Modular structure of Type II low energy expansion

Lecture III: Applications

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– Supermoduli –
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Bibliography

Based on

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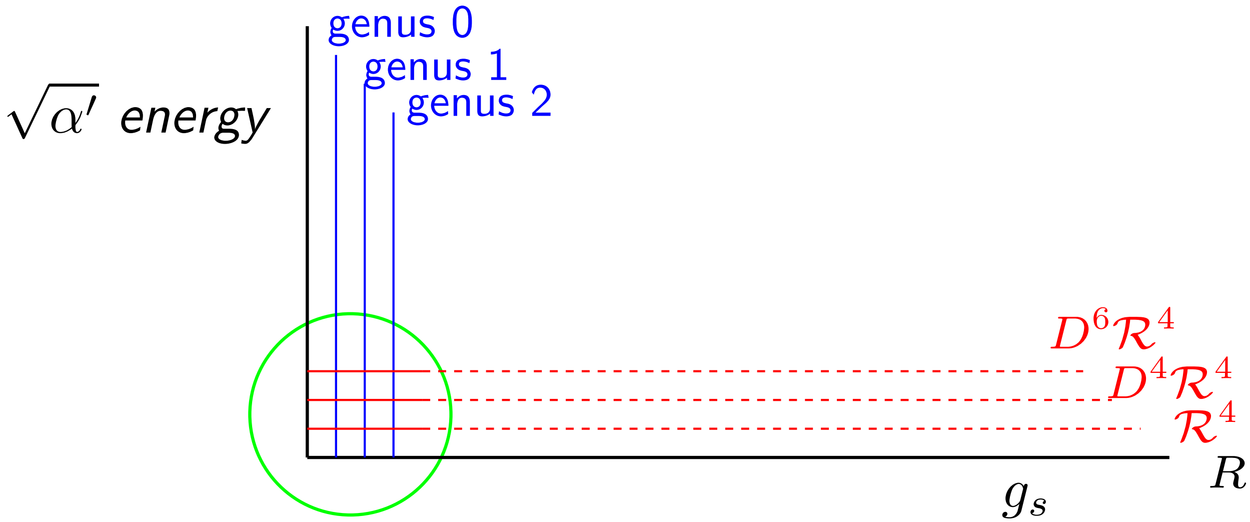
  On the modular structure of the genus-one Type II superstring low energy expansion
- ED, Michael Green, Boris Pioline, Rudolfo Russo, arXiv:1405.6226; JHEP 1501 (2015) 031,
  Matching the $D^6\mathcal{R}^4$ interaction at two-loops
- ED, Michael Green, arXiv:1308.4597; Journal of Number Theory, Vol 144 (2014) 111-150,
  Zhang-Kawazumi invariants and Superstring Amplitudes

Expansions of Type II Superstring Theory



- **Superstring Perturbation Theory in powers of** g_s
  - holds for weak coupling g_s
  - but for all energies
- **Classical supergravity** "R"
  - leading low energy expansion of string theory
  - holds for all couplings g_s
- **String induced effective interactions** $\mathcal{R}^4, D^4\mathcal{R}^4, D^6\mathcal{R}^4$
  - Evaluated in perturbation theory for $g_s \ll 1$

Effective Interactions

- **Four-graviton amplitude in Type II at genus 0,**

$$\mathcal{A}^{(0)} = \mathcal{R}^4 \, \frac{1}{stu} \, \frac{\Gamma(1-s)\,\Gamma(1-t)\,\Gamma(1-u)}{\Gamma(1+s)\,\Gamma(1+t)\,\Gamma(1+u)}$$

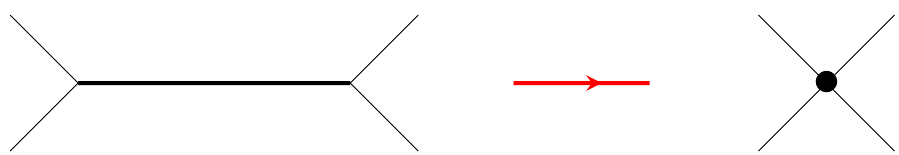
  - $\mathcal{R}^4 =$ unique maximally supersymmetric contraction of 4 Weyl tensors
  - External momenta k_i for $i = 1, 2, 3, 4$ with $k_i^2 = 0$ and $\sum_i k_i = 0$
  - Introduce dimensionless Lorentz-invariants $s_{ij} = -\alpha' k_i \cdot k_j / 2$
  - $s = s_{12} = s_{34}, \ t = s_{13} = s_{24}, \ u = s_{14} = s_{23}$ with $s + t + u = 0$

- **Low energy expansion corresponds to** $|s|, \, |t|, \, |u| \ll 1$

$$\frac{1}{stu} \ + \ 2\zeta(3) + \zeta(5)(s^2+t^2+u^2) + 2\zeta(3)^2 stu + \cdots$$

massless $\qquad \mathcal{R}^4 \qquad D^4\mathcal{R}^4 \qquad D^6\mathcal{R}^4$

Exchange of massive string states produces local effective interactions.



Focus of this talk

- **Effective interaction of the form $D^{2w}\mathcal{R}^4$, $w \in \mathbb{N}$**
  - accessible from the 4-graviton amplitudes
  - arising from surfaces of genus-one;
  - arising from surfaces of genus-two.

- **In both cases, we will find that the** *integrands* **on moduli space**
  - ⋆ of compact Riemann surfaces (without punctures),
  - ⋆ having integrated over all vertex operator positions,

  obey families of inhomogeneous Laplace-eigenvalue equations;

  - Genus-one: Kronecker-Eisenstein series;

  - Genus-two: "Zhang-Kawazumi-invariant" and generalizations thereof.

- **Remarks on S-duality in Type IIB and non-perturbative effects**

Genus-one effective interactions in Type II

- **Start with the Type II genus-one four-graviton amplitude,**

$$\mathcal{A}_4^{(1)}(s,t,u) = \mathcal{R}^4 \int_{\mathcal{M}_1} \frac{d^2\tau}{\tau_2^2} \mathcal{B}^{(1)}(s,t,u|\tau)$$

$$\mathcal{M}_1 = \{\tau = \tau_1 + i\tau_2,\ \tau_{1,2} \in \mathbb{R},\ \tau_2 > 0,\ |\tau| \geq 1, |\tau_1| \leq \tfrac{1}{2}\}$$

- **$\mathcal{B}^{(1)}$ is an integral over four copies of the torus Σ of modulus τ,**

$$\mathcal{B}^{(1)}(s,t,u|\tau) = \left(\prod_{i=1}^4 \int_\Sigma \frac{d^2 z_i}{\tau_2} \right) \exp\Big\{ \sum_{1 \leq i < j \leq 4} s_{ij}\, G(z_i - z_j|\tau) \Big\}$$

$G(z|\tau)$ is the scalar Green function on Σ with modulus τ
- parametrizing the torus by $z = \alpha + \beta\tau$ with $\alpha, \beta \in \mathbb{R}/\mathbb{Z}$
- G is given by a Fourier sum of torus momenta $(m,n) \in \mathbb{Z}^2$

$$G(z|\tau) = \sum_{(m,n) \neq (0,0)} \frac{\tau_2}{\pi |m\tau + n|^2} e^{2\pi i(m\alpha - n\beta)}$$

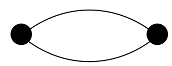
Worldsheet Feynman diagrams

- **Expansion of $\mathcal{B}^{(1)}$ for fixed τ in powers of s_{ij}**
  - finite radius of convergence $|s_{ij}| < 1$
  - integrals $\int_{\Sigma^4} G^n$ are convergent

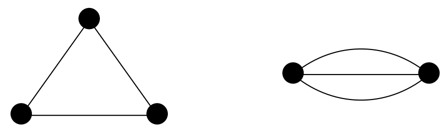
- **Expansion organized in "worldsheet Feynman diagrams"**
  - Each integration point z_i on Σ is represented by a vertex;
  - Each Green function $G(z_i - z_j|\tau)$ by a line —— between z_i and z_j;
  - One-line reducible diagrams (disconnected upon cutting one line) vanish by $\int_\Sigma d^2z\, G(z|\tau) = 0$
  - A diagram with w lines of G,
    - $\star$ has *weight* w;
    - $\star$ contributes to $D^{2w}\mathcal{R}^4$.

Worldsheet Feynman diagrams (connected, 1-line irreducible)

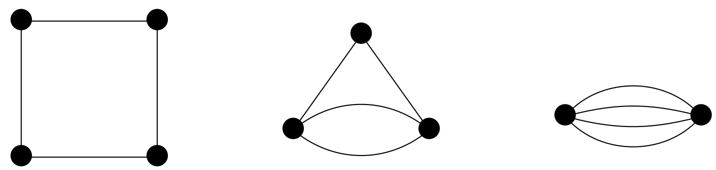
$D^4\mathcal{R}^4$



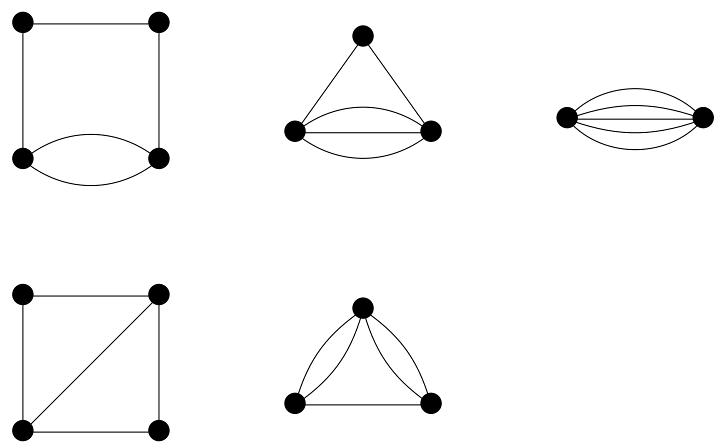
$D^6\mathcal{R}^4$



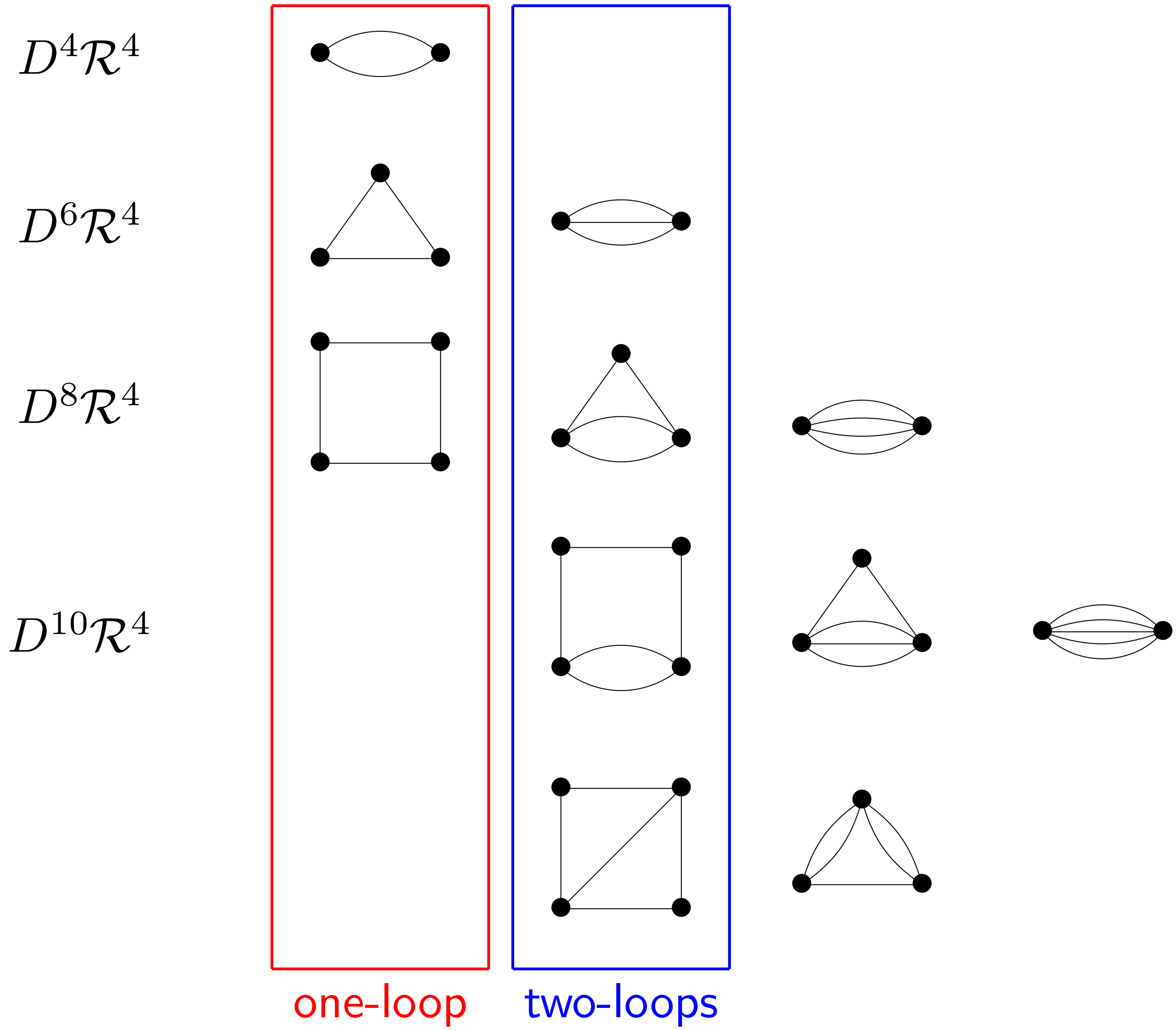
$D^8\mathcal{R}^4$



$D^{10}\mathcal{R}^4$



Worldsheet Feynman diagrams (connected, 1-line irreducible)



One-loop : Eisenstein series

- **One-loop worldsheet Feynman diagram with k-vertices**

$$\prod_{i=1}^{k} \int_\Sigma \frac{d^2 z_i}{\tau_2} G(z_i - z_{i+1}|\tau) = \sum_{(m,n)\neq(0,0)} \frac{\tau_2^k}{\pi^k |m\tau + n|^{2k}}$$

- **Non-holomorphic Eisenstein series** are defined by,

$$E_s(\tau) = \sum_{(m,n)\neq(0,0)} \frac{\tau_2^s}{\pi^s |m\tau + n|^{2s}}$$

- **Properties**

  - absolutely convergent for $\mathrm{Re}(s) > 1$; analytically continue to $s \in \mathbb{C}$
  - reflection relation $\Gamma(s)E_s(\tau) = \Gamma(1-s)E_{1-s}(\tau)$
  - modular invariant under $SL(2,\mathbb{Z})$, $E_s(\tau') = E_s(\tau)$ with $\tau' = \frac{a\tau+b}{c\tau+d}$
  - Laplace-eigenvalue equation,

$$\Big(\Delta - s(s-1)\Big)E_s = 0 \qquad\qquad \Delta = 4\tau_2^2 \partial_\tau \partial_{\bar\tau}$$

Two-loops : Kronecker-Eisenstein series

- **Two-loop Feynman diagrams evaluate to the series**
  (all worldsheet two-loop diagrams are of this type)

$$C_{a_1,a_2,a_3}(\tau) = \sum_{(m_r,n_r)\neq(0,0)} \delta_{m,0}\,\delta_{n,0} \prod_{r=1}^{3} \left(\frac{\tau_2}{\pi|m_r\tau+n_r|^2} \right)^{a_r}$$

  – The total worldsheet momenta must vanish by $\delta_{m,0}\,\delta_{n,0}$
  $m = m_1 + m_2 + m_3 = 0$,
  $n = n_1 + n_2 + n_3 = 0$

  – the *weight* is $w = a_1 + a_2 + a_3$;

  – For our diagrams we have integer $a_r \geq 1$ and the sums converge;

  – $C_{a_1,a_2,a_3}(\tau)$ is a modular function under $SL(2,\mathbb{Z})$;

  – $C_{a_1,a_2,a_3}(\tau)$ is invariant under permutations of a_1, a_2, a_3.

Laplacian on moduli space

- **Structure of the space of Kronecker-Eisenstein series $C_{a,b,c}(\tau)$?**

- Tool : The Laplacian $\Delta = 4\tau_2^2 \partial_\tau \partial_{\bar\tau}$ acts algebraically on the space of $C_{a,b,c}$.

$$\begin{aligned}\Delta C_{a,b,c} &= ab\, C_{a+1,b-1,c} + \frac{1}{2} ab\, C_{a+1,b+1,c-2} - 2ab\, C_{a+1,b,c-1} \\ &\quad + \frac{1}{2} a(a-1)\, C_{a,b,c} + \text{ 5 permutations of } (a,b,c)\end{aligned}$$

  – proven by differentiating term by term and using algebraic rearrangements;
  – Δ preserves the "weight" $w = a + b + c$.

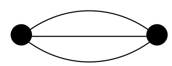
- **One of the subscript indices on the right side may equal 0 or -1,**

$$\begin{aligned}C_{a,b,0} &= E_a E_b - E_{a+b} & a+b \geq 3 \\ C_{a,b,-1} &= E_{a-1} E_b + E_a E_{b-1} & a,b \geq 2\end{aligned}$$

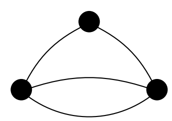
  – E_1 diverges logarithmically, but cancels out in $\Delta C_{a,b,c}$.

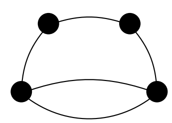
Examples at low weight w

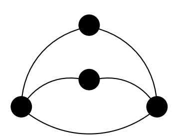
- **We find inhomogeneous Laplace-eigenvalue equations,**

$w = 3$ $\qquad C_{1,1,1} =$  $\qquad \Delta C_{1,1,1} = 6E_3$

  - Use $\Delta E_3 = 6E_3$ to get $\Delta(C_{1,1,1} - E_3) = 0$;
  - constant determined from asymptotics $C_{1,1,1} = E_3 + \zeta(3)$
    (obtained earlier by Zagier using direct calculation of sums)

$w = 4$ $\qquad C_{2,1,1} =$  $\qquad (\Delta - 2)C_{2,1,1} = 9E_4 - E_2^2$

$w = 5$ $\qquad C_{3,1,1} =$  $\qquad (\Delta - 6)C_{3,1,1} = 3C_{2,2,1} + 16E_5 - 4E_2E_3$

$w = 5$ $\qquad C_{2,2,1} =$  $\qquad (\Delta - 0)C_{2,2,1} = 8E_5$

  - Note eigenvalues of the form $s(s-1)$ for $s = 1, 2, 3$;

Structure Theorem for $C_{a,b,c}$ modular functions

- $C_{a,b,c}(\tau)$ **are linear combinations of modular functions** $\mathfrak{C}_{w;s;\mathfrak{p}}(\tau)$ **satisfying**

$$\big(\Delta - s(s-1)\big)\mathfrak{C}_{w;s;\mathfrak{p}} = \mathfrak{F}_{w;s;\mathfrak{p}}\big(E_{s'}\big)$$

  - $\mathfrak{C}_{w;s;\mathfrak{p}}$ and $\mathfrak{F}_{w;s;\mathfrak{p}}$ of weight $w = a+b+c$ (with $E_{s'}$ assigned weight s');
  - $\mathfrak{F}$ is a polynomial of total degree 2 in $E_{s'}$ with $2 \leq s' \leq w$;

$$s = w - 2\mathfrak{m} \qquad \mathfrak{m} = 1, \cdots, \left[\frac{w-1}{2}\right] \qquad \mathfrak{p} = 0, \cdots, \left[\frac{s-1}{3}\right]$$

- **Examples at low weight**

$w=3$	$s=1$	$0^{(1)}$
$w=4$	$s=2$	$2^{(1)}$
$w=5$	$s=1,3$	$0^{(1)} \oplus 6^{(1)}$
$w=6$	$s=2,4$	$2^{(1)} \oplus 12^{(2)}$
$w=7$	$s=1,3,5$	$0^{(1)} \oplus 6^{(1)} \oplus 20^{(2)}$
$w=8$	$s=2,4,6$	$2^{(1)} \oplus 12^{(2)} \oplus 30^{(2)}$
$w=9$	$s=1,3,5,7$	$0^{(1)} \oplus 6^{(1)} \oplus 20^{(2)} \oplus 42^{(3)}$

The generating function

- **A natural generating function is given by,**

$$\mathcal{W}(t_1,t_2,t_2|\tau) = \sum_{a,b,c=1}^{\infty} t_1^{a-1} t_2^{b-1} t_3^{c-1} C_{a,b,c}(\tau)$$

Summing gives a Feynman diagram for three scalars with masses $M_r^2 = -t_r \tau_2$,

$$\mathcal{W}(t_1,t_2,t_2|\tau) = \sum_{(m_r,n_r)\neq(0,0)} \delta_{m,0}\, \delta_{n,0} \prod_{r=1}^{3} \left(\frac{\tau_2}{\pi |m_r \tau + n_r|^2 - t_r \tau_2} \right)$$

- **Algebraic representation of Laplacian induces differential action on $\mathcal{W}$,**

$$\Delta \mathcal{W} - \mathfrak{L}^2 \mathcal{W} = \mathfrak{R}$$

$$\begin{aligned}
\mathfrak{D} &= t_1\partial_1 + t_2\partial_2 + t_3\partial_3 \\
\mathfrak{L}^2 &= \mathfrak{D}^2 + \mathfrak{D} + (t_1^2+t_2^2+t_3^2 - 2t_1t_2 - 2t_2t_3 - 2t_3t_1)(\partial_1\partial_2 + \partial_2\partial_3 + \partial_3\partial_1) \\
\mathfrak{R} &= \text{quadratic polynomial in the Eisenstein series } E_{s'}
\end{aligned}$$

Proof via generating function

- **Permutations of** (a,b,c) **induces permutations of** (t_1,t_2,t_3)
  - $\mathfrak{S}_3$ adapted coordinates,

$$\begin{aligned} u &= t_1+t_2+t_3 & \varepsilon &= e^{2\pi i/3} \\ v/\sqrt{2} &= t_1+\varepsilon t_2+\varepsilon^2 t_3 & (t_1,t_3,t_2)(u,v,\bar v) &= (u,\bar v,v) \\ \bar v/\sqrt{2} &= t_1+\varepsilon^2 t_2+\varepsilon t_3 & (t_2,t_3,t_1)(u,v,\bar v) &= (u,\varepsilon^2 v,\varepsilon \bar v) \end{aligned}$$

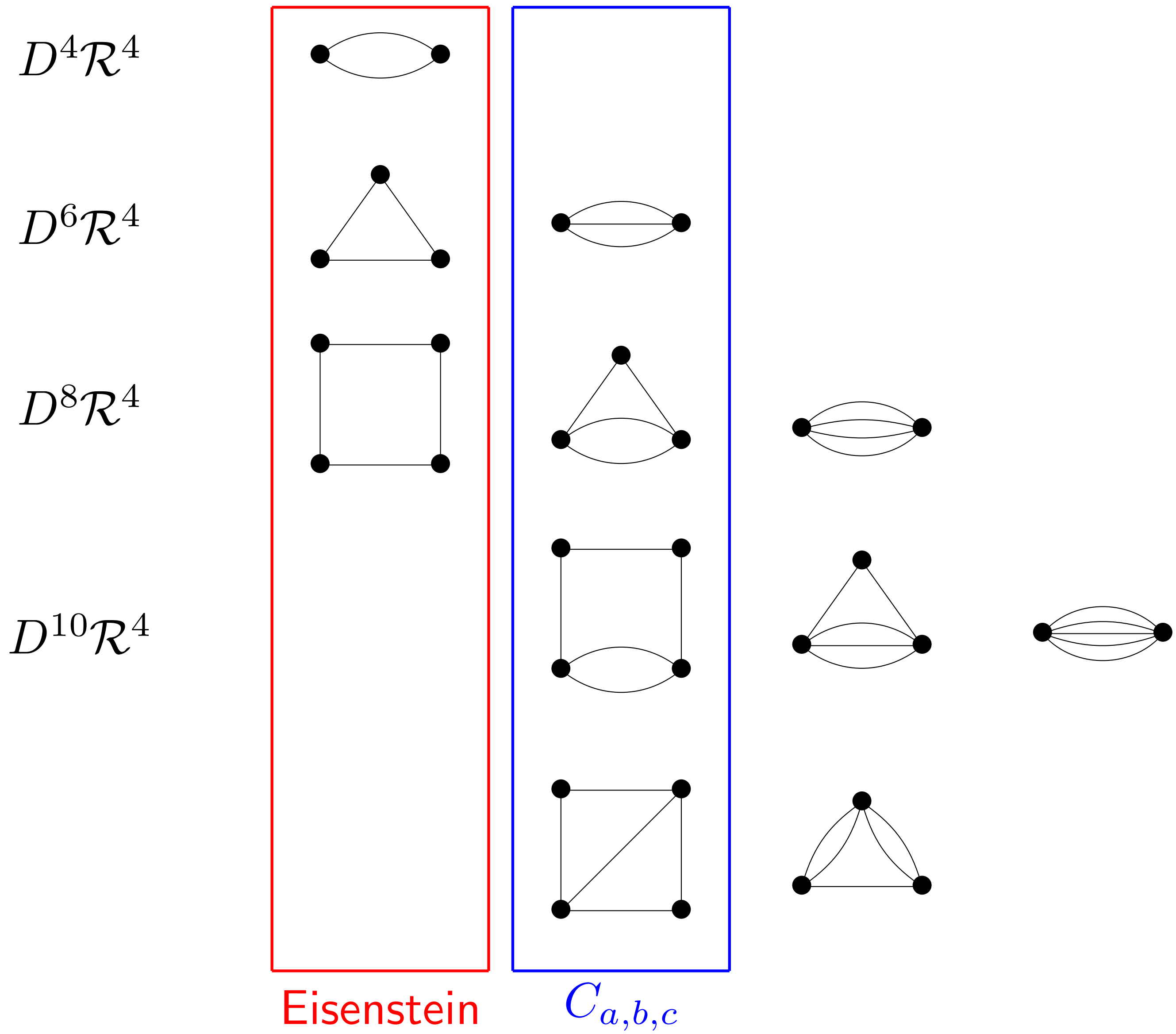
  - $\mathfrak{L}^2=\mathfrak{L}_0^2-\mathfrak{L}_1^2-\mathfrak{L}_2^2$ Casimir of $SO(1,2)$ generated by $\mathfrak{L}_0,\mathfrak{L}_1,\mathfrak{L}_2$;
  - Simultaneously diagonalize the $\mathfrak{S}_3$-invariant operators $\mathfrak{D}$, $\mathfrak{L}_0^2$, and $\mathfrak{L}^2$

$$\begin{aligned} \mathfrak{D}\mathcal{W}_{w;s;\mathfrak{p}} &= (w-3)\mathcal{W}_{w;s;\mathfrak{p}} & \mathfrak{D} &= t_1\partial_1+t_2\partial_2+t_3\partial_3 \\ \mathfrak{L}^2\mathcal{W}_{w;s;\mathfrak{p}} &= s(s-1)\mathcal{W}_{w;s;\mathfrak{p}} & \mathfrak{L}^2 &= -(u^2-2v\bar v)(\partial_u^2-2\partial_v\partial_{\bar v}) \\ \mathfrak{L}_0^2\mathcal{W}_{w;s;\mathfrak{p}} &= -9\mathfrak{p}^2\mathcal{W}_{w;s;\mathfrak{p}} & \mathfrak{L}_0 &= iv\partial_v-i\bar v\partial_{\bar v} \end{aligned}$$

  - $\mathfrak{S}_3$-invariance of eigenfunctions requires $\mathfrak{p}$ to be integer;
  - which explains multiplicities $[(s-1)/3]$.

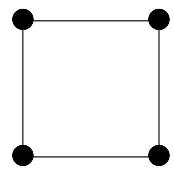
$\Longrightarrow$ constructive proof of Structure Theorem.

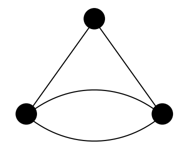
Recall ....

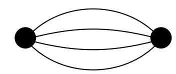

$D^4\mathcal{R}^4$
$D^6\mathcal{R}^4$
$D^8\mathcal{R}^4$
$D^{10}\mathcal{R}^4$
Eisenstein
$C_{a,b,c}$

Modular functions in $D^8\mathcal{R}^4$

- **Modular function $D^8\mathcal{R}^4$ requires**







$= D_4$

  – The modular function D_4 is of the form $C_{1,1,1,1}$ for $\rho = 4$ with

$$C_{a_1,\cdots,a_\rho}(\tau) = \sum_{(m_r,n_r)\neq(0,0)} \delta_{m,0}\,\delta_{n,0} \prod_{r=1}^{\rho} \left(\frac{\tau_2}{\pi|m_r\tau + n_r|^2} \right)^{a_r}$$

  – No useful algebraic representation of the Laplacian is available

Conjectured relation for D_4

- **Tools: take an educated guess + check asymptotic behavior near cusp.**
  - Relations for $C_{a,b,c}$ involved linear combinations of C for given weight;
  - Consider combinations of D_4, E_4, and E_2^2

$$(\Delta - 2)\left(5D_4 - 15E_2^2 - 18E_4\right) = -120E_2^2$$

  - Eliminate right side using $(\Delta - 2)C_{2,1,1} = 9E_4 - E_2^2$ gives

$$(\Delta - 2)\left(D_4 - 24C_{2,1,1} - 3E_2^2 + 18E_4\right) = 0$$

- **Asymptotics near the cusp $\tau_2 \to \infty$ fixes homogeneous solution,**

$$D_4 = 24C_{2,1,1} + 3E_2^2 - 18E_4$$

  - conjectured as an *exact relation* between modular functions;
  - conjectured relation between Feynman diagrams of *different loop orders;*
  - Additional support from direct numerical evaluation of the multiple sums.

Structure of the asymptotics near the cusp

- **The expansion near the cusp $\tau_2 \to \infty$ takes the following form,**

$$D_4(\tau) = \sum_{k,\bar{k}=0}^{\infty} \mathcal{D}_4^{(k,\bar{k})}(\pi\tau_2)\, q^k \bar{q}^{\bar{k}} \qquad\qquad q = e^{2\pi i \tau}$$

- **We have checked the following asymptotics** (similarly for $C_{2,1,1}$, E_4, E_2^2)

$$\begin{aligned}
\mathcal{D}_4^{(0,0)}(y) &= \frac{y^4}{945} + \frac{2\zeta(3)y}{3} + \frac{10\zeta(5)}{y} - \frac{3\zeta(3)^2}{y^2} + \frac{9\zeta(7)}{4y^3} \\
\mathcal{D}_4^{(0,1)}(y) &= \frac{4y^2}{15} + \frac{2y}{3} + 2 + \frac{4}{y} + \frac{12\zeta(3)}{y} - \frac{6\zeta(3)}{y^2} + \frac{9}{2y^2} + \frac{9}{4y^3} \\
\mathcal{D}_4^{(1,0)}(y) &= \mathcal{D}_4^{(0,1)}(y)
\end{aligned}$$

How could the conjecture fail ?

- Consider the difference $F = D_4 - 24 C_{2,1,1} - 3E_2^2 + 18 E_4$
  - the conjecture states $F = 0$

- If the conjecture were to fail, then $F \neq 0$ and its properties are,
  - modular function under $SL(2,\mathbb{Z})$;
  - its pure power part in the expansion near the cusp vanishes;
    - $\Longrightarrow$ F is a *cuspidal function*
  - Vanishing of leading exponential restricts it further.

Progress towards a full proof

- **Inspired by a calculation of Zagier for C_{111}, we first perform n-sums**

$$D_4(\tau) = \sum_{(m_r,n_r)\neq(0,0)} \delta_{m,0}\, \delta_{n,0} \prod_{r=1}^{4} \frac{\tau_2}{\pi |m_r \tau + n_r|^2}$$

  - partition sum according to the number of vanishing m_r;
  - solve for n_4; decompose in partial fractions in n_3; sum over n_3,
  - for $m_3 \neq 0$, sum using the formula (and its derivatives in z)

$$\sum_{n_3 \in \mathbb{Z}} \frac{1}{z+n_3} = -i\pi \, \frac{1+e^{2\pi i z}}{1-e^{2\pi i z}}$$

- **Explicit calculation (using MAPLE) shows the following structure**

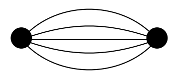
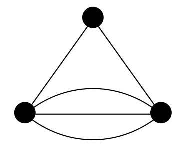
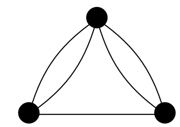
$$D_4(\tau) = \sum_{k=-3}^{4} (\tau_2)^k \, \mathfrak{D}_4^{(k)}(q,\bar q) \qquad\qquad q = e^{2\pi i \tau}$$

  - where $\mathfrak{D}_4^{(k)}(q,\bar q)$ is a function analytic in $q, \bar q$ near the cusp
  - similar expansions for C_{211}, E_2^2, E_4 have *the same range for* k.

- **Conjecture proven for $k=-3,1,2,3,4$;**

  ... in progress for remaining values $k=-2,-1,0$ ...

Conjectured relations for modular functions in $D^{10}\mathcal{R}^4$

$D^{10}\mathcal{R}^4$ requires $D_5 =$  $D_{3,1,1} =$  $D_{2,2,1} =$ 

in addition to E_5, $C_{3,1,1}$, E_2E_3, and $E_2C_{1,1,1}$ functions of weight 5.

– $D_5 = C_{1,1,1,1,1}$ and $D_{3,1,1} = C_{2,1,1,1}$ but $D_{2,2,1}$ not of the form C

- **Educated guesses and asymptotics near the cusp lead us to conjecture,**

$$\begin{aligned} D_5 &= 60C_{3,1,1} + 10E_2C_{1,1,1} - 48E_5 + 16\zeta(5) \\ 40D_{3,1,1} &= 300C_{3,1,1} + 120E_2E_3 - 276E_5 + 7\zeta(5) \\ 10D_{2,2,1} &= 20C_{3,1,1} - 4E_5 + 3\zeta(5) \end{aligned}$$

- **Pattern expected to continue for higher $D^{2w}\mathcal{R}^4$ interactions with $w > 5$.**

Generalizations and Multi-zeta-values

- **Generalized infinite families entering genus-one diagrams** (but not all)

$$C_{a_1,\cdots,a_\rho}(\tau) = \sum_{(m_r,n_r)\neq(0,0)} \delta_{m,0}\,\delta_{n,0} \prod_{r=1}^{\rho} \left(\frac{\tau_2}{|m_r\tau+n_r|^2} \right)^{a_r}$$

  – for integers $a_r \geq 1$ enter diagrams of weight $w = a_1 + \cdots + a_\rho$

- **Multi-zeta-functions**

$$\zeta_{s_1,\cdots,s_n} = \sum_{m_1>\cdots>m_n\geq 1} \frac{1}{m_1^{s_1}\cdots m_n^{s_n}}$$

  – generalize the standard Riemann zeta-function for $n=1$

- **The $C_{a_1,\cdots,a_\rho}$ provide a modular generalization of $\zeta_{s_1,\cdots,s_n}$**
  – leading τ_2 behavior of $C_{a_1,\cdots,a_\rho}$ may be expressed in terms of MZV.
  – MZV naturally enter into open string amplitudes [Schlotterer, Stieberger]

Genus-one coefficients of $D^{2w}\mathcal{R}^4$ for $w \geq 4$

- **Integrating $\mathcal{B}^{(1)}$ over $\mathcal{M}_1$ produces non-analytic behavior in s, t, u;**
  - branch cuts due to loops with massless strings for $s, t, u, \ll 1$;
  - non-analytic parts may be isolated systematically,
    [ED, Phong 1993; Green, Russo, Vanhove 2008]
  - Analytic part is unique after non-analytic part has been specified.

- **Partition fundamental domain $\mathcal{M}_1$ at fixed large $L \gg 1$;** [Maass; Selberg]
  - $\tau_2 > L$ gives non-analytic contributions in s, t, u;
  - $\tau_2 < L$ gives analytic contributions in s, t, u;

- **For compactifications on flat $\mathbb{T}^d$, with moduli ρ_d,**

$$\mathcal{E}_{D^8\mathcal{R}^4}(\rho_d, L) = \frac{1}{2}\int_{\mathcal{M}_1}^{\tau_2<L} d\mu_1 \big(\Delta C_{2,1,1} - 5E_4 + E_2^2\big)\, \Gamma_{d,d,1}(\rho_d|\tau)$$

  - parts non-analytic at $s, t, u = 0$ cancel in comparing moduli ρ_d and ρ'_d;
  - “Differences” produce well-defined and unique effective interactions.

Genus two

- **Siegel Upper half space $\mathcal{S}_2$**

$$\mathcal{S}_2 = \{\Omega_{IJ} = \Omega_{JI} \in \mathbb{C} \text{ with } I, J = 1, 2 \text{ and } Y = \mathrm{Im}\Omega > 0\}$$

  – $Sp(4, \mathbb{R})$ acts by $\Omega \to (A\Omega + B)(C\Omega + D)^{-1}$

$$M^t J M = J \qquad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

  – $\mathcal{S}_2$ has $Sp(4, \mathbb{R})$-invariant metric ds_2^2 and volume form $d\mu_2$

$$ds_2^2 = \sum_{I,J,K,L=1,2} Y^{-1}_{IJ}\, d\bar{\Omega}_{JK}\, Y^{-1}_{KL}\, d\Omega_{LI}$$

- **Compact Riemann surface Σ**
  – A_I, B_I a choice of canonical homology basis for $H_1(\Sigma, \mathbb{Z})$,
  – ω_I dual holomorphic (1,0) forms,

$$\oint_{A_I} \omega_J = \delta_{IJ} \qquad \oint_{B_I} \omega_J = \Omega_{IJ}$$

  – $\Omega \in \mathcal{S}_2$
  – Genus-two modular group $Sp(4, \mathbb{Z})$
  – Genus-two moduli space $\mathcal{M}_2 = \mathcal{S}_2 / Sp(4, \mathbb{Z})$

Effective interactions at genus-two in Type II

- **Start with Type II four-graviton amplitude at genus 2,** [ED, Phong 2005]

$$\mathcal{A}^{(2)}(s,t,u) = \mathcal{R}^4 \int_{\mathcal{M}_2} d\mu_2 \, \mathcal{B}^{(2)}(s,t,u|\Omega)$$

$$\mathcal{B}^{(2)}(s,t,u|\Omega) = \int_{\Sigma^4} \mathcal{Y} \wedge \bar{\mathcal{Y}} \exp \sum_{i<j} s_{ij} \, G(z_i,z_j)$$

  – $G(z_i,z_j)$ is the scalar Green function;
  – $\mathcal{B}^{(2)}$ invariant under $G(z,w) \to G(z,w) + f(z) + f(w)$ for any f
  – $\mathcal{Y} = (s-t)\Delta(z_1,z_3) \wedge \Delta(z_4,z_2) + 2$ permutations;
  – $\Delta(z_i,z_j)$ is a holomorphic $(1,0)_i \otimes (1,0)_j$ form independent of s,t,u.

- **Contributions to local effective interactions,**
  – $\mathcal{R}^4$: zero, since $\mathcal{Y}$ vanishes for $s=t=u=0$;
  – $D^4\mathcal{R}^4$: non-zero, $\mathcal{B}^{(2)}$ constant on $\mathcal{M}_2$;
  – $D^6\mathcal{R}^4$: non-zero, one power of G brought down in integral over Σ^4;

$$\mathcal{B}^{(2)}(s,t,u|\Omega) = 32(s^2+t^2+u^2) + 192 \, stu \, \varphi(\Omega) + \mathcal{O}(s^4, \cdots)$$

  – $\varphi(\Omega)$ coincides with the Zhang -Kawazumi invariant [ED, Green 2013].

The Zhang-Kawazumi invariant for genus-two

- **Definition of the ZK-invariant**

The ZK-invariant takes the following form,

$$8\varphi(\Omega) = \sum_{I,J,K,L} \left(Y^{-1}_{IJ} Y^{-1}_{KL} - 2Y^{-1}_{IL} Y^{-1}_{JK}\right) \int_{\Sigma^2} G(x,y)\omega_I(x)\overline{\omega_J(x)}\omega_K(y)\overline{\omega_L(y)}$$

– φ invariant under $G(z,w) \to G(z,w) + f(z) + f(w)$ for any f
– equivalent to definition via Arakelov geometry [Zhang 2007, Kawazumi 2008]
– invariant under the modular group $Sp(4,\mathbb{Z})$; hence $\varphi(\Omega) = \varphi(\Sigma)$
– related to the genus-two Faltings invariant [De Jong 2010]

- **Coefficient of the genus-two $D^6\mathcal{R}^4$ interaction involves $\int_{\mathcal{M}_2} d\mu_2\, \varphi(\Omega)$**

- **Direct evaluation appeared out of reach ... until ...**

ZK satisfies a Laplace eigenvalue equation

- **Theorem : genus-two ZK invariant satisfies remarkable equation,**

$$(\Delta - 5)\varphi = -2\pi\delta_{SN}$$

  - Δ is the Laplace-Beltrami operator on $\mathcal{M}_2$ with Siegel metric ds_2^2;
  - δ_{SN} has support on separating node (into two genus-one surfaces)

[ED, Green, Pioline, R. Russo 2014]

- **Using Theorem, the integral over $\mathcal{M}_2$ reduces to an integral over $\partial\mathcal{M}_2$,**

$$\int_{\mathcal{M}_2} d\mu_2\,\varphi = \frac{1}{5}\int_{\mathcal{M}_2} d\mu_2\big(\Delta\varphi + 2\pi\delta_{SN}\big) = \frac{2\pi^3}{45}$$

  - meaning of this number is to be discussed in last part of lecture

Proof via deformations of complex structures

- **Laplacian Δ on genus-two moduli space $\mathcal{M}_2$**
  $=$ Laplace-Beltrami operator for the Siegel metric on $\mathcal{S}_2$
  – In terms of the period matrix $\Omega_{IJ} = X_{IJ} + iY_{IJ}$, with $I, J = 1, 2$

$$\Delta = \sum_{I \leq J} \sum_{K \leq L} Y_{IK} Y_{JL} \frac{\partial}{\partial \bar{\Omega}_{IJ}} \frac{\partial}{\partial \Omega_{KL}}$$

- **Variations in Ω_{IJ} result from variation by Beltrami differential μ**

$$\delta_\mu \phi = \frac{1}{2\pi} \int_\Sigma d^2 w \, \mu_{\bar{w}}{}^w \, \delta_{ww} \phi$$

  – $\delta_{ww}\phi$ is obtained by variation of $\bar{\partial}$ or insertion of the stress tensor T_{ww}

$$\begin{aligned}
\delta_{ww} \Omega_{IJ} &= 2\pi i \, \omega_I(w) \omega_J(w) \\
\delta_{ww} \omega_I(x) &= \omega_I(w) \partial_x \partial_w \ln E(x, w) \\
\delta_{ww} G(x, y) &= \frac{1}{2} \left(\partial_w G(w, x) - \partial_w G(w, y) \right)^2
\end{aligned}$$

- **Calculation of mixed derivatives proves $(\Delta - 5)\varphi = 0$ inside $\mathcal{M}_2$**
  – contribution from separating node results from asymptotics of φ

[ED, Green, Pioline, R. Russo 2014]

Generalizations of ZK-invariant

- **The ZK-invariant exists for all genera $h \geq 2$** [Zhang 2007, Kawazumi 2008]
  - but does not satisfy a simple Laplace-eigenvalue equation for $h \geq 3$;
  - most likely is not the correct object for string theory at $h \geq 3$.
- **But the *integrands* on $\mathcal{M}_2$ for the coefficients of $D^8\mathcal{R}^4, D^{10}\mathcal{R}^4, \cdots$**
  - do naturally emerge from string theory;
  - are modular invariants which generalize ZK;
  - satisfy more complicated Laplace-type equations

    ... in progress ...
- **Lowest order generalization of ZK at genus 2, contributing to $D^8\mathcal{R}^4$**

$$\mathcal{B}^{(2)}_{(2,0)} = \int_{\Sigma^4} \Big| \Delta(1,2) \wedge \Delta(3,4) \Big|^2 \Big(G(1,4) + G(2,3) - G(1,3) - G(2,4) \Big)^2$$

S-duality of Type IIB and non-perturbative effects

- **S-duality = $SL(2,\mathbb{Z})$ symmetry of Type IIB superstring theory**
  - axion T_1 and dilaton T_2 are real fields of Type IIB, $T = T_1 + T_2$
  - Type IIB supergravity invariant under $SL(2,\mathbb{R})$ acting on T by Möbius transf.
  - Type IIB string theory invariant under $SL(2,\mathbb{Z})$ (cfr axion anomaly)

- **Effective interactions must respect $SL(2,\mathbb{Z})$ symmetry**
  - for $\mathcal{R}^4$ effective interaction with no derivatives on T, (in string frame)

$$\int_M d^{10}x\sqrt{G}\,(T_2)^{\frac{1}{2}}\,\mathcal{E}(T)\,\mathcal{R}^4 \qquad\qquad T_2 = \frac{1}{g_s}$$

  - $\mathcal{E}(T)$ real-valued modular function under $SL(2,\mathbb{Z})$

- $\mathcal{E}(T) = E_{\frac{3}{2}}(T)$ **conjectured from single D-instanton calculation** [Green Gutperle]
  - Using asymptotics of non-holomorphic Eisenstein near the cusp $T_2 \to \infty$,

$$E_s(T) = \frac{2\zeta(2s)}{\pi^s}T_2^s + \frac{2\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)\pi^{s-\frac{1}{2}}}T_2^{1-s} + \mathcal{O}(e^{-2\pi T_2})$$

  - Perturbative contributions to $\mathcal{R}^4$ arise from genus 0 and 1 **only**.

Supersymmetry and S-duality

- **Laplace-eigenvalue eq from space-time supersymmetry** [Green, Sethi, 1998]
  - Eisenstein series = unique modular solution with polynomial growth at cusp

- **Predicts vanishing contributions for high enough loop order,**

$\mathcal{R}^4$	1/2 BPS	$h \geq 2$	$E_{\frac{3}{2}}$
$D^4\mathcal{R}^4$	1/4 BPS	$h \geq 3$	$E_{\frac{5}{2}}$
$D^6\mathcal{R}^4$	1/8 BPS	$h \geq 4$	$(\Delta - 12)\mathcal{E}_{D^6\mathcal{R}^4} = (E_{\frac{3}{2}})^2$

[Green, Gutperle, Vanhove 1997; Green, Vanhove 2005]

- **Predicts relations between non-vanishing contributions** (e.g. with tree-level),

$\mathcal{R}^4$	$h = 1$	[Green, Gutperle 1997]
$D^4\mathcal{R}^4$	$h = 2$	[ED, Gutperle, Phong 2005]
$D^6\mathcal{R}^4$	$h = 2 \quad (\mathrm{ZK})$	[ED, Green, Pioline, Russo 2014]
	$h = 3$	[Gomez, Mafra 2013]

Summary and outlook

- **Low energy expansion of string theory has revealed a rich structure of**
  - non-holomorphic Kronecker-Eisenstein series on genus-one Riemann surfaces;
  - Zhang-Kawazumi modular invariant on genus-two Riemann surfaces;
  - differential and algebraic interrelations;
  - concrete analytic evaluation of higher local effective interactions.

- **Extensions at genus-one**
  - Understand general interrelations of Kronecker-Eisenstein series beyond $C_{a,b,c}$
  - Identify structure of the ring of all such non-holomorphic modular forms.
  - Equations obeyed by entire string integrand ? [ED, Green] ... in progress ...

- **Extensions at genus-two**
  - ZK-invariant as a ϑ-lift of complete degeneration limit [Pioline 2015]
  - Differential relations obeyed by higher order generalizations of Zhang-Kawazumi invariants [ED, Green, Pioline] ... in progress ...

- **Significance for number theory ?**