Modular structure of Type II low energy expansion

Lecture III: Applications

Eric D’Hoker

Simons Center for Geometry and Physics
– Supermoduli –
2015
Bibliography

Based on

• ED, Michael Green, Pierre Vanhove, arXiv:1502.06698,
  *On the modular structure of the genus-one Type II superstring low energy expansion*

• ED, Michael Green, Boris Pioline, Rudolfo Russo, arXiv:1405.6226; JHEP 1501 (2015) 031,
  *Matching the $D^6R^4$ interaction at two-loops*

• ED, Michael Green, arXiv:1308.4597; Journal of Number Theory, Vol 144 (2014) 111-150,
  *Zhang-Kawazumi invariants and Superstring Amplitudes*
Expansions of Type II Superstring Theory

- **Superstring Perturbation Theory in powers of** $g_s$
  - holds for weak coupling $g_s$
  - but for all energies

- **Classical supergravity** “$R$”
  - leading low energy expansion of string theory
  - holds for all couplings $g_s$

- **String induced effective interactions** $\mathcal{R}^4, D^4\mathcal{R}^4, D^6\mathcal{R}^4$
  - Evaluated in perturbation theory for $g_s \ll 1$
Effective Interactions

• Four-graviton amplitude in Type II at genus 0,

\[ \mathcal{A}^{(0)} = \mathcal{R}^4 \frac{1}{stu} \frac{\Gamma(1-s) \Gamma(1-t) \Gamma(1-u)}{\Gamma(1+s) \Gamma(1+t) \Gamma(1+u)} \]

- \( \mathcal{R}^4 \) = unique maximally supersymmetric contraction of 4 Weyl tensors
- External momenta \( k_i \) for \( i = 1, 2, 3, 4 \) with \( k_i^2 = 0 \) and \( \sum_i k_i = 0 \)
- Introduce dimensionless Lorentz-invariants \( s_{ij} = -\alpha' k_i \cdot k_j / 2 \)
- \( s = s_{12} = s_{34}, \; t = s_{13} = s_{24}, \; u = s_{14} = s_{23} \) with \( s + t + u = 0 \)

• Low energy expansion corresponds to \( |s|, |t|, |u| \ll 1 \)

\[ \frac{1}{stu} + 2\zeta(3) + \zeta(5)(s^2 + t^2 + u^2) + 2\zeta(3)^2stu + \cdots \]

massless \( \mathcal{R}^4 \), \( D^4\mathcal{R}^4 \), \( D^6\mathcal{R}^4 \)

Exchange of massive string states produces local effective interactions.
Focus of this talk

• Effective interaction of the form $D^{2w} R^4$, $w \in \mathbb{N}$
  – accessible from the 4-graviton amplitudes
  – arising from surfaces of genus-one;
  – arising from surfaces of genus-two.

• In both cases, we will find that the integrands on moduli space
  ★ of compact Riemann surfaces (without punctures),
  ★ having integrated over all vertex operator positions,
  obey families of inhomogeneous Laplace-eigenvalue equations;
  – Genus-one: Kronecker-Eisenstein series;

• Remarks on S-duality in Type IIB and non-perturbative effects
Genus-one effective interactions in Type II

• Start with the Type II genus-one four-graviton amplitude,

\[ A^{(1)}_4(s, t, u) = R^4 \int_{\mathcal{M}_1} \frac{d^2 \tau}{\tau_2^2} B^{(1)}(s, t, u|\tau) \]

\[ \mathcal{M}_1 = \{ \tau = \tau_1 + i\tau_2, \tau_{1,2} \in \mathbb{R}, \tau_2 > 0, |\tau| \geq 1, |\tau_1| \leq \frac{1}{2} \} \]

• \( B^{(1)} \) is an integral over four copies of the torus \( \Sigma \) of modulus \( \tau \),

\[ B^{(1)}(s, t, u|\tau) = \left( \prod_{i=1}^{4} \int_{\Sigma} \frac{d^2 z_i}{\tau_2} \right) \exp \left\{ \sum_{1 \leq i < j \leq 4} s_{ij} G(z_i - z_j|\tau) \right\} \]

\( G(z|\tau) \) is the scalar Green function on \( \Sigma \) with modulus \( \tau \)

– parametrizing the torus by \( z = \alpha + \beta \tau \) with \( \alpha, \beta \in \mathbb{R}/\mathbb{Z} \)

– \( G \) is given by a Fourier sum of torus momenta \( (m, n) \in \mathbb{Z}^2 \)

\[ G(z|\tau) = \sum_{(m,n) \neq (0,0)} \frac{\tau_2}{|m\tau + n|^2} e^{2\pi i (m\alpha - n\beta)} \]
Worldsheet Feynman diagrams

- Expansion of $B^{(1)}$ for fixed $\tau$ in powers of $s_{ij}$
  - finite radius of convergence $|s_{ij}| < 1$
  - integrals $\int_{\Sigma^4} G^n$ are convergent

- Expansion organized in “worldsheet Feynman diagrams”
  - Each integration point $z_i$ on $\Sigma$ is represented by a vertex;
  - Each Green function $G(z_i - z_j|\tau)$ by a line —— between $z_i$ and $z_j$;
  - One-line reducible diagrams (disconnected upon cutting one line)
    vanish by $\int_{\Sigma} d^2z G(z|\tau) = 0$
  - A diagram with $w$ lines of $G$,
    $\star$ has weight $w$;
    $\star$ contributes to $D^{2w}R^4$. 
Worldsheet Feynman diagrams (connected, 1-line irreducible)
Worldsheet Feynman diagrams (connected, 1-line irreducible)

\[ D^4 \mathcal{R}^4 \]

\[ D^6 \mathcal{R}^4 \]

\[ D^8 \mathcal{R}^4 \]

\[ D^{10} \mathcal{R}^4 \]

one-loop

two-loops
One-loop : Eisenstein series

- One-loop worldsheet Feynman diagram with $k$-vertices

$$\prod_{i=1}^{k} \int_{\Sigma} \frac{d^2 z_i}{\tau_2} G(z_i - z_{i+1} | \tau) = \sum_{(m,n) \neq (0,0)} \frac{\tau_2^k}{\pi^k |m\tau + n|^{2k}}$$

- Non-holomorphic Eisenstein series are defined by,

$$E_s(\tau) = \sum_{(m,n) \neq (0,0)} \frac{\tau_2^s}{\pi^s |m\tau + n|^{2s}}$$

- Properties

  - absolutely convergent for $\text{Re}(s) > 1$; analytically continue to $s \in \mathbb{C}$
  - reflection relation $\Gamma(s) E_s(\tau) = \Gamma(1 - s) E_{1-s}(\tau)$
  - modular invariant under $SL(2, \mathbb{Z})$, $E_s(\tau') = E_s(\tau)$ with $\tau' = \frac{a\tau + b}{c\tau + d}$
  - Laplace-eigenvalue equation,

$$\left( \Delta - s(s - 1) \right) E_s = 0 \quad \Delta = 4\tau_2^2 \partial_\tau \partial_{\bar{\tau}}$$
Two-loops : Kronecker-Eisenstein series

- Two-loop Feynman diagrams evaluate to the series
  (all worldsheet two-loop diagrams are of this type)

\[ C_{a_1,a_2,a_3}(\tau) = \sum_{(m_r,n_r) \neq (0,0)} \delta_{m,0} \delta_{n,0} \prod_{r=1}^{3} \left( \frac{\tau_2}{\pi |m_r \tau + n_r|^2} \right)^{a_r} \]

- The total worldsheet momenta must vanish by \( \delta_{m,0} \delta_{n,0} \)
  \[ m = m_1 + m_2 + m_3 = 0, \]
  \[ n = n_1 + n_2 + n_3 = 0 \]

- the weight is \( w = a_1 + a_2 + a_3 \);

- For our diagrams we have integer \( a_r \geq 1 \) and the sums converge;

- \( C_{a_1,a_2,a_3}(\tau) \) is a modular function under \( SL(2,\mathbb{Z}) \);

- \( C_{a_1,a_2,a_3}(\tau) \) is invariant under permutations of \( a_1, a_2, a_3 \).
Laplacian on moduli space

• **Structure of the space of Kronecker-Eisenstein series** $C_{a,b,c}(\tau)$?

• **Tool**: The Laplacian $\Delta = 4\tau^2 \partial_\tau \partial_{\bar{\tau}}$ acts algebraically on the space of $C_{a,b,c}$.

\[
\Delta C_{a,b,c} = ab C_{a+1,b-1,c} + \frac{1}{2} ab C_{a+1,b+1,c-2} - 2ab C_{a+1,b,c-1} \\
+ \frac{1}{2} a(a-1) C_{a,b,c} + 5 \text{ permutations of } (a,b,c)
\]

– proven by differentiating term by term and using algebraic rearrangements;
– $\Delta$ preserves the “weight” $w = a + b + c$.

• **One of the subscript indices on the right side may equal 0 or −1,**

\[
C_{a,b,0} = E_a E_b - E_{a+b} \quad a + b \geq 3 \\
C_{a,b,-1} = E_{a-1} E_b + E_a E_{b-1} \quad a, b \geq 2
\]

– $E_1$ diverges logarithmically, but cancels out in $\Delta C_{a,b,c}$. 
Examples at low weight \( w \)

- We find inhomogeneous Laplace-eigenvalue equations,

\[
\begin{align*}
  w = 3 & \quad C_{1,1,1} = \includegraphics[width=0.1\textwidth]{example1} & \quad \Delta C_{1,1,1} = 6E_3 \\
  w = 4 & \quad C_{2,1,1} = \includegraphics[width=0.15\textwidth]{example2} & \quad (\Delta - 2)C_{2,1,1} = 9E_4 - E_2^2 \\
  w = 5 & \quad C_{3,1,1} = \includegraphics[width=0.15\textwidth]{example3} & \quad (\Delta - 6)C_{3,1,1} = 3C_{2,2,1} + 16E_5 - 4E_2E_3 \\
  w = 5 & \quad C_{2,2,1} = \includegraphics[width=0.15\textwidth]{example4} & \quad (\Delta - 0)C_{2,2,1} = 8E_5
\end{align*}
\]

- Use \( \Delta E_3 = 6E_3 \) to get \( \Delta (C_{1,1,1} - E_3) = 0 \);
- constant determined from asymptotics \( C_{1,1,1} = E_3 + \zeta(3) \) (obtained earlier by Zagier using direct calculation of sums)

- Note eigenvalues of the form \( s(s - 1) \) for \( s = 1, 2, 3 \);
**Structure Theorem for $C_{a,b,c}$ modular functions**

- $C_{a,b,c}(\tau)$ are linear combinations of modular functions $C_{w;s;p}(\tau)$ satisfying
  \[
  \left( \Delta - s(s - 1) \right) C_{w;s;p} = F_{w;s;p}(E_{s'})
  \]
  - $C_{w;s;p}$ and $F_{w;s;p}$ of weight $w = a + b + c$ (with $E_{s'}$ assigned weight $s'$);
  - $F$ is a polynomial of total degree 2 in $E_{s'}$ with $2 \leq s' \leq w$;
  - $s = w - 2m$, $m = 1, \ldots, \left[ \frac{w - 1}{2} \right]$, $p = 0, \ldots, \left[ \frac{s - 1}{3} \right]$

- **Examples at low weight**
  
<table>
<thead>
<tr>
<th>$w$</th>
<th>$s$</th>
<th>$0^{(1)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1</td>
<td>$0^{(1)}$</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>$2^{(1)}$</td>
</tr>
<tr>
<td>5</td>
<td>1, 3</td>
<td>$0^{(1)} \oplus 6^{(1)}$</td>
</tr>
<tr>
<td>6</td>
<td>2, 4</td>
<td>$2^{(1)} \oplus 12^{(2)}$</td>
</tr>
<tr>
<td>7</td>
<td>1, 3, 5</td>
<td>$0^{(1)} \oplus 6^{(1)} \oplus 20^{(2)}$</td>
</tr>
<tr>
<td>8</td>
<td>2, 4, 6</td>
<td>$2^{(1)} \oplus 12^{(2)} \oplus 30^{(2)}$</td>
</tr>
<tr>
<td>9</td>
<td>1, 3, 5, 7</td>
<td>$0^{(1)} \oplus 6^{(1)} \oplus 20^{(2)} \oplus 42^{(3)}$</td>
</tr>
</tbody>
</table>
The generating function

- A natural generating function is given by,

\[ \mathcal{W}(t_1, t_2, t_2|\tau) = \sum_{a,b,c=1}^{\infty} t_1^{a-1} t_2^{b-1} t_3^{c-1} C_{a,b,c}(\tau) \]

Summing gives a Feynman diagram for three scalars with masses \( M_r^2 = -t_r \tau_2 \),

\[ \mathcal{W}(t_1, t_2, t_2|\tau) = \sum_{(m_r,n_r)\neq(0,0)} \delta_{m,0} \delta_{n,0} \prod_{r=1}^{3} \left( \frac{\tau_2}{\pi |m_r \tau + n_r|^2 - t_r \tau_2} \right) \]

- Algebraic representation of Laplacian induces differential action on \( \mathcal{W} \),

\[ \Delta \mathcal{W} - \mathcal{L}^2 \mathcal{W} = \mathcal{R} \]

\[ \mathcal{D} = t_1 \partial_1 + t_2 \partial_2 + t_3 \partial_3 \]

\[ \mathcal{L}^2 = \mathcal{D}^2 + \mathcal{D} + (t_1^2 + t_2^2 + t_3^2 - 2t_1 t_2 - 2t_2 t_3 - 2t_3 t_1)(\partial_1 \partial_2 + \partial_2 \partial_3 + \partial_3 \partial_1) \]

\[ \mathcal{R} = \text{quadratic polynomial in the Eisenstein series } E_s \]
Proof via generating function

- **Permutations of** \((a, b, c)\) **induces permutations of** \((t_1, t_2, t_3)\)
  - \(S_3\) adapted coordinates,
    \[
    u = t_1 + t_2 + t_3, \quad \varepsilon = e^{2\pi i/3}
    \]
    \[
    v/\sqrt{2} = t_1 + \varepsilon t_2 + \varepsilon^2 t_3 \quad \text{(}t_1, t_3, t_2\text{)}(u, v, \bar{v}) = (u, \bar{v}, v)
    \]
    \[
    \bar{v}/\sqrt{2} = t_1 + \varepsilon^2 t_2 + \varepsilon t_3 \quad \text{(}t_2, t_3, t_1\text{)}(u, v, \bar{v}) = (u, \varepsilon^2 v, \varepsilon \bar{v})
    \]
  - \(L^2 = L_0^2 - L_1^2 - L_2^2\) Casimir of \(SO(1, 2)\) generated by \(L_0, L_1, L_2\);
  - Simultaneously diagonalize the \(S_3\)-invariant operators \(\mathcal{D}, L_0^2, \) and \(L^2\)
    \[
    \mathcal{D} \mathcal{W}_{w;s;p} = (w - 3) \mathcal{W}_{w;s;p} \quad \mathcal{D} = t_1 \partial_1 + t_2 \partial_2 + t_3 \partial_3
    \]
    \[
    L^2 \mathcal{W}_{w;s;p} = s(s - 1) \mathcal{W}_{w;s;p} \quad L^2 = -(u^2 - 2v\bar{v})(\partial_u^2 - 2\partial_v \partial_{\bar{v}})
    \]
    \[
    L_0^2 \mathcal{W}_{w;s;p} = -9p^2 \mathcal{W}_{w;s;p} \quad L_0 = iv\partial_v - i\bar{v}\partial_{\bar{v}}
    \]
  - \(S_3\)-invariance of eigenfunctions requires \(p\) to be integer;
  - which explains multiplicities \([s - 1]/3\).

\(\implies\) constructive proof of Structure Theorem.
Recall ....

\[ \begin{align*}
& D^4 \mathcal{R}^4 \\
& D^6 \mathcal{R}^4 \\
& D^8 \mathcal{R}^4 \\
& D^{10} \mathcal{R}^4 \\
\end{align*} \]

Eisenstein $C_{a,b,c}$
Modular functions in $D^8 \mathcal{R}^4$

- Modular function $D^8 \mathcal{R}^4$ requires

- The modular function $D_4$ is of the form $C_{1,1,1,1}$ for $\rho = 4$ with

$$C_{a_1, \ldots, a_\rho}(\tau) = \sum_{(m_r, n_r) \neq (0,0)} \delta_{m,0} \delta_{n,0} \prod_{r=1}^{\rho} \left( \frac{\tau_2}{\pi |m_r \tau + n_r|^2} \right)^{a_r}$$

- No useful algebraic representation of the Laplacian is available
**Conjectured relation for** $D_4$

- **Tools:** take an educated guess + check asymptotic behavior near cusp.
  - Relations for $C_{a,b,c}$ involved linear combinations of $C$ for given weight;
  - Consider combinations of $D_4$, $E_4$, and $E_2$
    \[(\Delta - 2) \left(5D_4 - 15E_2^2 - 18E_4\right) = -120E_2^2\]
    
    - Eliminate right side using $(\Delta - 2)C_{2,1,1} = 9E_4 - E_2^2$ gives
      \[(\Delta - 2) \left(D_4 - 24C_{2,1,1} - 3E_2^2 + 18E_4\right) = 0\]

- **Asymptotics near the cusp** $\tau_2 \to \infty$ fixes homogeneous solution,
  \[D_4 = 24C_{2,1,1} + 3E_2^2 - 18E_4\]

  - conjectured as an *exact relation* between modular functions;
  - conjectured relation between Feynman diagrams of *different loop orders*;
  - Additional support from direct numerical evaluation of the multiple sums.
Structure of the asymptotics near the cusp

• The expansion near the cusp $\tau_2 \rightarrow \infty$ takes the following form,

$$D_4(\tau) = \sum_{k, \bar{k}=0}^{\infty} D_4^{(k, \bar{k})}(\pi \tau_2) \ q^k \bar{q}^{\bar{k}} \quad q = e^{2\pi i \tau}$$

• We have checked the following asymptotics (similarly for $C_{2,1,1}$, $E_4$, $E_{2}^2$)

$$D_4^{(0,0)}(y) = \frac{y^4}{945} + \frac{2\zeta(3)}{3} + \frac{10\zeta(5)}{y} - \frac{3\zeta(3)^2}{y^2} + \frac{9\zeta(7)}{4y^3}$$

$$D_4^{(0,1)}(y) = \frac{4y^2}{15} + \frac{2y}{3} + 2 + \frac{4}{y} + \frac{12\zeta(3)}{y} - \frac{6\zeta(3)}{y^2} + \frac{9}{2y^2} + \frac{9}{4y^3}$$

$$D_4^{(1,0)}(y) = D_4^{(0,1)}(y)$$
How could the conjecture fail?

- Consider the difference \( F = D_4 - 24C_{2,1,1} - 3E_2^2 + 18E_4 \)
  - the conjecture states \( F = 0 \)

- If the conjecture were to fail, then \( F \neq 0 \) and its properties are,
  - modular function under \( SL(2,\mathbb{Z}) \);
  - its pure power part in the expansion near the cusp vanishes;
    \( \implies F \) is a \textbf{cuspidal function}
  - Vanishing of leading exponential restricts it further.
Progress towards a full proof

• Inspired by a calculation of Zagier for $C_{111}$, we first perform $n$-sums

$$D_4(\tau) = \sum_{(m_r, n_r) \neq (0,0)} \delta_{m, 0} \delta_{n, 0} \prod_{r=1}^{4} \frac{\tau_2}{\pi |m_r \tau + n_r|^2}$$

– partition sum according to the number of vanishing $m_r$;
– solve for $n_4$; decompose in partial fractions in $n_3$; sum over $n_3$,
– for $m_3 \neq 0$, sum using the formula (and its derivatives in $z$)

$$\sum_{n_3 \in \mathbb{Z}} \frac{1}{z + n_3} = -i\pi \frac{1 + e^{2\pi iz}}{1 - e^{2\pi iz}}$$

• Explicit calculation (using MAPLE) shows the following structure

$$D_4(\tau) = \sum_{k=-3}^{4} (\tau_2)^k \mathcal{D}_4^{(k)}(q, \bar{q}) \quad q = e^{2\pi i \tau}$$

– where $\mathcal{D}_4^{(k)}(q, \bar{q})$ is a function analytic in $q, \bar{q}$ near the cusp
– similar expansions for $C_{211}, E_2^3, E_4$ have the same range for $k$.

• Conjecture proven for $k = -3, 1, 2, 3, 4$;
  … in progress for remaining values $k = -2, -1, 0$ …
Conjectured relations for modular functions in $D^{10}R^4$

$D^{10}R^4$ requires $D_5 = \bullet \circ \bullet$ and

\[
D_{3,1,1} = \bullet \circ \bullet \quad D_{2,2,1} = \bullet \circ \bullet
\]

in addition to $E_5$, $C_{3,1,1}$, $E_2E_3$, and $E_2C_{1,1,1}$ functions of weight 5.

- $D_5 = C_{1,1,1,1,1}$ and $D_{3,1,1} = C_{2,1,1,1}$ but $D_{2,2,1}$ not of the form $C$

**Educated guesses and asymptotics near the cusp lead us to conjecture,**

\[
\begin{align*}
D_5 &= 60C_{3,1,1} + 10E_2C_{1,1,1} - 48E_5 + 16\zeta(5) \\
40D_{3,1,1} &= 300C_{3,1,1} + 120E_2E_3 - 276E_5 + 7\zeta(5) \\
10D_{2,2,1} &= 20C_{3,1,1} - 4E_5 + 3\zeta(5)
\end{align*}
\]

**Pattern expected to continue for higher $D^{2w}R^4$ interactions with $w > 5$.**
Generalizations and Multi-zeta-values

- **Generalized infinite families entering genus-one diagrams** (but not all)

\[ C_{a_1, \ldots, a_\rho}(\tau) = \sum_{(m_r, n_r) \neq (0,0)} \delta_{m,0} \delta_{n,0} \prod_{r=1}^{\rho} \left( \frac{\tau_2}{|m_r \tau + n_r|^2} \right)^{a_r} \]

- for integers \( a_r \geq 1 \) enter diagrams of weight \( w = a_1 + \cdots + a_\rho \)

- **Multi-zeta-functions**

\[ \zeta_{s_1, \ldots, s_n} = \sum_{m_1 > \cdots > m_n \geq 1} \frac{1}{m_1^{s_1} \cdots m_n^{s_n}} \]

- generalize the standard Riemann zeta-function for \( n = 1 \)

- **The** \( C_{a_1, \ldots, a_\rho} \) **provide a modular generalization of** \( \zeta_{s_1, \ldots, s_n} \)

  - leading \( \tau_2 \) behavior of \( C_{a_1, \ldots, a_\rho} \) may be expressed in terms of MZV.
  - MZV naturally enter into open string amplitudes [Schlotterer, Stieberger]
Genus-one coefficients of $D^2 w R^4$ for $w \geq 4$

- **Integrating $B^{(1)}$ over $M_1$ produces non-analytic behavior in $s, t, u$;**
  - branch cuts due to loops with massless strings for $s, t, u, \ll 1$;
  - non-analytic parts may be isolated systematically,
    - [ED, Phong 1993; Green, Russo, Vanhove 2008]
    - Analytic part is unique after non-analytic part has been specified.

- **Partition fundamental domain $M_1$ at fixed large $L \gg 1$;** [Maass; Selberg]
  - $\tau_2 > L$ gives non-analytic contributions in $s, t, u$;
  - $\tau_2 < L$ gives analytic contributions in $s, t, u$;

- **For compactifications on flat $T^d$, with moduli $\rho_d$,**
  \[
  E_{D^8 R^4}(\rho_d, L) = \frac{1}{2} \int_{M_1}^{\tau_2 < L} d\mu_1 \left( \Delta C_{2,1,1} - 5E_4 + E_2^2 \right) \Gamma_{d,d,1}(\rho_d|\tau)
  \]
  - parts non-analytic at $s, t, u = 0$ cancel in comparing moduli $\rho_d$ and $\rho'_d$;
  - “Differences” produce well-defined and unique effective interactions.
**Genus two**

- **Siegel Upper half space** $\mathcal{S}_2$
  
  $$\mathcal{S}_2 = \{ \Omega_{IJ} = \Omega_{JI} \in \mathbb{C} \text{ with } I, J = 1, 2 \text{ and } Y = \text{Im} \Omega > 0 \}$$

  - $Sp(4, \mathbb{R})$ acts by $\Omega \rightarrow (A\Omega + B)(C\Omega + D)^{-1}$

  $$M^tJM = J \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

  - $\mathcal{S}_2$ has $Sp(4, \mathbb{R})$-invariant metric $ds_2^2$ and volume form $d\mu_2$

  $$ds_2^2 = \sum_{I, J, K, L=1,2} Y^{-1}_{IJ} d\bar{\Omega}_{JK} Y^{-1}_{KL} d\Omega_{LI}$$

- **Compact Riemann surface** $\Sigma$

  - $A_I, B_I$ a choice of canonical homology basis for $H_1(\Sigma, \mathbb{Z})$,

  - $\omega_I$ dual holomorphic $(1,0)$ forms,

  $$\int_{A_I} \omega_J = \delta_{IJ} \quad \int_{B_I} \omega_J = \Omega_{IJ}$$

  - $\Omega \in \mathcal{S}_2$

  - Genus-two modular group $Sp(4, \mathbb{Z})$

  - Genus-two moduli space $\mathcal{M}_2 = \mathcal{S}_2/Sp(4, \mathbb{Z})$
Effective interactions at genus-two in Type II

- Start with Type II four-graviton amplitude at genus 2, \[\text{[ED, Phong 2005]}\]

\[
A^{(2)}(s, t, u) = \mathcal{R}^4 \int_{\mathcal{M}_2} d\mu_2 B^{(2)}(s, t, u|\Omega)
\]

\[
B^{(2)}(s, t, u|\Omega) = \int_{\Sigma^4} \mathcal{Y} \wedge \bar{\mathcal{Y}} \exp \sum_{i<j} s_{ij} G(z_i, z_j)
\]

- \(G(z_i, z_j)\) is the scalar Green function;
- \(B^{(2)}\) invariant under \(G(z, w) \rightarrow G(z, w) + f(z) + f(w)\) for any \(f\)
- \(\mathcal{Y} = (s - t) \Delta(z_1, z_3) \wedge \Delta(z_4, z_2) + 2\) permutations;
- \(\Delta(z_i, z_j)\) is a holomorphic \((1, 0)_i \otimes (1, 0)_j\) form independent of \(s, t, u\).

- Contributions to local effective interactions,
  - \(\mathcal{R}^4\) : zero, since \(\mathcal{Y}\) vanishes for \(s = t = u = 0\);
  - \(D^4\mathcal{R}^4\) : non-zero, \(B^{(2)}\) constant on \(\mathcal{M}_2\);
  - \(D^6\mathcal{R}^4\) : non-zero, one power of \(G\) brought down in integral over \(\Sigma^4\);

\[
B^{(2)}(s, t, u|\Omega) = 32(s^2 + t^2 + u^2) + 192 stu \varphi(\Omega) + \mathcal{O}(s^4, \cdots)
\]

- \(\varphi(\Omega)\) coincides with the Zhang-Kawazumi invariant \[\text{[ED, Green 2013]}\].
The Zhang-Kawazumi invariant for genus-two

• Definition of the ZK-invariant

The ZK-invariant takes the following form,

$$8\varphi(\Omega) = \sum_{I,J,K,L} (Y^{-1}_{IJ} Y^{-1}_{KL} - 2Y^{-1}_{IL} Y^{-1}_{JK}) \int_{\Sigma^2} G(x,y) \omega_I(x) \omega_J(x) \omega_K(y) \omega_L(y)$$

• $\varphi$ invariant under $G(z,w) \rightarrow G(z,w) + f(z) + f(w)$ for any $f$
• equivalent to definition via Arakelov geometry [Zhang 2007, Kawazumi 2008]
• invariant under the modular group $Sp(4,\mathbb{Z})$; hence $\varphi(\Omega) = \varphi(\Sigma)$
• related to the genus-two Faltings invariant [De Jong 2010]

• Coefficient of the genus-two $D^6R^4$ interaction involves $\int_{\mathcal{M}_2} d\mu_2 \varphi(\Omega)$

• Direct evaluation appeared out of reach ... until ...
ZK satisfies a Laplace eigenvalue equation

- **Theorem**: genus-two ZK invariant satisfies remarkable equation,

  \[(\Delta - 5)\varphi = -2\pi \delta_{SN}\]

  - \(\Delta\) is the Laplace-Beltrami operator on \(\mathcal{M}_2\) with Siegel metric \(ds_2^2\);
  - \(\delta_{SN}\) has support on separating node (into two genus-one surfaces)

  \([\text{ED, Green, Pioline, R. Russo 2014}]\)

- **Using Theorem**, the integral over \(\mathcal{M}_2\) reduces to an integral over \(\partial \mathcal{M}_2\),

  \[\int_{\mathcal{M}_2} d\mu_2 \varphi = \frac{1}{5} \int_{\mathcal{M}_2} d\mu_2 (\Delta \varphi + 2\pi \delta_{SN}) = \frac{2\pi^3}{45}\]

  - meaning of this number is to be discussed in last part of lecture
Proof via deformations of complex structures

• \textbf{Laplacian} $\Delta$ on genus-two moduli space $M_2$
  
  = Laplace-Beltrami operator for the Siegel metric on $S_2$
  
  – In terms of the period matrix $\Omega_{IJ} = X_{IJ} + iY_{IJ}$, with $I, J = 1, 2$

  $$
  \Delta = \sum_{I \leq J} \sum_{K \leq L} Y_{IK} Y_{JL} \frac{\partial}{\partial \Omega_{IJ}} \frac{\partial}{\partial \Omega_{KL}},
  $$

• \textbf{Variations in} $\Omega_{IJ}$ result from variation by Beltrami differential $\mu$

  $$
  \delta_\mu \phi = \frac{1}{2\pi} \int_{\Sigma} d^2 w \mu_w \delta_{ww} \phi
  $$

  – \(\delta_{ww} \phi\) is obtained by variation of $\bar{\partial}$ or insertion of the stress tensor $T_{ww}$

  $$
  \delta_{ww} \Omega_{IJ} = 2\pi i \omega_I(w) \omega_J(w)
  $$

  $$
  \delta_{ww} \omega_I(x) = \omega_I(w) \partial_x \partial_w \ln E(x, w)
  $$

  $$
  \delta_{ww} G(x, y) = \frac{1}{2} (\partial_w G(w, x) - \partial_w G(w, y))^2
  $$

• \textbf{Calculation of mixed derivatives proves} $(\Delta - 5) \varphi = 0$ inside $M_2$

  – contribution from separating node results from asymptotics of $\varphi$

  [ED, Green, Pioline, R. Russo 2014]
Generalizations of ZK-invariant

• The ZK-invariant exists for all genera $h \geq 2$ [Zhang 2007, Kawazumi 2008]
  – but does not satisfy a simple Laplace-eigenvalue equation for $h \geq 3$;
  – most likely is not the correct object for string theory at $h \geq 3$.

• But the integrands on $M_2$ for the coefficients of $D^8 R^4$, $D^{10} R^4$, …
  – do naturally emerge from string theory;
  – are modular invariants which generalize ZK;
  – satisfy more complicated Laplace-type equations
    ... in progress ...

• Lowest order generalization of ZK at genus 2, contributing to $D^8 R^4$

$$B_{(2,0)}^{(2)} = \int_{\Sigma^4} \left| \Delta(1, 2) \wedge \Delta(3, 4) \right|^2 \left( G(1, 4) + G(2, 3) - G(1, 3) - G(2, 4) \right)^2$$
S-duality of Type IIB and non-perturbative effects

- **S-duality** = $SL(2, \mathbb{Z})$ symmetry of Type IIB superstring theory
  - axion $T_1$ and dilaton $T_2$ are real fields of Type IIB, $T = T_1 + T_2$
  - Type IIB supergravity invariant under $SL(2, \mathbb{R})$ acting on $T$ by Möbius transf.
  - Type IIB string theory invariant under $SL(2, \mathbb{Z})$ (cfr axion anomaly)

- **Effective interactions must respect** $SL(2, \mathbb{Z})$ symmetry
  - for $R^4$ effective interaction with no derivatives on $T$, (in string frame)
    $$\int_M d^{10}x \sqrt{G} \left(T_2^{\frac{1}{2}} \mathcal{E}(T)\right) R^4$$
    $$T_2 = \frac{1}{g_s}$$
  - $\mathcal{E}(T)$ real-valued modular function under $SL(2, \mathbb{Z})$

- **$\mathcal{E}(T) = E_{\frac{3}{2}}(T)$ conjectured from single D-instanton calculation** [Green Gutperle]
  - Using asymptotics of non-holomorphic Eisenstein near the cusp $T_2 \to \infty$,
    $$E_s(T) = \frac{2\zeta(2s)}{\pi^s} T_2^s + \frac{2\Gamma(s - \frac{1}{2})\zeta(2s - 1)}{\Gamma(s) \pi^{s - \frac{1}{2}}} T_2^{1-s} + \mathcal{O}(e^{-2\pi T_2})$$
  - Perturbative contributions to $R^4$ arise from genus 0 and 1 only.
Supersymmetry and S-duality

- Laplace-eigenvalue eq from space-time supersymmetry [Green, Sethi, 1998]
  - Eisenstein series = unique modular solution with polynomial growth at cusp

- Predicts vanishing contributions for high enough loop order,
  \[
  \begin{align*}
  \mathcal{R}^4 & \quad 1/2 \text{ BPS} \quad h \geq 2 \quad E_3^{\frac{3}{2}} \\
  D^4\mathcal{R}^4 & \quad 1/4 \text{ BPS} \quad h \geq 3 \quad E_5^{\frac{5}{2}} \\
  D^6\mathcal{R}^4 & \quad 1/8 \text{ BPS} \quad h \geq 4 \quad (\Delta - 12)E_{D^6\mathcal{R}^4} = (E_3^{\frac{3}{2}})^2
  \end{align*}
  \]
  [Green, Gutperle, Vanhove 1997; Green, Vanhove 2005]

- Predicts relations between non-vanishing contributions (e.g. with tree-level),
  \[
  \begin{align*}
  \mathcal{R}^4 & \quad h = 1 \quad [Green, Gutperle 1997] \\
  D^4\mathcal{R}^4 & \quad h = 2 \quad [ED, Gutperle, Phong 2005] \\
  D^6\mathcal{R}^4 & \quad h = 2 \quad (ZK) \quad [ED, Green, Pioline, Russo 2014] \\
  & \quad h = 3 \quad [Gomez, Mafra 2013]
  \end{align*}
  \]
Summary and outlook

• **Low energy expansion of string theory has revealed a rich structure of**
  – non-holomorphic Kronecker-Eisenstein series on genus-one Riemann surfaces;
  – Zhang-Kawazumi modular invariant on genus-two Riemann surfaces;
  – differential and algebraic interrelations;
  – concrete analytic evaluation of higher local effective interactions.

• **Extensions at genus-one**
  – Understand general interrelations of Kronecker-Eisenstein series beyond $C_{a,b,c}$
  – Identify structure of the ring of all such non-holomorphic modular forms.
  – Equations obeyed by entire string integrand? [ED, Green] ... in progress ...

• **Extensions at genus-two**
  – ZK-invariant as a $\vartheta$-lift of complete degeneration limit [Pioline 2015]
  – Differential relations obeyed by higher order generalizations of Zhang-Kawazumi invariants [ED, Green, Pioline] ... in progress ...

• **Significance for number theory?**