ For external gravitons reduces to

$$T_{1,2|3,4} = (k_1 - k_3) \cdot (k_2 - k_4)\, t_8(f_1, f_2, f_3, f_4)$$

Two-loop five-string amplitude

- **Counting of zero modes shows** $\mathcal{K}_5 = \mathcal{K}_5^V + \mathcal{K}_5^S$
  - $\mathcal{K}_5^V$ linear in p
  - $\mathcal{K}_5^S$ independent of p

  - $\mathcal{K}_5^V$ evaluated using zero mode integrations and BRST closure
  - $\mathcal{K}_5^S$ involves Wick contractions that are not available

    [Gomez, Mafra, Sclotterer 2015]

- **Use pure spinor cohomology to evaluate** $\mathcal{K}_5^V$

$$\mathcal{K}_5^V = 2\pi i p_I^\mu T^\mu_{1,2,3|4,5} \omega_I(2)\Delta(3,4)\Delta(5,1) + \text{cycl}(1,2,3,4,5)$$

⋆ construct $T^\mu_{1,2,3|4,5}$ by using factorization

$$T^\mu_{1,2,3|4,5} = A_1^\mu T_{2,3|4,5} + A_2^\mu T_{3,1|4,5} + A_3^\mu T_{1,2|4,5} + W^\mu_{1,2,3|4,5}$$

⋆ determine W by vanishing BRST modulo gauge transformations

$$QT^\mu_{1,2,3|4,5} = ik_1^\mu T_{2,3|4,5} + ik_2^\mu T_{3,1|4,5} + ik_3^\mu T_{1,2|4,5}$$

Two-loop five-string amplitude (cont'd)

- **While $\mathcal{K}_5^V$ is BRST invariant, it is not invariant under homology shits**
  - ⋆ Consider the derivative $\partial_{z_i} \ln \mathcal{I}_5 = ik_i \cdot \mathcal{P}(z_i)$ of chiral Koba-Nielsen

$$\mathcal{P}(z_i) = 2\pi i p_I \omega(z_i) + \sum_{j \neq i} k_j \, \partial_i \ln E(z_i, z_j)$$

  - ⋆ $\mathcal{P}(z_i)$ invariant under all homology shifts

- **Promote $2\pi i p_I \omega(z_i) \to \mathcal{P}(z_i)$ in the expression for $\mathcal{K}_5^V$**

$$\mathcal{K}_5^V \to \tilde{\mathcal{K}}_5^V = \mathcal{P}(z_2) T^{\mu}_{1,2,3|4,5} \Delta(3,4)\Delta(5,1) + \text{cycl}(1,2,3,4,5)$$

  - ⋆ While $\tilde{\mathcal{K}}_5^V$ is homology shift invariant, it is no longer BRST-invariant

- **Determine $\mathcal{K}_5^S$ by requiring BRST closure of $\mathcal{K}_5 = \mathcal{K}_5^V + \mathcal{K}_5^S = \tilde{\mathcal{K}}_5^V + \tilde{\mathcal{K}}_5^S$**
  - ⋆ Require that $\tilde{\mathcal{K}}_5^S$ satisfy
    - – independence of p
    - – BRST closure $Q\tilde{\mathcal{K}}_5^V + Q\tilde{\mathcal{K}}_5^S = 0$

$\Longrightarrow$ unique solution [ED, Mafra, Pioline, Schlotterer 2020]

Space-time structure of the amplitude

- **Chiral blocks of the amplitude**

$$\mathcal{F}_5 = \mathcal{K}_5 \, \mathcal{I}_5 \qquad\qquad \mathcal{K}_5 = \mathcal{K}_5^V + \mathcal{K}_5^S$$

- **The difference between Type IIA and Type IIB**

$$\begin{array}{lll} \text{IIA}: & \mathcal{K}_5 = \mathcal{K}_5^{\text{even}} + \mathcal{K}_5^{\text{odd}} & \tilde{\mathcal{K}}_5 = \tilde{\mathcal{K}}_5^{\text{even}} - \tilde{\mathcal{K}}_5^{\text{odd}} \\ \text{IIB}: & \mathcal{K}_5 = \mathcal{K}_5^{\text{even}} + \mathcal{K}_5^{\text{odd}} & \tilde{\mathcal{K}}_5 = \tilde{\mathcal{K}}_5^{\text{even}} + \tilde{\mathcal{K}}_5^{\text{odd}} \end{array}$$

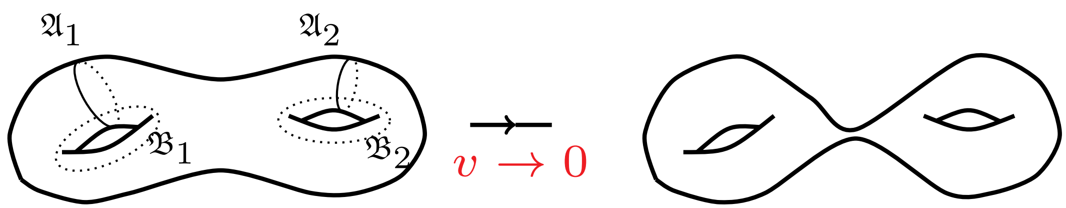
- **Kinematic structure of NS-NS bosonic part** $\{i, j, \ell, m, n\} = \{1, 2, 3, 4, 5\}$

$$\begin{array}{rcl} \mathcal{K}_5^{V\,\text{even}} & \supset & (p_I \varepsilon_i k_i k_j) \, t_8(k_j, k_\ell, k_m, k_n) \\ \mathcal{K}_5^{V\,\text{odd}} & \supset & p_I \cdot \epsilon_{10}(\varepsilon_i, f_j, f_\ell, f_m, f_n) \, (k_m \cdot k_n) \\ \mathcal{K}_5^S & \supset & f_i^{\mu\nu} k_j^\mu k_\ell^\nu \, t_8(f_j, f_\ell, f_m, f_n), \\ & & t_8([f_i, f_j], f_\ell, f_m, f_n) \end{array}$$

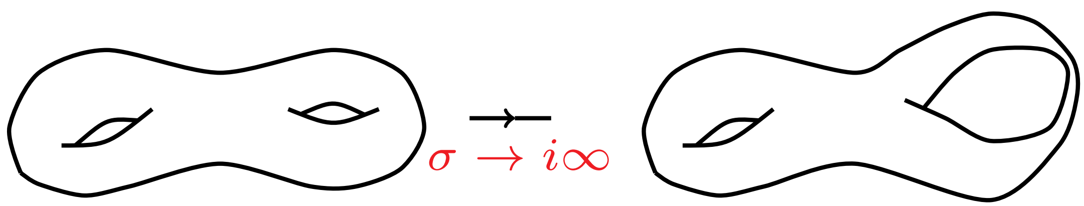
$\Longrightarrow$ confirms our guess for $\mathcal{K}_5^{\text{odd P}}$

Degenerations of genus-two Riemann surfaces

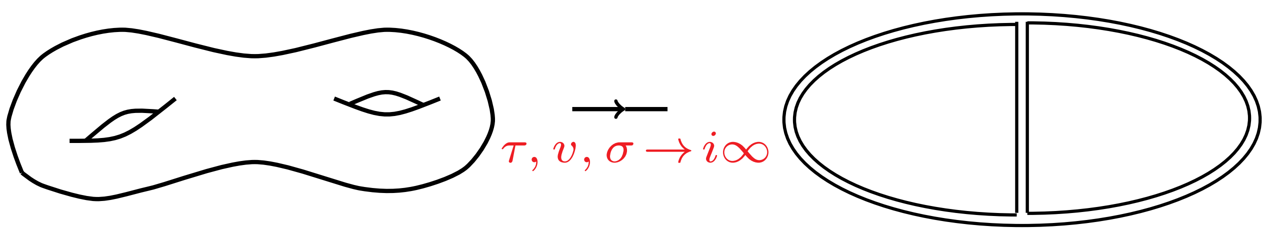
- **Locally parametrize** $\mathcal{M}_2$ **by period matrix** $\Omega = \begin{pmatrix} \tau & v \\ v & \sigma \end{pmatrix}$ **with** $\tau, v, \sigma \in \mathbb{C}$
  - *Separating degeneration*



  - *Non-separating degeneration*



  - *Maximal degeneration (or "tropical limit") to QFT*



Non-separating degeneration

- **Σ degenerates to torus Σ_1 of modulus τ with punctures p_a, p_b**
  - $\star$ keep the cycles $\mathfrak{A}_1, \mathfrak{B}_1, \mathfrak{A}_2$ fixed, and let $\mathfrak{B}_2 \to \infty$ as $\mathrm{Im}\,(\sigma) \to \infty$

- **Modular group $Sp(4,\mathbb{Z})$ reduces to $SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^3$ Fourier-Jacobi group**
  $=$ the subgroup that leaves $\mathfrak{B}_2$ invariant

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{Z});\; m,n,r \in \mathbb{Z} \qquad \begin{cases} \tau \to \tau' = (\alpha\tau+\beta)/(\gamma\tau+\delta) \\ v \to v' = (v + m\tau + n)/(\gamma\tau+\delta) \\ \sigma \to \sigma' = \sigma + r - \gamma v^2/(\gamma\tau+\delta) \end{cases}$$

  - $\star$ The degeneration parameter σ is not invariant under $SL(2,\mathbb{Z})$
  - $\star$ Siegel modular forms degenerate to Jacobi forms (Eichler & Zagier 1985)

- **There exists a real-valued $SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^3$ invariant parameter $t > 0$**

$$t = \frac{\det(\mathrm{Im}\ \Omega)}{\mathrm{Im}\ \tau} = \mathrm{Im}\,\sigma - \frac{(\mathrm{Im}\,v)^2}{\mathrm{Im}\,\tau} \qquad \Omega = \begin{pmatrix} \tau & v \\ v & \sigma \end{pmatrix}$$

  - $\star$ the non-separating node is characterized by $t \to \infty$
    (an analogous invariant parameter exists for arbitrary genus)

Degeneration of the integrand on moduli

- **Recall e.g. the integrand of genus-two 4-string amplitude**

$$\int_{\Sigma^4} \frac{\mathcal{Y} \wedge \overline{\mathcal{Y}}}{(\det Y)^2} \exp\left\{ \sum_{i<j} s_{ij} G(z_i, z_j) \right\} = \sum_{w=0}^{\infty} \frac{1}{w!} \mathcal{B}_w(s_{ij}|\Omega)$$

$\star$ Taylor series in s_{ij} into terms of degree w

$$\mathcal{B}_w(s_{ij}|\Omega) = \int_{\Sigma^4} \frac{\mathcal{Y} \wedge \overline{\mathcal{Y}}}{(\det Y)^2} \left(\sum_{i<j} s_{ij} G(z_i, z_j) \right)^w$$

- The expansion of $\mathcal{B}_w(s_{ij}|\Sigma)$ near the non-separating node is given by *a Laurent polynomial of* **finite** *degree* $(w, -w)$ *in* t [ED, Green, Pioline 2017]

$$\mathcal{B}_w(s_{ij}|\Omega) = \sum_{k=-w}^{w} \mathcal{B}_w^{(k)}(s_{ij}|v, \tau)\, t^k + \mathcal{O}(e^{-2\pi t})$$

The coefficients are invariant under $SL(2, \mathbb{Z}) \ltimes \mathbb{Z}^2 \subset Sp(4, \mathbb{Z})$

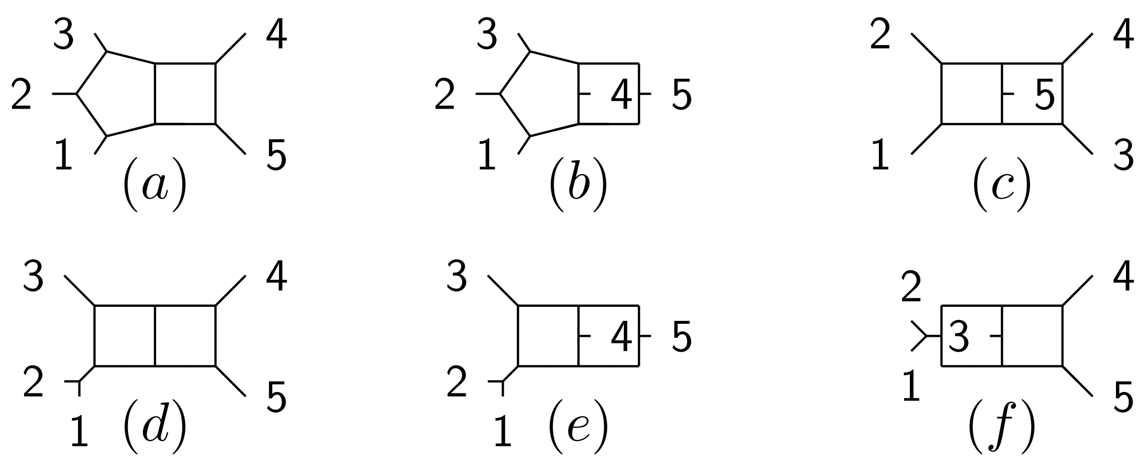
$$\mathcal{B}_w^{(k)}(s_{ij}|v', \tau') = \mathcal{B}_w^{(k)}(s_{ij}|v, \tau)$$

These are "elliptic modular graph functions". [ED, Kleinschmidt, Schlotterer 2020]

- **Readily generalizes to the 5-string amplitude**

The supergravity limit of the five-string amplitude

- **Graphs contributing in maximally supersymmetric QFTs**
  - ⋆ $\mathcal{N}=4$ SYM and $\mathcal{N}=8$ supergravity [Carrasco, Johansson 2011]
    - – double copy and color kinematic duality [Bern, Carrasco, Johansson 2008, 2010]
    - – pure spinor methods [Mafra, Schlotterer 2015]



  - ⋆ Perfect match with the tropical limit of the string amplitude

- **Type IIB string kinematics decomposes onto SYM field theory kinematics**
  - ⋆ for $U(1)_Y$-conserving amplitudes only

$$\mathcal{A}_5^{(2)}\Big|_{U(1)_Y} = \begin{pmatrix} \tilde{A}_{\rm SYM}(1,2,3,5,4) \\ \tilde{A}_{\rm SYM}(1,3,2,5,4) \end{pmatrix}^t \mathcal{M} \begin{pmatrix} A_{\rm SYM}(1,2,3,4,5) \\ A_{\rm SYM}(1,3,2,4,5) \end{pmatrix}$$

Summary and outlook

- **Effective Interactions induced on supergravity** [ED, Mafra, Pioline, Schlotterer 2020]
  - ⋆ Implications of (non-linear) supersymmetry
    - – coefficient of $D^2\mathcal{R}^5$ and $D^4\mathcal{R}^4$ match, as at tree-level and one-loop
    - – coefficient of $D^4\mathcal{R}^5$ and $D^6\mathcal{R}^4$ match, as at tree-level and one-loop
  - ⋆ Appearance of $U(1)_Y$ violating effective interactions
    - – absent at tree-level since Type IIB supergravity is $U(1)_Y$ invariant
    - – generated at one and two loops e.g. $\phi\mathcal{R}^4$ and $\mathcal{G}^2\mathcal{R}^3$
  - $\Longrightarrow$ much remains to be done ! (we only scratched the surface)

- **The odd-parity 5-string amplitude**
  - ⋆ in RNS involves super-moduli space for odd spin structures
    $=$ very little understood, but now we have the final answer !

- **Modular graph tensors** [Kawazumi 2010], [ED, Schlotterer 2020]
  - ⋆ generalize modular graph forms from the genus-one case
  - ⋆ new types of identities relating them

- **Toroidal compactifications**
  - ⋆ In chiral splitting formulas $\int dp \to \sum_p$ over space-time torus momenta