# Introduction to Superstring Perturbation Theory

## Part I: the bosonic string

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# Strings

- **A string is a 1-dimensional object**
  - open string = topology of an interval;
  - closed string = topology of a circle;
  - physical size Planck length $\ell_P \approx 10^{-33}\mathrm{cm} \approx 10^{-19}\times$ size of the proton.

- **Goal: unified theory of particle physics and gravity**
  - elementary particles correspond to strings and their excited states;
  - consistently with quantum mechanics and general relativity;
  - remarkably unique structure.

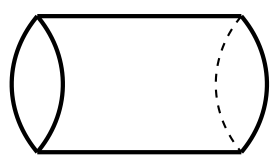
# String Topology

- **Consistent interacting string theories**
  - closed and open strings
  - only closed strings (focus of these lectures)

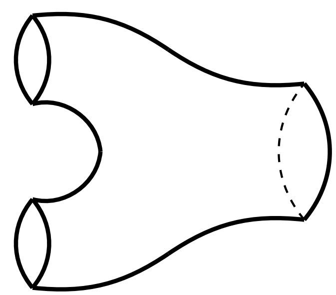
    (note: Type II closed string theories have open strings in the presence of D-branes)

- **Assume strings live in a physical space-time $M$**
  - $M$ may be a manifold or an orbifold
  - space-time visible to us is 4-dim.
  - but superstring theory predicts 10-dim.

- **Under time-evolution strings sweep out a 2-dim. surface**



time-evolution
(freely propagating)



basic interaction
(purely topological)

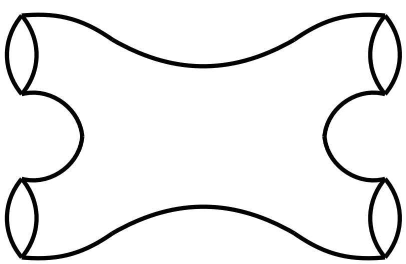
# Perturbative Quantum Probability Amplitudes

- **Quantum theory predicts the probability amplitude for a process**
  (whose norm square is the probability)

- **Closed oriented string perturbation theory**

  = sum over all oriented surfaces
  with given boundary components for initial and final strings

  – The only remaining topological characterization is the genus $h \geq 0$
  - probability amplitude includes sum over all genera
  - weighed by a factor $g_s^{2h-2}$ where $g_s$ is the "string coupling"

$g_s^{-2}$

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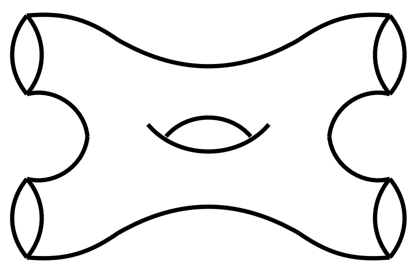
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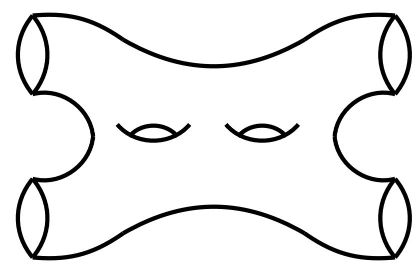

$+ \cdots$

# Perturbative Probability Amplitudes (cont'd)

- **Probability amplitude is schematically given by**

$$A = \sum_{h=0}^{\infty} g_s^{2h-2} \times \text{partial amplitude at given genus h}$$

  - The perturbative expansion in $g_s$ is only asymptotic
  - Instantons contribute $\approx e^{-1/g_s^2}$.

# String Data (closed oriented strings)

- **Assume fixed space-time $M$, with fixed metric $G$**
  - Physical space-time has Minkowski signature metric
  - Starting point for string theory is a Riemannian metric
    (if needed to be analytically continued to Minkowski signature)

- **The 2-dimensional worldsheet $\Sigma$ is mapped into space-time $M$**
  - The space of all such maps $x : \Sigma \to M$ is denoted $\mathrm{Map}(\Sigma)$.

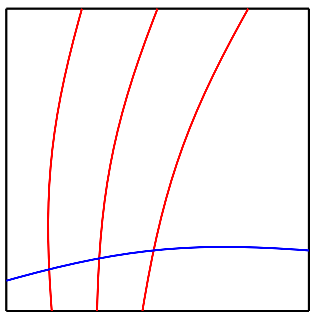
- **Riemannian metric $G$ induces a Riemannian metric $x^*(G)$ on $\Sigma$**
  - Hence $\Sigma$ is a Riemann surface

- **Polyakov formulation invokes an independent metric**
  - Riemannian metric $g$ on $\Sigma$
  - Denote the $\infty$-dim. Riemannian manifold of such metrics by $\mathrm{Met}(\Sigma)$
  - Probability amplitude obtained by weighed sum over $h, g, x$

$$A = \sum_{h=0}^{\infty} g_s^{2h-2} \int_{\mathrm{Met}(\Sigma)} Dg \int_{\mathrm{Map}(\Sigma)} Dx \; e^{-I_x[x,g]}$$

# Group Action of Diff($\Sigma$) $\ltimes$ Weyl($\Sigma$)

- **Fix topology of $\Sigma$ with genus $h$ and no boundaries**
  - Diff$^+$($\Sigma$)  orientation preserving diffeomorphisms of $\Sigma$
  - Weyl ($\Sigma$)  $g \to g' = e^\sigma g$ with $\sigma : \Sigma \to \mathbb{R}$
- **Both act on the space $\mathrm{Met}(\Sigma)$**



$\mathrm{Met}(\Sigma)$

$\mathrm{Met}(\Sigma)$/Diff($\Sigma$) $\ltimes$ Weyl($\Sigma$)$= \mathcal{M}_h$

orbits of Diff($\Sigma$) $\ltimes$ Weyl($\Sigma$)

- $\mathcal{M}_h$ **= moduli space of compact Riemann surfaces (genus $h$, no boundaries)**
  $=$ space of complex structures
  $=$ space of conformal classes

$$\dim_{\mathbb{C}} \mathcal{M}_h = \begin{cases} 0 & h = 0 \\ 1 & h = 1 \\ 3h-3 & h \geq 2 \end{cases}$$

# The worldsheet action $I_x$ and the measure $Dx$

- **Basic Criteria**
  - Intrinsic = invariant under Diff$^+(\Sigma)$
  - lead to a well-defined QFT (renormalizable)

- **Basic Example** in space-time $M$ with fixed Riemannian metric $( \, , \, )_G$

$$I_x[x,g] = \int_\Sigma (dx, *dx)_G = \int_\Sigma d\mu_g \, g^{mn} \partial_m x^\mu \partial_n x^\nu G_{\mu\nu}(x)$$

$$( \, g = g_{mn} d\xi^m d\xi^n, \quad \partial_m = \partial / \partial \xi^m, \quad d\mu_g = \sqrt{\det g_{mn}} d^2 \xi \, )$$

  - "non-linear sigma-model" in physics terminology
  - It's stationary points are harmonic maps $x : \Sigma \to M$
  - The measure is governed by the $L^2$-norm

$$(\delta x, \delta x)_G = \int_\Sigma d\mu_g \, \delta x^\mu \, \delta x^\nu \, G_{\mu\nu}(x)$$

  - manifestly intrinsic
  - renormalizable in a generalized sense

# Weyl($\Sigma$)-invariance

- **The action $I_x$ is Weyl-invariant**
  - but the measure $Dx$ is not invariant
  - which gives rise to a "Weyl-anomaly"
    $=$ symmetry of classical action not preserved by quantization

- **Also $Dg$ is not Weyl-invariant, but the combination in the amplitude**

$$Dg\, e^{-W[g]} \qquad\qquad e^{-W[g]} = \int_{\mathrm{Map}(\Sigma)} Dx \; e^{-I_x[x,g]}$$

**will be Weyl-invariant provided $W[g]$ obeys**

$$\frac{dW[g_t]}{dt} = \frac{c}{24} \int_\Sigma d\mu_{g_t}\, R_{g_t}\, \sigma$$

- for family of Weyl transforms $g_t = e^{t\sigma} g$ of $g$ and given $\sigma : \Sigma \to \mathbb{R}$
- with $c = 26$

**namely when $I_x$ defined a CFT with central charge $c = 26$**

# Weyl($\Sigma$)-invariance (cont'd)

- **Equivalently $I_x$ defines a CFT invariant under** $\mathrm{Vir} \oplus \tilde{\mathrm{Vir}}$
  - with central charges $c = \tilde{c} = 26$
  - generators of $\mathrm{Vir}$ satisfy the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0}$$

- **Basic Examples**
  - Flat $M = \mathbb{R}^d$, $G_{\mu\nu} = \delta_{\mu\nu}$ with $c = d = 26$
  - Compactification on a torus $M = \mathbb{R}^{26-n} \times T^n$ with constant metric $G$
  - Metric $G_{\mu\nu}$ slowly varying on scale $\ell_P$, approximately $R_{\mu\nu} = 0$

- **Consequences**
  - Integral on $\mathrm{Met}(\Sigma)$ will project to integral on $\mathcal{M}_h$;
  - Boundary components of $\Sigma$ (representing $N$ initial and final states)
    - ⋆ may be mapped to $N$ marked points on compact $\Sigma$
    - ⋆ upon inserting **vertex operators** $\mathcal{V}_i[x, g]$ representing physical states

$$A = \sum_{h=0}^{\infty} g_s^{2h-2} \int_{\mathrm{Met}(\Sigma)} Dg \int_{\mathrm{Map}(\Sigma)} Dx \; \mathcal{V}_1[x,g] \cdots \mathcal{V}_N[x,g] \, e^{-I_x[x,g]}$$

# Decomposing the measure $Dg$

- **At any point $g \in \mathrm{Met}(\Sigma)$ the measure $Dg$ factors**

$$\begin{array}{ccccccccc} Dg & = & Z_g & \times & D\sigma & \times & Dv & \times & d\mu_{\mathcal{M}_h} \\ & & \text{Jacobian} & & \text{Weyl} & & \text{Diff}_0 & & \mathcal{M}_h \end{array}$$

  - Infinitesimal Weyl $\quad \delta g_{mn} = \delta\sigma \, g_{mn} \quad \delta\sigma : \Sigma \to \mathbb{R}$
  - Infinitesimal Diff$_0$ $\quad \delta g_{mn} = \nabla_m \delta v_n + \nabla_n \delta v_m \quad \delta v \in T(\Sigma)$

- **Goal**
  - compute $Z_g$
  - formulate $Z_g$ in terms of ghosts
  - omit volume factors $D\sigma \, Dv$ of the group Diff$^+(\Sigma) \ltimes$ Weyl$(\Sigma)$

# Tensor Spaces

- **Orthogonal decomposition of tensor spaces**
  - Denote by $K$ the canonical bundle of $\Sigma$ (holó part of $T^*\Sigma$)
  - Choose local complex coordinates $(z, \bar z)$ for which $g = g_{z\bar z}|dz|^2$
    the sections of $K^m \otimes \bar K^n$ are tensors of the type $t(z,\bar z) dz^m d\bar z^n$
    Using the metric, we identify $K^m \otimes \bar K^n \approx K^{m-n}$
  - On sections $\phi_1, \phi_2 \in K^m$ we have the $L^2$-inner product

$$(\phi_1^*, \phi_2) = \int_\Sigma d\mu_g (g_{z\bar z})^{-m} \phi_1^* \phi_2$$

  - The spaces $K^m$ and $K^n$ with $m \neq n$ are mutually orthogonal

# Tensor Spaces (cont'd)

- **The covariant derivative $\nabla$ on $\phi \in K^m$ decomposes into**

$$\nabla \phi = \nabla_z^{(m)} \phi + \nabla_{\bar z}^{(m)} \phi$$

  - $\nabla_z^{(m)} : K^m \to K^{m+1}$
  - Using the metric $\nabla_{\bar z}^{(m)}$ may be identified with $\nabla^z_{(m)} : K^m \to K^{m-1}$
  - Adjoint operators:

$$\left(\nabla_z^{(m)}\right)^\dagger = -\nabla^z_{(m+1)} \qquad\qquad \left(\nabla^z_{(m)}\right)^\dagger = -\nabla_z^{(m-1)}$$

- **Riemann-Roch theorem**

$$\dim \mathrm{Ker} \nabla_z^{(m)} - \dim \mathrm{Ker} \nabla^z_{(m+1)} = (2m+1)(1-h)$$

- **Vanishing Theorems**

  - $\mathrm{Ker} \nabla_{\bar z}^{(m)} = 0$ for $h \geq 2$ and $m \leq -1$ (no holó vector fields for $h \geq 2$)
  - $\mathrm{Ker} \nabla_{\bar z}^{(m)} = 0$ for $h = 0$ and $m \geq 1$ (no holó forms on the sphere)

# Moduli deformations

- **Orthogonal decomposition of $T_g(\mathrm{Met}(\Sigma))$**

$$T_g(\mathrm{Met}(\Sigma)) = \{\delta\sigma\, g_{mn}\} \oplus \{\text{ traceless } \delta g_{mn}\} = K^0 \oplus K^2 \oplus \bar K^2$$

  - traceless variations : $\delta g = g_{z\bar z}\left(\delta\eta_z{}^{\bar z} dz^2 + \delta\eta_{\bar z}{}^z d\bar z^2\right)$
  - $\mathrm{Diff}_0$ acts on $\delta\eta_z{}^{\bar z}$ by $\delta v^{\bar z} \in \bar K^{-1} \approx K$ via

$$\delta\eta_z{}^{\bar z} = \nabla_z^{(1)} \delta v^{\bar z} \qquad\qquad \nabla_z^{(1)} : \bar K^{-1} \to K \otimes \bar K^{-1} \approx K^2$$

- **The complement of the range of $\nabla_z^{(1)}$ is determined by**

$$\mathrm{Range}\, \nabla_z^{(1)} \oplus \mathrm{Ker}\left(\nabla_z^{(1)}\right)^\dagger = K^2 \qquad\qquad \mathrm{Ker}\left(\nabla_z^{(1)}\right)^\dagger = \mathrm{Ker}\nabla_{\bar z}^{(2)}$$

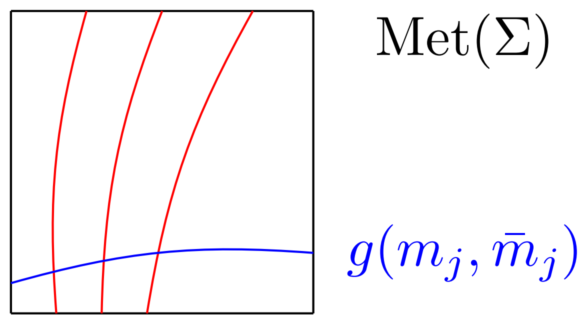
- **Holomorphic quadratic differentials $\phi_j \in \mathrm{Ker}\nabla_{\bar z}^{(2)}$**
  - provide linear forms on the space of $\delta\eta_z{}^{\bar z}$ (using the metric $\approx K^2$)

$$\delta m_j = (\delta\eta, \phi_j) = \int_\Sigma dz d\bar z\, \delta\eta_{\bar z}{}^z\, \phi_{j\ zz}$$

  - Weyl-invariant pairing and vanishes on $\delta\eta \in \mathrm{Range}\nabla_z^{(1)}$
  - Hence we may identify $\mathrm{Ker}\nabla_{\bar z}^{(2)} = T^*_{(1,0)}(\mathcal{M}_h)$.

# Decomposing the measure $Dg$ (cont'd)

- **Parametrize $\mathcal{M}_h$ by a slice in $\mathrm{Met}(\Sigma)$ transverse to Weyl $\ltimes$ Diff$_0$**




orbits of Diff$(\Sigma)$ $\ltimes$ Weyl$(\Sigma)$

# Decomposing the measure $Dg$ (cont'd)

- From the orthogonal decomposition,

$$T_g(\mathrm{Met}(\Sigma)) = \left\{ \delta\sigma \, g_{z\bar z} |dz|^2 \right\} \oplus \left\{ \delta\eta_z{}^{\bar z} dz^2 \right\} \oplus \left\{ \delta\eta_{\bar z}{}^z d\bar z^2 \right\}$$

  - we have the factorization $Dg = D\sigma \, D\eta \, D\bar\eta$
  - $\delta v^z$ together with $\delta m_j$ provide a complete parametrization of $\delta\eta_{\bar z}{}^z$
  - $\delta v^{\bar z}$ together with $\delta \bar m_j$ provide a complete parametrization of $\delta\eta_z{}^{\bar z}$

$$\delta\eta_{\bar z}{}^z = \nabla_{\bar z}^{(-1)} \delta v^z + \sum_j g^{z\bar z} \frac{\partial g_{\bar z \bar z}}{\partial m_j} \delta m_j$$

- Define the linear operator $P_g$ at the metric $g$ by

$$P_g \quad \begin{cases} (\delta v^z, \delta m_j) & \rightarrow \quad \delta\eta_{\bar z}{}^z \\ (\delta v^{\bar z}, \delta \bar m_j) & \rightarrow \quad \delta\eta_z{}^{\bar z} \end{cases}$$

  - For simplicity assume $\chi(\Sigma) < 0$, so that $\mathrm{Ker}\, \nabla_{\bar z}^{(-1)} = 0$
  - Then $P_g$ is invertible, and the Jacobian is given by $Z_g = \mathrm{Det}(P_g)$

$$Dg = \mathrm{Det}(P_g) \times D\sigma \times Dv \times d\mu_{\mathcal{M}_h} \qquad\qquad d\mu_{\mathcal{M}_h} = \prod_j dm_j \, d\bar m_j$$

# Determinants by Grassmann integrals

- Let $M$ be an $n \times n$ complex matrix.
- Introduce Grassmann odd variables $b^i, c^i$, $i = 1, \cdots, n$

$$\{b^i, b^j\} = \{b^i, c^j\} = \{c^i, c^j\} = 0 \text{ for all } i, j = 1, \cdots, n$$

- Berezin integrals $\int db^i \, b^j = \int dc^i \, c^j = \delta_i^j$ with all others vanishing

$$\det M = \int dc^1 \cdots dc^n \, db^1 \cdots db^n \left( \exp \sum_{i,j=1}^{n} b^i M_{ij} c^j \right)$$

# Ghosts

- **Introduce ghost fields as Grassmann odd variables**

$$(\delta v^z, \delta v^{\bar z}) = \delta v^m \quad \rightarrow \quad c^m \in K^{-1} \oplus \bar K^{-1}$$

$$(\delta m_j, \delta \bar m_j) = \delta m_p \quad \rightarrow \quad a_p \quad p = 1, \cdots, \dim_{\mathbb{R}}(\mathcal{M}_h) \qquad \text{Grassmann odd}$$

$$(\delta g_{zz}, \delta g_{\bar z \bar z}) = \delta g_{mn} \quad \rightarrow \quad b_{mn} \in K^2 \oplus \bar K^2 \qquad \text{traceless}$$

- **Represent determinant of the operator $P_g$ by Berezin integral**
  - by immediate generalization of the finite-dimensional case

$$Z_g = \mathrm{Det}(P_g) = \int Db \int Dc \int \prod_p da_p \, e^{-I_{\rm gh}[b,c,g,f]}$$

  - The ghost action is given by

$$I_{\rm gh}[b,c,g,f] = \frac{1}{2\pi} \int_\Sigma d\mu_g \, b_{mn} \nabla^m c^n + \frac{1}{4\pi} \int_\Sigma d\mu_g \, b_{mn} \, f^{mn}$$

  - The slice function $f^{mn}$ is given by

$$f^{mn} = \sum_p a_p \, \mu_p^{mn} \qquad\qquad \mu_p^{mn} = \frac{\partial g^{mn}}{\partial m_p}$$

  - $b, c$ is CFT with $c_{\rm gh} = -26$ : Weyl-invariant requires $c_x = 26$

# Measure on $\mathcal{M}_h$

- **Integration over $a_p$ produces $b$-ghost insertion**

$$\int \prod_p da_p \, e^{-\sum_p a_p(b,\mu_p)} = \prod_p (b,\mu_p)$$

– produces an intrinsic volume form on $\mathcal{M}_h$

$$Dg \int Dx \, e^{-I_x} = D\sigma \, Dv \int D(xbc) \prod_p (b,\mu_p) dm_p \, e^{-I_x[x,g]-I_{\rm gh}[b,c,g,f=0]}$$

- Alternatively: take $xbc$ system with arbitrary $f=\delta g \in T^*{\rm Met}(\Sigma)$ [Witten]

$$F(g|\delta g) = \int D(xbc) \, e^{-I_x[x,g]-I_{\rm gh}[b,c,g,\delta g]}$$

– $F(g|\delta g)$ is a pullback to ${\rm Met}(\Sigma)$ of the top form on $\mathcal{M}_h$
  - ⋆ Invariant under Diff$^+(\Sigma)$
  - ⋆ Depends only on $\delta g \in K^2 \oplus \bar K^2$ since $b_{mn}$ is traceless
  - ⋆ Vanishes upon contraction with $\delta g_{mn} = \nabla_m \delta v^m + \nabla_n \delta v_m$
    (since measure $Dc$ invariant under shifting $c^m \to c^m + \delta v^m$)

– $a_p$ may be interpreted geometrically as $a_p = dm_p \in T^*(\mathcal{M}_h)$

# BRST symmetry

- **The combined $x, b, c$ action and measure have BRST symmetry**
  - generated by a Grassmann odd charge $Q_B$
  - $Q_B^2 = 0$ in Weyl-invariant theory
  - acting by Diff$_0$ with $\delta v^m = c^m$ on non-ghost fields

$$[Q_B, x^\mu] = c^m \partial_m x^\mu \qquad\qquad [Q_B, c^m] = c^n \nabla_n c^m$$

$$[Q_B, b_{mn}] = T_{mn}$$

  - $T_{mn}$ is the stress tensor for $x, b, c$

$$\delta I_x + \delta I_{\rm gh} = \frac{1}{4\pi} \int_\Sigma d\mu_g \, \delta g^{mn} \, T_{mn}$$

- Extended BRST also transforms the metric

$$[Q_B, g_{mn}] = \delta g_{mn} \qquad\qquad [Q_B, \delta g_{mn}] = 0$$

  - $Q_B$ acts on functions of $g$ and $\delta g$ by total differential

$$Q_B = \int_\Sigma d\mu_g \, \delta g_{mn} \frac{\delta}{\delta g_{mn}}$$

  - The form $F(g|\delta g)$ is BRST-invariant $[Q_B, F(g|\delta g)] = 0$

# Vertex operators

- **Probability amplitude in the presence of vertex operators**

$$A = \sum_{h=0}^{\infty} g_s^{2h-2} \int_{\mathrm{Met}(\Sigma)} Dg \int_{\mathrm{Map}(\Sigma)} Dx \, \mathcal{V}_1[x,g] \cdots \mathcal{V}_N[x,g] \, e^{-I_x[x,g]}$$

  - Vertex operators $\mathcal{V}_i[x,g]$ must preserve Diff($\Sigma$) $\ltimes$ Weyl($\Sigma$)
  - Integrated vertex operator of local function $V_i(x,g)$

$$\mathcal{V}_i[x,g] = \int_\Sigma d\mu_g \, V_i(x,g)$$

- **Basic examples in flat $M = \mathbb{R}^{26}$**
  - string state with momentum $k^\mu$ requires factor of $e^{ik\cdot x}$ in $V$
  - $\mathcal{V}$ transforms under representation of $SO(26)$ of string state
  - e.g. graviton vertex operator

$$V(x,g) = \varepsilon_{\mu\nu}(k) \, g^{mn} \partial_m x^\mu \partial_n x^\nu \, e^{ik\cdot x}$$

  - Weyl-invariance requires $k^2 = 0$ and $k^\mu \varepsilon_{\mu\nu}(k) = k^\nu \varepsilon_{\mu\nu}(k) = 0$

  $\approx$ perturbation of metric $G$ by a plane wave $\delta G_{\mu\nu}(x) = \varepsilon_{\mu\nu}(k) \, e^{ik\cdot x}$

# Vertex operators via BRST

- **Gauge-fixing replaces Diff($\Sigma$) $\ltimes$ Weyl($\Sigma$) by BRST**
  - In formulation with ghosts and extended BRST [Witten]

$$F(g|\delta g)_\Omega = \int D(xbc)\, \Omega\, e^{-I_x[x,g]-I_{\rm gh}[b,c,g,\delta g]}$$

  - For any $\Omega$, BRST symmetry of action and measure give

$$dF(g|\delta g)_\Omega + F(g|\delta g)_{[Q_B,\Omega]} = 0$$

  - If $\Omega$ is BRST invariant $[Q_B, \Omega = 0$, then $dF(g|\delta g)_\Omega = 0$

# Simplest vertex operators

- **For $N$ external strings states**

$$\Omega = \mathcal{V}_1(x,b,c,p_1)\cdots\mathcal{V}_N(x,b,c,p_N)$$

– inserted at marked points (or punctures) $p_i$ on $\Sigma$.
  - ⋆ In the presence of punctures, $\mathrm{Diff}_0(\Sigma)$ restricted to $\delta v^m(p_i)=0$.
  - ⋆ Gauge-fixing $\delta v^m \to c^m$ results in requiring $c^m(p_i)=0$
  - ⋆ Since $c^m$ is odd, we have $\delta(c^m(p_i)) = c^z(p_i)c^{\bar z}(p_i)$

– Hence the simplest vertex operators have the form

$$\mathcal{V}_i(x,b,c,p_i) = c^z c^{\bar z} V_i(x,p_i)$$

  - ⋆ where $V_i(x,p_i)$ is independent of $b,c$ ghosts
    (including derivatives of $c$ would make derivatives of $\delta v^m$ vanish)
  - ⋆ BRST-invariance $[Q_B,\mathcal{V}_i]=0$ requires $V_i(x,p_i)\in K\otimes\bar K$

– $F(g|\delta g)_\Omega$ gives string measure on moduli space $\mathcal{M}_{h,N}$
of surfaces of genus $h$ with $N$ punctures.

# Concluding remarks

- **The bosonic string is unphysical**
  - no physical fermions (such as electrons, protons, quarks)
  - tachyonic state (faster than light) : internally inconsistent

# Introduction to Superstring Perturbation Theory

## Lecture II: Superstrings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– Supermoduli –
2015

# Strings with Fermions

- **The bosonic string is not physical**
  - contains no fermions (i.e. space-time spinors)
  - contains a tachyonic state (propagates faster than light)

- **Space-time spinors introduced alternatively by**

  $M = \mathbb{R}^{10}$ Minkowski space-time; $SO(1,9)$ Lorentz group

  ⋆ The Green-Schwarz formulation (and related "pure spinor formulation")
  - Add field $\theta$ which is a space-time-spinor, and scalar on $\Sigma$
  - no supermoduli

  ⋆ The Ramond-Neveu-Schwarz (RNS) formulation
  - Add field $\psi^\mu$ which is a space-time-vector, and a spinor on $\Sigma$
    - ⋆ Two sectors : NS bosons (tensor reps of $SO(1,9)$)
      R fermions (spinor reps of $SO(1,9)$)
    - ⋆ left and right chiralities independent
    - ⋆ summation over spin structures on $\Sigma$
    - ⋆ superconformal symmetry, super Riemann surfaces
    - ⋆ supermoduli !

# Spinor fields on $\Sigma$ and spin structures

- **Let $\Sigma$ be a compact Riemann surface of genus $h$**
  - spin bundle $S$ with $S^2 \approx K$ ($K$ canonical bundle)
  - $2^{2h}$ distinct spin structures

- **A spinor field is a section of $S$**
  - the spinor field $\psi^\mu$ transforms under the vector representation of $SO(1,9)$
  - hence all components of $\psi^\mu$ with $\mu = 1, \cdots, 10$
    are sections of the same $S$, with the same spin structure

# Reducibility of spinor representations

- **Reducibility of spinor representations of $SO(d)$ and $SO(1,d-1)$**
  - depends on the dimension $d$ mod 8
  - depends on the signature of the metric

- **Space-time $M = \mathbb{R}^{10}$ Minkowski $SO(1,9)$**
  - Dirac spinors = 32-dim complex
  - Weyl spinors = 16-dim complex
  - Majorana-Weyl = Weyl + reality condition

- **Worldsheet with Minkowski signature $SO(1,1)$**
  - Weyl = 1-dim complex
  - Majorana-Weyl = 1-dim real

- **Worldsheet with Euclidean signature $SO(2)$**
  - Weyl = 1-dim complex
  - there is no Majorana-Weyl condition

# Independence of worldsheet chiralities

- **With Minkowski signature**
  - worldsheet spinors with opposite chirality $\psi_\pm^\mu$ are *independent*
  - Dirac equation in local coordinates $(\tau, \sigma)$ solved by $\psi_+^\mu(\tau - \sigma)$ and $\psi_-^\mu(\tau + \sigma)$

- **With Euclidean signature (Riemann surfaces)**

  Left (-movers) $\tau + \sigma \to \tilde{z}$

  Right (-movers) $\tau - \sigma \to z$

  - worldsheet spinors $\psi_\pm^\mu$ are complex
  - Dirac equation in local coordinates $(z, \tilde{z})$ solved by $\psi_+^\mu(z)$ and $\psi_-^\mu(\tilde{z})$
  - should be considered as independent (not complex conjugates of one another)

    (cfr "cs" or complex supersymmetric supermanifolds)

- **Different (closed) string theories result from different pairings**

  ⋆ Type 0 has $\psi_-^\mu = (\psi_+^\mu)^*$ : tachyon + no space-time spinors

  ⋆ Heterotic has $\psi_+^\mu(z)$, but no $\psi_-^\mu(\tilde{z})$

  (instead internal degrees of freedom $\lambda_-^\alpha$ with $\alpha = 1, \cdots, 32$)

  ⋆ Type II has independent $\psi_\pm^\mu$, with independent spin structures

# Quantization of worldsheet spinor fields

- **Illustrate**
  - Ramond and Neveu-Schwarz sectors
  - independence of chiralities

- **Dirac action and equation for flat $M = \mathbb{R}^{10}$ with metric $\eta$**
  - All components of $\psi_+^\mu$ are sections of the same spin bundle $S$
  - Let $J$ be the complex structure of $\Sigma$, and $(z, \tilde{z})$ local complex coordinates
  - Dirac action,

$$I_\psi[\psi, J] = \frac{1}{2\pi} \int_\Sigma d\tilde{z} dz \, \psi_+^\mu \partial_{\tilde{z}} \psi_+^\nu \eta_{\mu\nu}$$

  - Dirac equation $\partial_{\tilde{z}} \psi_+^\mu = 0$ locally,
  - but products of operators produce singularities

$$\psi_+^\mu(z) \psi_+^\nu(w) = \frac{\eta^{\mu\nu}}{z - w} + \text{regular}$$

  - each component $\psi^\mu$ generates a CFT with central charge $c = \frac{1}{2}$.

# Quantization of worldsheet spinor fields (cont'd)

- **Quantization on flat cylinder or conformal equivalent on flat annulus**
  - cylinder $w = \tau + i\sigma$ with identification $\sigma \approx \sigma + 2\pi$
  - annulus centered at $z = 0$, conformally mapped by $z = e^w$
  - one-forms related by $dz = e^w \, dw$, spinors by $(dz)^{\frac{1}{2}} = e^{w/2} \, (dw)^{\frac{1}{2}}$
  - fields related by conformal transformation $\psi_{\rm cyl}(z) = e^{w/2} \, \psi_{\rm ann}(w)$

- **Two possible spin structures**

NS $\quad \psi^\mu_{\rm cyl}(\tau, \sigma + 2\pi) = -\, \psi^\mu_{\rm cyl}(\tau, \sigma) \quad$ or $\quad \psi^\mu_{\rm ann}(e^{2\pi i} \, z) \ = +\, \psi^\mu_{\rm ann}(z)$

R $\quad \psi^\mu_{\rm cyl}(\tau, \sigma + 2\pi) = +\, \psi^\mu_{\rm cyl}(\tau, \sigma) \quad$ or $\quad \psi^\mu_{\rm ann}(e^{2\pi i} \, z) \ = -\, \psi^\mu_{\rm ann}(z)$

- **Free field quantization in annulus representation**

NS $\quad \psi^\mu(z) = \sum_{r \in \frac{1}{2} + \mathbb{Z}} b^\mu_r \, z^{-\frac{1}{2} - r} \qquad \{b^\mu_r, b^\nu_s\} = \eta^{\mu\nu} \delta_{r+s,0}$

R $\quad \psi^\mu(z) = \sum_{n \in \mathbb{Z}} d^\mu_n \, z^{-\frac{1}{2} - n} \qquad \{d^\mu_m, d^\nu_n\} = \eta^{\mu\nu} \delta_{m+n,0}$

# Quantization of worldsheet spinor fields (cont'd)

- **Lorentz generators of $SO(1,9)$ :** $[J^{\mu\nu}, \psi^\kappa(z)] = \eta^{\nu\kappa}\psi^\mu(z) - \eta^{\mu\kappa}\psi^\nu(z)$

$$J^{\mu\nu}_{\rm NS} = \sum_{r \in \mathbb{N}-\frac{1}{2}} \left( b^\mu_{-r} b^\nu_r - b^\nu_{-r} b^\mu_r \right)$$

$$J^{\mu\nu}_{\rm R} = \frac{1}{2}[d^\mu_0, d^\nu_0] + \sum_{n \in \mathbb{N}} \left( d^\mu_{-n} d^\nu_n - d^\nu_{-n} d^\mu_n \right)$$

- **Fock space construction produces two sectors**

  ⋆ NS ground state defined by $b^\mu_r |0;{\rm NS}\rangle = 0$ for all $r > 0$
  - $|0;{\rm NS}\rangle$ is unique and in $SO(1,9)$ trivial rep.
  - Fock space generated by linear combinations of $b^{\mu_1}_{-r_1} \cdots b^{\mu_p}_{-r_p} |0;{\rm NS}\rangle$, $r_i > 0$
  - All states in tensor rep. of $SO(1,9)$, and thus all space-time bosons.

  ⋆ R ground state is degenerate $d^\mu_n |0,\alpha;{\rm R}\rangle = 0$ for all $n > 0$
  - $|0,\alpha;{\rm R}\rangle$ transforms under spinor rep. of $SO(1,9)$, states labelled by $\alpha$
  - Fock space generated by linear combinations of $d^{\mu_1}_{-n_1} \cdots d^{\mu_p}_{-n_p} |0,\alpha;{\rm R}\rangle$, $n_i > 0$
  - All states in spinor reps of $SO(1,9)$ are space-time fermions.

# Summation over spin structures

- **Theory with both bosons and fermions requires both NS and R sectors**
  - to include both, one must sum over spin structures of the annulus
- **Type II spin structures of $\psi_\pm^\mu$ are independent of one another**
  - space-time fermions are in the R $\otimes$ NS and NS $\otimes$ R sectors
    which could never arise if spin structures for both chiralities coincides
- **On the torus, viewed as cylinder + identification**
  - spin structures along cycle of cylinder produce R and NS sectors
  - sum over spin structures along conjugate cycle produces GSO-projection
    - ⋆ reduces to half the states in both R and NS sectors
    - ⋆ R-sector: space-time spinor of definite chirality
    - ⋆ NS-sector: eliminates the tachyon
  - natural sum over *all* spin structures
- **On higher genus $h$ surface, sum over all spin structures**
  - with choice of canonical basis $A_I, B_I$ with $I = 1, \cdots, h$ for $H_1(\Sigma, \mathbb{Z})$
  - along A-cycles produces R and NS sectors
  - along B-cycles produces GSO-projection

# Super-conformal Symmetry

- **Why was conformal symmetry needed in the bosonic string ?**
  - Take flat Minkowski $M = \mathbb{R}^{26}$, metric $\eta = \mathrm{diag}(- + \cdots +)$
  - $\Sigma$ with local complex coordinates $(z, \tilde{z})$
  - Maps $x : \Sigma \to M$ satisfy Laplace equation $\partial_{\tilde{z}} \partial_z x^\mu = 0$, $\mu = 1, \cdots, 26$
  - Concentrate on holomorphic field

$$\partial_z x^\mu = \sum_{m \in \mathbb{Z}} x_m^\mu \, z^{-m-1} \qquad [x_m^\mu, x_n^\nu] = m \, \delta_{m+n,0} \, \eta^{\mu\nu} \qquad (x_n^\mu)^\dagger = x_{-n}^\mu$$

  - Ground state satisfies $x_0^\mu |0, k\rangle = k^\mu |0, k\rangle$ and $x_m^\mu |0, k\rangle = 0$ for $m > 0$
  - Fock space (holo sector) generated by $x_{-m_1}^{\mu_1} \cdots x_{-m_p}^{\mu_p} |0, k\rangle$
  - Lowest level $\varepsilon_\mu(k) x_{-1}^\mu |0, k\rangle$ has norm $||\varepsilon_\mu(k) x_{-1}^\mu |0, k\rangle||^2 = \varepsilon^\mu(k) \varepsilon^\nu(k) \eta_{\mu\nu}$
  - component $\varepsilon^\mu = \delta^{\mu,0}$ produces negative norm state
    $=$ inconsistent with quantum mechanics

# Superconformal Symmetry (cont'd)

- **Conformal symmetry guarantees the existence of Virasoro algebra**

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}$$

  - for the bosonic string $c = 26$ and

$$L_m = \sum_{n \in \mathbb{Z}} \frac{1}{2} x_{m-n} \cdot x_n \qquad\qquad L_0 = \frac{1}{2}x_0^2 + \sum_{n \in \mathbb{N}} x_{-n} \cdot x_n$$

  - Physical states $|\psi\rangle$ are subject to $(L_0 - 1)|\psi\rangle = L_m|\psi\rangle = 0$ for $m \in \mathbb{N}$
  - Eliminates all negative norm states;
  - Null states, associated with space-time gauge invariance, decouple;
  - e.g. on lowest level states $\varepsilon(k) \cdot x_{-1}|0, k\rangle$
    - ⋆ $L_1$ constraint imposes $k \cdot \varepsilon(k) = 0$
    - ⋆ $L_0$ constraint imposes $k^2 = 0$
    - ⋆ $L_m$ constraints are automatic for $m \geq 2$

- **Additional negative norm states arise from the time-component of $\psi^\mu$**
  - both in the R and NS sectors
  - requires an additional local worldsheet fermionic symmetry

# Superconformal Symmetry (cont'd)

- **Superconformal algebra extends Virasoro $L_m$ with generator $G_r$**
  - additional structure relations

$$[L_m, G_r] = (\frac{1}{2}m - r)G_{r+m}$$

$$\{G_r, G_s\} = 2L_{r+s} + \frac{c}{3}(r^2 - \varepsilon)\delta_{r+s,0}$$

  - with central charge $c = 15$, with $\varepsilon = 0$ for R, $\varepsilon = 1/2$ for NS

  - e.g. For R-sector in flat $\mathbb{R}^{10}$

$$L_m = \sum_{n \in \mathbb{Z}} \frac{1}{2} (x_{m-n} \cdot x_n + d_{m-n} \cdot d_n) \qquad G_m = \sum_{n \in \mathbb{Z}} x_{-n} \cdot d_{m+n}$$

  - Physical R-states $|\psi; R\rangle$ obey $L_m|\psi; R\rangle = G_m|\psi; R\rangle = 0$ for $m \geq 0$
  - Eliminates all negative norm states and decouples null-states

  - e.g. on lowest level R-states $\sum_\alpha u^\alpha(k)|0, k\rangle \otimes |0, \alpha; R\rangle$
    - ⋆ $G_0$ constraint imposes Dirac equation $k_\mu \Gamma^\mu u(k) = 0$ since $d_0^\mu = \Gamma^\mu$
    - ⋆ $L_m, G_m$ constraints are automatic for $m \geq 1$

# Superconformal symmetry and super Riemann surfaces

- **Conformal symmetry** (with zero central charge)
  - realized on ordinary Riemann surface $\Sigma$ by vector fields
    $L_m = -z^{m+1}\partial_z$, with $m \in \mathbb{Z}$
  - transition functions on $\Sigma$ are holomorphic functions of $z$

- **Superconformal symmetry** (with zero central charge)
  - complex supermanifold $\Sigma$ of dimension $1|1$ locally isomorphic to $\mathbb{C}^{1|1}$
  - local complex coordinates $z|\theta$
  - tangent bundle $T\Sigma$ generated by $\partial_z$ and $\partial_\theta$
  - general vector field takes the form

$$V(z,\theta)\partial_z + W(z,\theta)\partial_\theta \qquad\qquad \begin{aligned} V(z,\theta) &= V_0(z) + \theta V_\theta(z) \\ W(z,\theta) &= W_0(z) + \theta W_\theta(z) \end{aligned}$$

  - twice as many components as needed for superconformal symmetry
  - need a restriction to superconformal transition functions

# Superconformal symmetry and super Riemann surfaces

- **Superconformal transformations on $z|\theta$ viewed as**
  - leaving the superderivative $D_\theta = \partial_\theta + \theta \partial_z$ invariant up to scaling

$$f : (z|\theta) \to (\hat z|\hat\theta) \qquad\qquad D_\theta = \partial_\theta + \theta\partial_z$$
$$D_{\hat\theta} = \partial_{\hat\theta} + \hat\theta \partial_{\hat z}$$

  - then $f$ is superconformal provided $D_\theta = F(z,\theta) D_{\hat\theta}$ for some $F$
  - Explicit condition for superconformal $f$ is given by $D_\theta \hat z = \hat\theta D_\theta \hat\theta$
  - Two vector fields out of four become dependent on the other two
  - Vector fields generate superconformal symmetry group (NS)

$$\begin{aligned} L_m &= -z^{m+1}\partial_z - \frac{1}{2}(m+1) z^m \,\theta\, \partial_\theta \\ G_r &= z^{r+\frac{1}{2}}(\partial_\theta - \theta\partial_z) \end{aligned}$$

- $T\Sigma$ has a subbundle $\mathcal{D}$ of rank $0|1$ which is "completely non-integrable"
  - if $D$ is a section of $\mathcal{D}$, then $D^2$ is nowhere proportional to $D$
  - section here $D = D_\theta$ and $D_\theta^2 = \partial_z$ which is nowhere proportional to $D_\theta$

# Superstring worldsheets

- **Heterotic**

  - LEFT $\partial_{\tilde{z}} x^\mu$, no $\psi_-^\mu$, internal $\lambda_-^\alpha$, $\alpha = 1, \cdots, 32$
  - RIGHT $\partial_z x^\mu$ and $\psi_+^\mu$

    ⋆ left : conformal, ordinary Riemann surface $\Sigma_L$, coord. $\tilde{z}$
    ⋆ right : superconformal, super Riemann surface $\Sigma_R$, coord. $z|\theta$
    Super Riemann surface $\Sigma \subset \Sigma_L \times \Sigma_R$ of dimension $1|1$
    subject to $\Sigma_{\rm red} = {\rm diag}(\Sigma_{L\,\rm red} \times \Sigma_{R\,\rm red})$ : $\tilde{z} = \bar{z} + {\rm nilpotent}$

- **Type II**

  - LEFT $\partial_{\tilde{z}} x^\mu$ and $\psi_-^\mu$
  - RIGHT $\partial_z x^\mu$ and $\psi_+^\mu$

    ⋆ Left : superconformal, super Riemann surface $\Sigma_L$, coord. $\tilde{z}|\tilde{\theta}$
    ⋆ Right : superconformal, super Riemann surface $\Sigma_R$, coord. $z|\theta$
    Super Riemann surface $\Sigma \subset \Sigma_L \times \Sigma_R$
    subject to $\Sigma_{\rm red} = {\rm diag}(\Sigma_{L\,\rm red} \times \Sigma_{R\,\rm red})$ : $\tilde{z} = \bar{z} + {\rm nilpotent}$

# Worldsheet action for Heterotic strings

- Worldsheet is $\Sigma \subset \Sigma_L \times \Sigma_R$
  - conformal structure $\tilde J$ for $\Sigma_L$, with local coordinates $\tilde z$
  - superconformal structure $\mathcal{J}$ for $\Sigma_R$ with local coordinates $z|\theta$

$$\begin{aligned} X^\mu(\tilde z, z, \theta) &= x^\mu(\tilde z, z) + \theta \psi_+^\mu(\tilde z, z) \\ \Lambda^\alpha(\tilde z, z, \theta) &= \lambda_-^\alpha(\tilde z, z) + \theta \ell^\alpha(\tilde z, z) \end{aligned}$$

- Superconformal invariant action ("matter part")

$$I_M[X^\mu, \Lambda^\alpha, \tilde J, J] = \frac{1}{2\pi} \int_\Sigma [d\tilde z dz|d\theta] \Big[ \partial_{\tilde z} X^\mu D_\theta X^\nu \eta_{\mu\nu} + \sum_\alpha \Lambda^\alpha D_\theta \Lambda^\alpha \Big]$$

- Integrating out $\theta$, we recover familiar action,

$$I_M = \frac{1}{2\pi} \int d\tilde z dz \Big[ \partial_{\tilde z} x^\mu \partial_z x^\nu \eta_{\mu\nu} + \psi_+^\mu \partial_{\tilde z} \psi_+^\nu \eta_{\mu\nu} + \sum_\alpha (\lambda^\alpha \partial_z \lambda^\alpha + \ell^\alpha \ell^\alpha) \Big]$$

- Superconformal algebra $(L_m, G_r)$ on fields generated by modes of

$$\begin{aligned} T_{zz} &= -\frac{1}{2} \partial_z x^\mu \partial_z x^\nu \eta_{\mu\nu} + \frac{1}{2} \psi_+^\mu \partial_z \psi_+^\nu \eta_{\mu\nu} \\ S_{z\theta} &= \psi_+^\mu \partial_z x^\nu \eta_{\mu\nu} \end{aligned}$$

# Moduli space of super Riemann surfaces

- **Heterotic string**
  - starting point are left and right moduli spaces

$$\{\tilde{J}\}/\mathrm{Diff}(\Sigma_L) = \mathcal{M}_L \qquad \text{ordinary Riemann surfaces}$$
$$\{\mathcal{J}\}/\mathrm{Diff}(\Sigma_R) = \mathfrak{M}_R \qquad \text{super Riemann surfaces}$$

  - moduli space to be integrated is a cycle [Witten]

$$\Gamma \subset \mathcal{M}_L \times \mathfrak{M}_R$$

such that

$$\Gamma_{\mathrm{red}} = \mathrm{diag}(\mathcal{M}_L \times \mathfrak{M}_{R\,\mathrm{red}})$$

# Deformations of superconformal structures

- **Under deformation of the conformal structure $\tilde{J}$ on** $\Sigma_L$ (cfr bosonic string)

$$\delta I = \int d\tilde{z} dz \, \delta\eta_z{}^{\tilde{z}} \, T_{\tilde{z}\tilde{z}}$$

- **Under deformation of superconformal structure $\mathcal{J}$ on** $\Sigma_R$

$$\delta I = \int d\tilde{z} dz \, \left[ \delta\eta_{\tilde{z}}{}^z \, T_{zz} + \delta\chi_{\tilde{z}}{}^\theta \, S_{z\theta} \right]$$

  - $\chi_{\tilde{z}}{}^\theta$ is the "worldsheet gravitino" field
  - section of $K_R^{-\frac{1}{2}}$ with values in $\tilde{K}_L$ on $\Sigma_{\rm red}$

- Assemble deformations into superconformal invariant action

$$\begin{aligned}
\mathcal{S}_{z\theta} &= S_{z\theta} + \theta T_{zz} \\
\delta H_{\tilde{z}}{}^z &= \delta\eta_{\tilde{z}}{}^z + \theta \delta\chi_{\tilde{z}}{}^\theta \in T_L^*(\Sigma) \otimes \mathcal{D}^2 \\
\delta H_\theta{}^{\tilde{z}} &= \delta\tau_\theta{}^{\tilde{z}} + \theta \delta\eta_z{}^{\tilde{z}} \in T_L(\Sigma) \otimes \mathcal{D}^{-1} \\
\delta I &= \int_\Sigma [d\tilde{z} dz | d\theta] \left( \delta H_{\tilde{z}}{}^z \mathcal{S}_{z\theta} + \delta H_\theta{}^{\tilde{z}} T_{\tilde{z}\tilde{z}} \right)
\end{aligned}$$

# Supermoduli deformations

- $\mathrm{Diff}(\Sigma)$ **acts on** $\delta H$ **by**

$$\delta H_{\tilde z}{}^z = \partial_{\tilde z} V^z \qquad\qquad V^z = v^z + \theta \xi^\theta$$

$$\delta H_\theta{}^{\tilde z} = D_\theta V^{\tilde z} \qquad\qquad V^{\tilde z} = v^{\tilde z} + \theta \xi_\theta{}^{\tilde z}$$

Components transform as follows,

$$\delta \eta_{\tilde z}{}^z = \partial_{\tilde z} v^z \qquad\qquad \delta \chi_{\tilde z}{}^\theta = \partial_{\tilde z} \xi^\theta$$

$$\delta \eta_z{}^{\tilde z} = \partial_z v^{\tilde z} \qquad\qquad \delta \tau_\theta{}^{\tilde z} = \xi_\theta{}^{\tilde z}$$

- The space of all deformations $\delta H_\theta{}^{\tilde z}$ may be decomposed

$$\{\delta H_\theta{}^{\tilde z}\} = \mathrm{Range}\left( D_\theta \Big|_{V^{\tilde z}} \right) \oplus \mathrm{Ker}\left( D_\theta \Big|_{V^{\tilde z}} \right)^\dagger$$

  – The range of $\xi_\theta{}^{\tilde z}$ spans all of $\delta \tau_\theta{}^{\tilde z}$
  – Hence $D_\theta$ may be restricted to the action on $v^{\tilde z}$

$$\mathrm{Ker}\left( D_\theta \Big|_{V^{\tilde z}} \right)^\dagger = \mathrm{Ker}\left( \partial_z \Big|_{v^{\tilde z}} \right)^\dagger = T^*_{(1,0)}(\mathcal{M}_L)$$

  – $\tilde \phi_a$ basis for $\mathrm{Ker}\left( \partial_z \big|_{v^{\tilde z}} \right)^\dagger$
  – linear forms on space of $\delta H_\theta{}^{\tilde z}$ by $\delta \tilde m_a = \langle \delta H_\theta{}^{\tilde z}, \tilde \phi_a \rangle = \int_\Sigma d\tilde z dz d\theta \, \delta H_\theta{}^{\tilde z} \, \tilde \phi_a$

# Supermoduli deformations (cont'd)

- The space of all deformations $\delta H_{\bar z}{}^z$ may be decomposed

$$\{\delta H_{\bar z}{}^z\} = \mathrm{Range}\left(\partial_{\bar z}\Big|_{V^z}\right) \oplus \mathrm{Ker}\left(\partial_{\bar z}\Big|_{V^z}\right)^\dagger$$

– The complements to the ranges of both $v^z$ and $\xi^\theta$ contribute

$$\mathrm{Ker}\left(\partial_{\bar z}\Big|_{V^z}\right)^\dagger = T^*_{(1,0)}(\mathfrak{M}_R)$$

– Let $\Phi_{A\,zz}$ be a basis for $\mathrm{Ker}\left(\partial_{\bar z}\Big|_{V^z}\right)^\dagger$
– provide linear forms on space of $\delta H_{\bar z}{}^z$ by

$$\delta m_A = \langle \delta H_{\bar z}{}^z, \Phi_{A\,zz}\rangle = \int_\Sigma d\tilde z dz d\theta\, \delta H_{\bar z}{}^z\, \Phi_{A\,zz}$$

- **On $\Sigma_{\rm red}$, we have**

$$\mathrm{Ker}\left(\partial_{\bar z}\Big|_{V^z}\right)^\dagger = \mathrm{Ker}\ \partial_{\bar z}\Big|_{K^2} \oplus \mathrm{Ker}\ \partial_{\bar z}\Big|_{K\otimes S}$$

$$\dim_{\mathbb{C}}\mathrm{Ker}\left(\partial_{\bar z}\Big|_{V^z}\right)^\dagger = (3h-3|2h-2) \qquad h \geq 2$$

– giving $3h-3$ even moduli, and $2h-2$ odd moduli

# Ghosts

- **Parametrization of the spaces $\delta H_{\bar z}{}^z$ and $\delta H_\theta{}^{\tilde z}$ by slice in $\{\tilde J, \mathcal{J}\}$**

$$\delta H_{\bar z}{}^z = \partial_{\bar z} V^z + \sum_A H_A \delta m_A \qquad H_A = \frac{\partial \mathcal{J}_{\bar z}{}^z}{\partial m_A}$$

$$\delta H_\theta{}^{\tilde z} = D_\theta V^{\tilde z} + \sum_a \tilde H_a \delta \tilde m_a \qquad \tilde H_a = \frac{\partial \tilde J_\theta{}^{\tilde z}}{\partial \tilde m_a}$$

- **Introducing ghost fields**

$$V^z \to C^z = c^z + \theta \gamma^\theta \qquad \delta m_A \to dm_A \qquad \delta H_{\bar z}{}^z \to B_{z\theta} = \beta_{z\theta} + \theta b_{zz}$$

$$V^{\tilde z} \to \tilde C^{\tilde z} = c^z + \theta \gamma_\theta^{\tilde z} \qquad \delta \tilde m_a \to d\tilde m_a \qquad \delta H_\theta{}^{\tilde z} \to \tilde B_{\tilde z \tilde z} = b_{\tilde z \tilde z} + \theta \beta_{\tilde z \tilde z \theta}$$

– $b_{zz}, b_{\tilde z \tilde z}, c^z, c^{\tilde z}$ are anti-commuting ghosts, familiar from the bosonic string;
– $\beta_{z\theta}, \gamma^\theta$ are commuting ghosts;
– $\gamma_\theta^{\tilde z}, \beta_{\tilde z \tilde z \theta}$ auxiliary, non-dynamical ghosts (integrate out trivially).

– Superconformally invariant ghost action,

$$I_{\rm gh} = \int_\Sigma [d\tilde z dz | d\theta] \Big[ B_{z\theta} \partial_{\bar z} C^z + \tilde B_{\tilde z \tilde z} D_\theta C^{\tilde z} + B_{z\theta} \sum_A H_A dm_A + \tilde B_{\tilde z \tilde z} \sum_a \tilde H_a d\tilde m_a \Big]$$

# Heterotic string amplitudes

- **Integrations over finite-dimensional variables $dm_A$ and $d\tilde{m}_a$**
  - integrations may be carried out explicitly for $d\tilde{m}_a$ (cfr bosonic)

$$\int \prod_a d(d\tilde{m}_a)\, e^{-\sum_a \langle \tilde{B}, \tilde{H}_a \rangle d\tilde{m}_a} = \prod_a \delta\left(\langle \tilde{B}, \tilde{H}_a \rangle\right)$$

  - integration over $dm_A$ requires an algebraic definition,

$$\int \prod_A d(dm_A)\, e^{-\sum_A \langle B, H_A \rangle dm_A} = \prod_A \delta\left(\langle B, H_A \rangle\right)$$

- **Assembling all factors, we obtain the integrand on supermoduli space**

$$\int D(XB\tilde{B}C\tilde{C})\, \Omega \prod_a d\tilde{m}_a \delta\left(\langle \tilde{B}, \tilde{H}_a \rangle\right) \prod_A dm_A \delta\left(\langle B, H_A \rangle\right)\, e^{-I_M - I_{\rm gh}}$$

  - with vertex operators $\Omega = \mathcal{V}_1 \cdots \mathcal{V}_N$.
  - e.g. un-integrated graviton vertex operator (NS sector)

$$\mathcal{V} = \varepsilon_{\mu\nu}(k) c^z\, c^{\tilde{z}}\, \delta(\gamma^\theta)\, \partial_{\tilde{z}} X^\mu D_\theta X^\nu\, e^{ik_\mu X^\mu}$$

# Practical parametrization of supermoduli

- Superconformal structure was specified by choice of local coordinates $(z|\theta)$.
  - even and odd supermoduli specify transition functions
- Practical calculations mostly using parametrization by gravitino field ${\chi_{\tilde z}}^\theta$
  - originates in the worldsheet supergravity approach (not discussed here)
  - convenient interface with worldsheet CFT
- Parametrization of bosonic moduli (in Weyl-invariant theory)
  - Initial complex structure specified by local coordinates $(z, \tilde z)$ with $g = |dz|^2$
  - deformation of complex structure by Beltrami differential to $g' = |dz + \mu d\tilde z|^2$
  - realized in CFT by insertion of $\int_\Sigma d\tilde z dz \, {\mu_{\tilde z}}^z \, T_{zz}$ (to all orders in $\mu$)
- Parametrization of supermoduli (in Weyl-invariant Heterotic theory)
  - Start with $\Sigma_{\rm red}$ with complex structure given by local coordinates $(z, \tilde z)$
  - Deformation of super conformal structure realized by insertion of $T$ and $S$

$$\int_{\Sigma_{\rm red}} d\tilde z dz \, \left( {\mu_{\tilde z}}^z T_{zz} + {\chi_{\tilde z}}^\theta S_{z\theta} \right)$$

  - matches couplings predicted by worldsheet supergravity formulation
  - $\chi$ and $\mu$ parametrized by local odd coordinates on $\mathfrak{M}_h$

# Chiral splitting in Type II

- In Type II superstrings, both $\Sigma_L$ and $\Sigma_R$ are super Riemann surfaces
  - Deformation will now involve independent ${\chi_{\tilde z}}^\theta$ and ${\chi_z}^{\tilde\theta}$
  - ghosts $B, C$ couple only to ${\chi_{\tilde z}}^\theta$ and $\tilde B, \tilde C$ only to ${\chi_z}^{\tilde\theta}$
  - supergravity action for $X^\mu$ couples left to right chiralities : ${\chi_{\tilde z}}^\theta {\chi_z}^{\tilde\theta} \psi_+^\mu \psi_-^\nu \eta_{\mu\nu}$

- Chiral splitting is obtained at fixed internal loop momenta
  - Fix canonical homology basis $A_I, B_I, I = 1, \cdots, h$ on $\Sigma$ of genus $h$
  - $h$ independent internal loop momenta $p_I^\mu$ are defined across the cycles $A_I$

$$p_I^\mu = \oint_{A_I} dz\, \partial_z x^\mu + \oint_{A_I} d\tilde z\, \partial_{\tilde z} x^\mu$$

  - Amplitude is an integral over $p_I^\mu$ of a product of chiral amplitudes

$$\int_{\mathbb{R}^{10}} dp_I^\mu\, \mathcal{F}_L({\mu_z}^{\tilde z}, {\chi_z}^{\tilde\theta}, p_I^\mu)\, \mathcal{F}_R({\mu_{\tilde z}}^z, {\chi_{\tilde z}}^\theta, p_I^\mu)$$

# Chiral amplitudes

- Chiral amplitude $\mathcal{F}_R$ has supermoduli deformations of only $\Sigma_R$

  (similarly $\mathcal{F}_L$ has supermoduli deformations of only $\Sigma_L$)
  - $\mathcal{F}_R$ computed with effective rules for chiral fields $x_+, \psi_+^\mu$
  - and chiral vertex operators $\mathcal{V}_1^+ \cdots \mathcal{V}_N^+$

$$\mathcal{F}_R(\mu_{\bar z}{}^z, \chi_{\bar z}{}^\theta, p_I^\mu) = \left\langle \mathcal{V}_1^+ \cdots \mathcal{V}_N^+ \, e^{p_I^\mu \oint_{B_I} dz \partial_z x_+^\mu} \exp \int_{\Sigma_{\rm red}} d\tilde z dz (\mu_{\bar z}{}^z T_{zz} + \chi_{\bar z}{}^\theta S_{z\theta}) \right\rangle_{x_+, \psi_+}$$

- with stress tensor and supercurrent evaluated on chiral fields $x_+, \psi_+^\mu$

$$\begin{aligned} T_{zz} &= -\frac{1}{2}\partial_z x_+^\mu \partial_z x_+^\nu \eta_{\mu\nu} + \frac{1}{2}\psi_+^\mu \partial_z \psi_+^\nu \eta_{\mu\nu} \\ S_{z\theta} &= \psi_+^\mu \partial_z x_+^\nu \eta_{\mu\nu} \end{aligned}$$

- $\langle \cdots \rangle_{x_+, \psi_+}$ indicates Wick contractions of $x_+^\mu, \psi_+^\mu$ with
- effective chiral Green functions

$$\begin{aligned} \langle \psi_+^\mu(z) \psi_+^\nu(w) \rangle &= -\eta^{\mu\nu} S(z,w) && \text{Szegö kernel} \\ \langle x_+^\mu(z) x_+^\nu(w) \rangle &= -\eta^{\mu\nu} \ln E(z,w) && \text{prime form} \end{aligned}$$