Exact half-BPS Solutions to Type IIB supergravity

with John Estes & Michael Gutperle

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- Exact half-BPS Type IIB interface solutions I,
  Local solutions and supersymmetric Janus, arXiv:0705.0022

- Exact half-BPS Type IIB interface solutions II,
  Flux solutions and multi-Janus, arXiv:0705.0024

- Gravity duals of half-BPS Wilson loops, arXiv:0705.1004
• Construct solutions with 16 supersymmetries to Type IIB supergravity on the following spaces,

- $AdS_4 \times S^2 \times S^2 \times w\Sigma$ with $SO(2,3) \times SO(3) \times SO(3)$ symmetry
- $AdS_2 \times S^4 \times S^2 \times w\Sigma$ with $SO(2,1) \times SO(5) \times SO(3)$ symmetry

products warped over 2-dim parameter space $\Sigma$ (to be specified)

• General local solution: exactly in terms of two harmonic functions on $\Sigma$

• General global solution: all non-singular Type IIB solutions are obtained for geometries whose boundary is locally asymptotically $\sim AdS_5 \times S^5$

• These solutions have varying dilaton, and non-zero 3-form fields.
Motivation

• Construction of AdS dual to half-BPS states in $\mathcal{N} = 4$ SYM
• Closely related to Lin, Lunin and Maldacena (LLM)
  – Half-BPS states on $\mathbb{R} \times S^3$, $s$-wave, $SU(4)$-highest weight
  – equivalent to free fermion quantum mechanics (Berenstein)
  – LLM provide AdS duals to these states,
  – obtain all half-BPS solutions with $\mathbb{R} \times SO(4) \times SO(4)$ symmetry, with constant dilaton, and vanishing 3-form fields
• $AdS_4 \times S^2 \times S^2 \times w \Sigma$ geometry: $\mathcal{N} = 4$ SYM with a planar interface
• $AdS_2 \times S^4 \times S^2 \times w \Sigma$ geometry: $\mathcal{N} = 4$ SYM with a Wilson line
• AdS$_4$ slicings of AdS$_5$ have 2+1-dim planar interface CFT duals

\[ ds^2 = f(\mu)^2 (d\mu^2 + ds_{AdS_4}^2) + ds_{S^5}^2 \]

\[ \phi(\mu) \]

• AdS$_5 \times S^5$ has $f(\mu) = (\cos \mu)^{-1}$, with $\mu \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\phi$ constant. • More generally, $\phi$ may vary: Janus solution (Bak, Gutperle, Hirano)

• $SO(2,3)$ isometry group of AdS$_4 = \text{conformal group of planar interface}$
Let $x^\pi$ be a coordinate transverse to a planar interface at $x^\pi = 0$

Bulk Lagrangians on each side are supersymmetric $\delta \mathcal{L}_\pm = \partial_\mu X_\pm^\mu$

$$\mathcal{L} = \theta(x^\pi)\mathcal{L}_+ + \theta(-x^\pi)\mathcal{L}_-$$

$$\delta \mathcal{L} = \partial_\mu \left( \theta(x^\pi)X_+^\mu + \theta(-x^\pi)X_-^\mu \right) - \delta(x^\pi) \left( X_+^\pi - X_-^\pi \right)$$

Can one compensate for $X_+^\pi - X_-^\pi$ by interface Lagrangian $\mathcal{L}_I$?

Assume coupling varies across interface,

$$\mathcal{L} = \frac{1}{g(x^\pi)^2} \mathcal{L}_{\{N=4\}} + \mathcal{L}_I$$

Bulk susy transformations must also be modified on interface
Susy driven by $\psi^t Y \psi + cc$ term in $\mathcal{L}_I$, where $Y$ acts on $SU(4)$ indices

$$Y \rightarrow \text{diag}[d_1, d_2, d_3, d_4] \quad d_i \text{ real } \geq 0$$

- **0 supersymmetries** $\text{diag}[0 \ 0 \ 0 \ 0]$, global $SU(4)$ R-symmetry,
  - CFT dual to Janus solution of Bak, Gutperle, Hirano
- **4 supersymmetries** $\text{diag}[1 \ 0 \ 0 \ 0]$, global $SU(3)$ R-symmetry;
  - CFT: Clark, Freedman, Karch, Schnabl; AdS dual D’Hoker, Estes, Gutperle
- **8 supersymmetries** $\text{diag}[1 \ 1 \ 0 \ 0]$, global $SO(2) \times SO(3)$ R-symmetry;
- **16 supersymmetries** $\text{diag}[1 \ 1 \ 1 \ 1]$, global $SO(3) \times SO(3)$ R-symmetry;
  - AdS dual: This talk!
AdS side : Type IIB Supergravity

- The fields of Type IIB sugra are

\[
g_{MN} \quad \text{metric} \quad M, N = 0, 1, \cdots, 9
\]
\[
B \quad \text{axion/dilaton} \quad P \sim dB \quad (\text{contains } \chi, \phi)
\]
\[
B(2) \quad \text{antisymmm} \quad G(3) \sim (dB(2) - BdB^*(2))
\]
\[
C(4) \quad \text{antisymmm} \quad F(5) = dC(4) + \text{Im}(\bar{B}(2)dB(2))/8 \quad F(5) = *F(5)
\]
\[
\psi_M \quad \text{gravitino} \quad \text{Weyl spinor}
\]
\[
\lambda \quad \text{dilatino} \quad \text{Weyl spinor}
\]

- The susy variation equations for the spinors are (J.H. Schwarz, 1983)

\[
\delta \psi_M = D_M \epsilon + \frac{i}{4} \left( F(5) \cdot \Gamma \right) \Gamma M \epsilon - \frac{1}{16} \left( \Gamma_M (G(3) \cdot \Gamma) + 2 (G(3) \cdot \Gamma) \Gamma_M \right) B^{-1} \epsilon^*
\]
AdS dual to Interface with 16 susys

- Isometry group must be $SO(2,3) \times SO(3) \times SO(3)$;
- Space-time is $AdS_4 \times S^2_1 \times S^2_2$ warped over a 2-dim parameter space $\Sigma$

\[
e^{i_1} = f_1 \hat{e}^{i_1} \quad ds^2 = f_1^2 ds^2_{S^2_1} + f_2^2 ds^2_{S^2_2} + f_4^2 ds^2_{AdS^2_4} + ds^2_{\Sigma}
\]
\[
e^{i_2} = f_2 \hat{e}^{i_2} \quad G(3) = g_a e^{45a} + i h_a e^{67a}
\]
\[
e^m = f_4 \hat{e}^m \quad F(5) = f_a (-e^{0123a} + \epsilon^a_e e^{4567e})
\]

- index ranges: $m = 0, 1, 2, 3$; $i_1 = 4, 5$; $i_2 = 6, 7$; $a = 8, 9$; $\epsilon^8_9 = 1$
- orthonormal frames $\hat{e}^m, \hat{e}^{i_1}, \hat{e}^{i_2}$ resp. on $AdS_4, S^2_1, S^2_2$ with unit radius
- $e^a$ is the orthonormal frame on $\Sigma$ with $ds^2_{\Sigma} = e^a \otimes e^a$
- functions $f_1, f_2, f_4, f_a$ are real, $g_a, h_a, B$ complex
• Reduce BPS equations $\delta \lambda = \delta \psi = 0$ to the above Ansatz

• Method of Killing spinors/vectors
  (Gauntlett, Martelli, Pakis, Waldram – Pilch, Warner)
  – well-suited for problems with $G_{(3)} = 0$
  – less suited for varying dilaton and $G_{(3)} \neq 0$

• $AdS_4 \times S^2 \times S^2$ BPS eqs set up by Gomis and Rommelsberger
  $AdS_2 \times S^4 \times S^2$ BPS eqs set up by Lunin
  – identified one harmonic function,
  – but obtaining $f_1, f_2, f_4, \cdots$ still requires solving differential equations
  – which they did not succeed in doing.

• We work directly with the BPS equations;
  – we shall use Killing spinors as well,
  – but we shall not follow the standard methods of GMPW – PW
  – this will allow us to find the general solution, exactly.
• Solutions with 16 susys
  – axion/dilaton runs over a geodesic path

  – by $SL(2, \mathbb{R})$ symmetry of Type IIB, every solution to the BPS equations may be mapped to a solution with vanishing axion field, and real $g_a, h_a, B$.

  – Every solution to the BPS equations automatically solves the Bianchi and field equations
The role of Killing spinors

- The dilatino BPS equation, for non-constant dilaton, $\partial_M \phi \neq 0$, 
  
  \[ 0 = 4(\partial_M \phi) \Gamma^M B^{-1} \varepsilon^* - (G_{(3)} \cdot \Gamma) \varepsilon \]

  - allows for at most 16 independent solutions $\varepsilon$

- The gravitino BPS equation should impose no further restrictions on $\varepsilon$,
  
  \[ 0 = D_M \varepsilon + \frac{i}{4} (F(5) \cdot \Gamma) \Gamma_M \varepsilon - \frac{1}{16} \left( \Gamma_M (G_{(3)} \cdot \Gamma) + 2(G_{(3)} \cdot \Gamma) \Gamma_M \right) B^{-1} \varepsilon^* \]

- On any one of the maximally symmetric components, $AdS_4$, $S^2_1$, $S^2_2$
  
  - $\varepsilon$ should reduce to a “Killing spinor” (see e.g. Gomis and Rommelsberger)
  
  $\varepsilon = \text{spinor of maximal rank}$
Consider spheres $S^d = SO(d + 1)/SO(d)$ of even dimension $d$.

Maurer-Cartan connection $\omega$, in spinor representation, $V \in SO(d + 1)$,

$$\omega = V^{-1}dV = \frac{1}{4} \omega_{\bar{m}\bar{n}} \gamma^{\bar{m}\bar{n}} \quad \bar{m}, \bar{n} = 1, \ldots, d + 1$$

– decomposes into a $SO(d)$ connection $\omega_{mn}$, with $m, n = 1, \ldots, d$

and an orthonormal frame $e_m = \omega_m(d+1)$ on $S^d$

The parallel transport equation $(d + \omega)\chi = 0$ is solved by $\chi = V\chi_0$,

– No constraints on $\chi_0 \Rightarrow$ solution space always has maximal rank

– Coincides with the Killing spinor equation on $S^d$,

$$\left( d + \frac{1}{4} \omega_{mn} \gamma^{mn} - \frac{1}{2} \eta e_m \gamma^m \gamma^{d+1} \right) \chi_\eta = 0 \quad \eta = \pm 1$$

Analogously for $AdS_d = SO(2, d - 1)/SO(1, d - 1)$ spaces.
Reducing the BPS equations

• Use Killing spinors on $AdS_4 \times S^2 \times S^2$ as basis for the susy parameter $\varepsilon$,

$$\varepsilon = \sum_{\eta_1, \eta_2, \eta_3} \chi^{\eta_1, \eta_2, \eta_3} \otimes \zeta^{\eta_1, \eta_2, \eta_3}$$

- $\chi^{\eta_1, \eta_2, \eta_3}$ fixed basis, $\eta = \pm$ independently;
- $\zeta^{\eta_1, \eta_2, \eta_3}$ are 2-component spinors on $\Sigma$,

$\Rightarrow$ 2 complex algebraic reduced dilatino eqs
$\Rightarrow$ 6 complex algebraic reduced gravitino eqs along $AdS_4 \times S^2 \times S^2$;
$\Rightarrow$ 4 complex differential reduced gravitino eqs.

• Symmetries of reduced BPS eqs lead to only non-vanishing components,

$$\zeta^{\pm \pm \pm -} \sim \zeta^{* \pm \mp +} \sim \alpha$$

$$\zeta^{- \pm \mp -} \sim \zeta^{* \pm \pm +} \sim \beta$$
Reducing the BPS equations cont’d

- Introduce local complex coordinates $w, \bar{w}$ on $\Sigma$, $ds^2_\Sigma = 4\rho^2|dw|^2$

- The algebraic equations may be used to solve for $f_1, f_2, f_4, g_a, h_a, f_a$,

  \[
  f_1 = \alpha\bar{\beta} + \bar{\alpha}\beta \\
  f_2 = i\bar{\alpha}\beta - i\alpha\bar{\beta} \\
  f_4 = \alpha\bar{\alpha} + \beta\bar{\beta} \\
  g_z + ih_z = -4\alpha(\rho\beta)^{-1}\partial_w\phi \\
  g_z - ih_z = +4\beta(\rho\alpha)^{-1}\partial_w\phi \\
  f_z = \frac{i\nu}{2\alpha\beta} - \frac{\alpha^4 - \beta^4}{4\rho\alpha^2\beta^2}\partial_w\phi
  \]

- Then, eliminate these functions from the differential BPS equations,

- Two of the differential BPS eqs are equivalent to Cauchy-Riemann eqs,

  \[
  \partial_w(\rho\alpha^2) + \rho\beta^2\partial_w\phi = 0 \quad \Rightarrow \quad \alpha^2 + \beta^2 = \bar{\kappa} e^{-\bar{\lambda}} \rho^{-1} e^{-\phi} \\
  \partial_w(\rho\beta^2) + \rho\alpha^2\partial_w\phi = 0 \quad \Rightarrow \quad \alpha^2 - \beta^2 = \bar{\kappa} e^{+\bar{\lambda}} \rho^{-1} e^{+\phi}
  \]

- $\kappa, \lambda$ arbitrary holomorphic, respectively 1-form and 0-form.
Mapping to a new integrable system

● Eliminating $\alpha, \beta$ from the remaining 2 differential eqs
  – leaves 2 complex differential eqs for $\phi, \rho$ in terms of $\kappa, \lambda$.
  – Drastic simplification by changing variables to $i\mu \equiv \lambda - \bar{\lambda}$, and
    
    \[
    \frac{\text{sh}(2\phi + 2\lambda)}{\text{sh}(2\phi + 2\bar{\lambda})} = e^{2i\vartheta} \quad \rho^8 = \hat{\rho}^8 \left(\sin 2\mu\right)^2 \frac{\kappa^4 \bar{\kappa}^4}{16} \frac{\sin \vartheta + \sin \mu}{(\sin \vartheta - \sin \mu)^3}
    \]

● Two differential eqs for $\vartheta, \hat{\rho}$ in terms of $\kappa, \mu$ become,

    \[
    \partial_w \vartheta - \left(e^{-i\vartheta} + i \sin \mu \left(\cos \mu\right)^{-1} \partial_w \mu\right) = -i\kappa \hat{\rho}^2 e^{i\vartheta/2}
    \]

    \[
    \partial_w \vartheta - 2e^{-i\vartheta} \left(\cos \mu\right)^{-1} \partial_w \mu = -2i\partial_w \ln \hat{\rho}^2
    \]

    = System of Bäcklund transfs for the partial differential eq,

    \[
    \partial_{\bar{w}} \left(\partial_w \vartheta - 2(\cos \mu)^{-1}(\partial_w \mu)e^{-i\vartheta}\right) + \text{c.c.} = 0
    \]

● This system is automatically integrable.
• A final change of variables, $\psi \equiv \hat{\rho}^{-2}(\cos \mu)e^{-i\vartheta/2}$ maps to a linear system,

$$\begin{align*}
\partial_w \psi &= -\kappa \cos \mu \\
\partial_w \bar{\psi} &= (i\psi - \bar{\psi} \sin \mu)(\partial_w \mu)(\cos \mu)^{-1}
\end{align*}$$

• whose general solution is given via $\kappa, \mu$, or equivalently harmonic $h_1, h_2$,

$$\psi = ih_1e^{-\bar{\lambda}} + h_2e^{\bar{\lambda}} \quad e^{2\lambda} = i\partial_w h_1/\partial_w h_2 \quad \kappa^2 = 4i\partial_w h_1\partial_w h_2$$

• All fields of the Ansatz may be expressed in terms of $h_1, h_2$, e.g.

$$\begin{align*}
e^{4\phi} &= \frac{2h_1h_2|\partial_w h_2|^2 - h_2^2W}{2h_1h_2|\partial_w h_1|^2 - h_1^2W} \quad W \equiv \partial_w h_1\partial_{\bar{w}} h_2 + c.c. \\
\rho^8 &= \frac{W^2}{h_1^3h_2^3}\left(2h_1|\partial_w h_2|^2 - h_2W\right)\left(2h_2|\partial_w h_1|^2 - h_1W\right)
\end{align*}$$
**AdS$_5 \times S^5$ and Janus with 16 susys**

- We readily obtain a 2-parameter family of non-singular solutions,

\[
h_1 = \text{Im} \left( e^{w-\phi^+} - e^{-w-\phi^-} \right) \quad \quad h_2 = \text{Re} \left( e^{w+\phi^+} + e^{-w+\phi^-} \right)
\]

- For $\phi^+ = \phi^-$ gives $\text{AdS}_5 \times S^5$

- For $\phi^+ \neq \phi^-$, dilaton varies
  - in $\partial \Sigma$ : $h_1 h_2 = 0$
  - in $\Sigma$ : $W \leq 0, \ h_1, h_2 \geq 0$
More generally, regularity will require that inside $\Sigma$, we have

$$0 < e^{4\phi} = \frac{2h_1 h_2 |\partial_w h_2|^2 - h_2^2 W}{2h_1 h_2 |\partial_w h_1|^2 - h_1^2 W} \quad W \equiv \partial_w h_1 \partial_w h_2 + c.c.$$

$$0 \leq \rho^8 = \frac{W^2}{h_1^4 h_2^4} \left(2h_1 h_2 |\partial_w h_2|^2 - h_2^2 W \right) \left(2h_1 h_2 |\partial_w h_1|^2 - h_1^2 W \right)$$

Set of manifestly sufficient conditions inside $\Sigma$,

$$h_1 > 0 \quad h_2 > 0 \quad W \leq 0$$

- These are obeyed by $AdS_5 \times S^5$ and Janus with 16 susys

Still need boundary conditions on $\partial \Sigma$. 
General regularity conditions cont’d

• General solution manifold $S$ is specified by 3 conformal data: $\Sigma, h_1, h_2$.
  – Assume boundary of $S$ is locally asymptotic to $AdS_5 \times S^5$;
  – Asymptotic $AdS_5 \times S^5$ regions originate from isolated points on $\partial \Sigma$;
  $\Rightarrow$ Away from those points, $\partial \Sigma$ produces interior points of $S$;
  $\Rightarrow$ Either $S_1^2$ or $S_2^2$ must shrink to zero on $\partial \Sigma$ (but never $AdS_4$)
  $\Rightarrow$ Either $f_1 = 0$ or $f_2 = 0$ on $\partial \Sigma$ (but $f_4$ is never zero);

• The form of the solution then imposes boundary conditions on $h_1, h_2$,
  \[
  f_1^2 f_4^2 = 4 e^{2\phi} h_1^2 \quad f_1 = 0 \quad \Rightarrow \quad (h_1 = 0 \quad \& \quad \partial_w h_2 = 0) \\
  f_2^2 f_4^2 = 4 e^{-2\phi} h_2^2 \quad f_2 = 0 \quad \Rightarrow \quad (h_2 = 0 \quad \& \quad \partial_w h_1 = 0)
  \]

• Equivalent to two coupled electro-statics problems with
  – alternating Neumann and vanishing Dirichlet conditions on $\partial \Sigma$
  – $h_1, h_2 \geq 0$ in the interior of $\Sigma$
General regular solutions

- Map the domain $\Sigma$ onto the lower half-plane with complex coordinate $u$.
  - The boundary $\partial \Sigma$ is then the real axis $\mathbb{R}$.
  - Points $e_i$ on $\partial \Sigma$ where Dirichlet $\leftrightarrow$ Neumann, $i = 1, 2, \cdots, 2g + 2$.
- Construction of $h_1, h_2$ via hyperelliptic curve of genus $g$, defined by
  \[ s(u)^2 = (u - e_1)(u - e_2) \cdots (u - e_{2g+1}) \]
  \[ e_{2g+1} < \cdots < e_1 < e_0 = \infty \]
- The meromorphic differentials $\partial h_1, \partial h_2$ may be written down explicitly,
  \[ \partial h_1 = -i \frac{P_1(u)du}{s(u)^3} \]
  \[ \partial h_2 = -\frac{P_2(u)du}{s(u)^3} \]
  - for two real polynomials $P_1, P_2$ of degree $3g + 1$,
  - Neumann and Dirichlet conditions automatically satisfied,
  - behavior at branch points $du/(u-e_i)^{3/2}$ guarantees asymptotic $AdS_5 \times S^5$
• Regularity requires that
  – $P_1, P_2$ have $g$ common complex zeros $u_a, a = 1, \cdots, g,$
  – $P_1$ has $g + 1$ real zeros $\alpha_b, b = 1, \cdots, g + 1,$
  – $P_2$ has $g + 1$ real zeros $\beta_b, b = 1, \cdots, g + 1,$ satisfying the ordering
    \[
    \alpha_{g+1} < e_{2g+1} < \beta_{g+1} < e_{2g} < \cdots < e_2 < \alpha_1 < e_1 < \beta_1
    \]
• It only remains to ensure that the Dirichlet conditions VANISH,
  \[
  \text{Im} \int_{e_{2j}}^{e_{2j-1}} \partial h_1 = \text{Im} \int_{e_{2j+1}}^{e_{2j}} \partial h_2 = 0 \quad j = 1, \cdots, g
  \]
• Given the branch points $e_i$ and the ordered real zeros $\alpha_b, \beta_b,$
  – The above period relations determine the $g$ complex zeros $u_a.$
  – The geometry of the allowed moduli space is known explicitly for $g = 1,$
  – and is known locally for $g \geq 2$; analogous to instanton moduli space.
Topography of regular solutions

- $2g + 2$ branch points = different asymptotic boundary $AdS_5 \times S^5$ regions
  - each with its independent constant dilaton limit
- There are $g$ independent pairs of homology 3-spheres, $j = 1, \ldots, g$
  - $S^3_{1j} = [e_{2j}, e_{2j-1}] \times f$ $S^2_1$ NSNS 3-form charges $\mathcal{H}_j = \int_{S^3_{1j}} H_{(3)}$
  - $S^3_{2j} = [e_{2j+1}, e_{2j}] \times f$ $S^2_2$ RR 3-form charges $\mathcal{F}_j = \int_{S^3_{2j}} F_{(3)}$
  - e.g. genus 1
• For any genus $g$, solutions have $2g + 2$ asymptotic $AdS_5 \times S^5$
• Number of free parameters of solution is $4g + 6$,
  – including: restoring the axion by $SL(2, \mathbb{R})$, and overall radius
Topo change: a collapsing branch cut

\[ \partial h_1 = \frac{(u - u_i)(u - \bar{u}_i)(u - \alpha_i)}{(u - e_{2i})^{3/2}(u - e_{2i-1})^{3/2}}(\partial h_1)_{g-1} \]
\[ \partial h_2 = \frac{(u - u_i)(u - \bar{u}_i)(u - \beta_{i+1})}{(u - e_{2i})^{3/2}(u - e_{2i-1})^{3/2}}(\partial h_2)_{g-1} \]

- As \( e_{2i-1} \to e_{2i} \) we must have \( \alpha_i \to e_{2i} \), and \( \text{Im} \int \partial h_1 = 0 \) forces \( u_i \to e_{2i} \)
- Two possibilities
  - (A) \( \beta_{i+1} \to e_{2i} \) gives topology change \( (\partial h_{1,2})_g \to (\partial h_{1,2})_{g-1} \)
  - (B) \( \beta_{i+1} \not\to e_{2i} \) gives \( \partial h_1 \to (\partial h_1)_{g-1} \) but leaves a singular \( \partial h_2 \)

\( \sim \) the probe limit: a D5 (or NS5) brane remains
Total branch cut collapse

- Collapse of all branch cuts produces probe brane limit,
  - $m_R$ D5 branes and $m_{NS}$ NS5 branes with $m_R + m_{NS} = g$
  - leads to a simple explicit solution,
    \[
    h_1 = -2i(w - \bar{w}) \left(1 + \frac{C_0}{|w|^2}\right) + \sum_{j=1}^{m_R} \frac{C_j}{\ell_j} \ln \left|\frac{w + i\ell_j}{w - i\ell_j}\right|^{2}
    \]
    \[
    h_2 = -2(w + \bar{w}) \left(1 + \frac{D_0}{|w|^2}\right) - \sum_{i=1}^{m_{NS}} \frac{D_i}{k_i} \ln \left|\frac{w + k_i}{w - k_i}\right|^{2}
    \]
- for real $k_i, \ell_j, C_j, D_i, C_0, D_0$, and positive
- RR and NSNS 3-form charges given by
  \[
  \mathcal{F}_j = C_j/\ell_j \quad \quad \mathcal{H}_i = D_i/k_i
  \]
CFT dual to $AdS_4$ solutions (in progress)

- The $AdS_4$ factor indicates the presence of an interface.
- For $g = 0$, CFT dual has interface operators (built from bulk fields).
- For $g \geq 1$, several gauge groups
  - different species of $\mathcal{N}=4$, decoupled away from interface
  - interact only via the interface
  - are coupled via extra massless fields on the interface
  - On AdS side, extra massless fields arise from $S^3$ shrinking to zero
- For $g \geq 1$, as branch cuts collapse,
  - and we approach the limit with probe branes,
  - recover massless string excitations from probe D5 branes
    of De Wolfe, Freedman, Ooguri – Skenderis, Taylor
- Our solutions are fully back-reacted geometries with D5 and NS5 branes
The Wilson loop with 16 supersymmetries

- On CFT side, half-BPS Wilson loop in $\mathcal{N}=4$, $SU(N)$ SYM,

$$\mathcal{W} = \mathcal{W}_{16} = \mathrm{Tr}_R \exp i \int dt \left( A_0 + n^I \phi_I \right) \mid n \mid = 1$$

- Invariant under $SO(2,1) \times SO(3) \times SO(5)$

- On AdS side, quantum numbers can be realized via probe branes
  - fundamental rep $\Box$ via D3 probe (Maldacena – Rey, Yee)
  - symmetrizations of $\Box$ via D5 probes (Drukker, Fiol)

- General representation $R$, (Gomis, Passerini)
  - probe D5 branes $j = 1, \cdots, M$, with $m_j$ units of F1 dissolved
  OR probe D3 branes $i = 1, \cdots, N$, with $n_i$ units of F1 dissolved
Wilson loop AdS dual supergravity solution

- CFT symmetry $SO(2, 1) \times SO(3) \times SO(5)$ requires
- AdS dual geometry $AdS_2 \times S^2 \times S^4 \times w \Sigma$
  - Formally, obtained by analytic continuation from $AdS_4 \times S^2 \times S^2 \times w \Sigma$
  - Analytic continuation may send regular to singular, & change # susys
    $\Rightarrow$ Topologies and # moduli different
- Our methods: general solution via harmonic functions $h_1, h_2$ on $\Sigma$. 
e.g. The genus 1 solution for $AdS_2$

- For any genus $g$, solutions have only a single asymptotic $AdS_5 \times S^5$
- There are $g$ independent homology $S^3$ carrying RR 3-form charges
- Number of free parameters of solution is $2g + 5$ for $g \geq 1$ (3 for $g = 0$).
Can one construct all solutions with 16 susys to Type IIB sugra?
Can one construct all solutions with 16 susys which have a CFT dual?
  – View as AdS duals to deformations of $\mathcal{N} = 4$ SYM
  – Expect a subgroup $H$ of $SU(2, 2|4)$ with 16 susys to be preserved
  – Semi-simple $H$ first, with maximal bosonic subgroup $H_B$

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<td>2)\times SU(2</td>
<td>2)$</td>
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<td>$OSp(4</td>
<td>4^*)$</td>
<td>$SO(2, 3)\times SO(3)\times SO(3)$</td>
<td>$AdS_4\times S^2\times S^2\times \Sigma$</td>
</tr>
<tr>
<td>$OSp(4^*</td>
<td>4)$</td>
<td>$SO(2, 1)\times SO(3)\times SO(5)$</td>
<td>$AdS_2\times S^2\times S^4\times \Sigma$</td>
</tr>
<tr>
<td>$SU(2</td>
<td>4)$</td>
<td>$SO(3)\times SO(5)$</td>
<td>$M_3\times S^2\times S^4$</td>
</tr>
<tr>
<td>$SU(1, 1</td>
<td>4)$</td>
<td>$SO(2, 1)\times SO(5)$</td>
<td>$AdS_2\times S^5\times E_3$</td>
</tr>
<tr>
<td>$SU(2, 2</td>
<td>2)$</td>
<td>$SO(2, 4)\times SO(3)$</td>
<td>$AdS_5\times S^2\times E_3$</td>
</tr>
</tbody>
</table>
Further open problems

- Half-BPS solutions to Type IIB supergravity are surprisingly manageable;

- Regular solutions to $AdS_2$ and $AdS_4$ problems with other topologies?

- Can one derive a reduced quantization of only the half-BPS states
  - Free fermion/matrix reduction may be derived directly from LLM
    (Maoz, Rychkov – Grant, Maoz, Marsano, Papadodimas, Rychkov)

- Unified approach to 16 susy solutions from subgroups of $SU(2,2|4)$?
The End
• Using the same requirements as for the interface transformations,

\[
\mathcal{L}_\psi = \frac{\partial_\pi g}{g^3} \text{tr} \left( y_1 \bar{\psi} \gamma^\pi \psi + \frac{i}{4} y_2 i^{ij} \bar{\psi} \gamma^\pi \rho^{ij} \psi - \frac{i}{2} y_3 i^{ijk} \psi^t \rho^{ijk} \psi + cc \right)
\]

\[
\mathcal{L}_\phi = \frac{\partial_\pi g}{2g^3} \text{tr} \left( z_1^{ij} \partial_\pi (\phi^i \phi^j) + 2 z_2^{ij} \phi^{[i} D_\pi \phi^{j]} - i z_3^{ijk} \phi^i [\phi^j, \phi^k] \right)
\]

\[
\mathcal{L}_{\phi^2} = \frac{(\partial_\pi g)^2}{2g^4} \text{tr} \left( z_4^{ij} \phi^i \phi^j \right)
\]

• The interface terms have the following $SU(4)_R$ representations,

\[
\left\{ \begin{array}{c}
y_1 \\
z_1, z_4
\end{array} \right\} \begin{array}{c} 1 \\ 1 \oplus 20' \end{array}
\left\{ \begin{array}{c}
y_2, z_2 \\
y_3, z_3
\end{array} \right\} \begin{array}{c} 15 \\ 10 \oplus 10^* \end{array}
\]

• The $10, 10^*, 15, 20'$, couple to sugra fields; $1$ couples to strings.
The maximally supersymmetric iCFT

- $D = \text{diag}[1\ 1\ 1\ 1]$, 8 Poincaré (16 conf) susy: global $SO(3) \times SO(3)$;

$$\mathcal{L}_I = \frac{\partial \pi g}{2g^3} \text{tr} \left( i \psi^t C \psi + i \psi^\dagger C \psi^* - 4i \phi^2 [\phi^4, \phi^6] \right)$$

$$\quad + (g^{-3}(\partial_\pi g)\partial_\pi - 2g^{-4}(\partial_\pi g)^2) \text{tr} \left( (\phi^2)^2 + (\phi^4)^2 + (\phi^6)^2 \right)$$

- Last term may be absorbed into kinetic term for $\phi^i$ by rescaling $\phi^i \rightarrow g^2 \phi^i$ for $i = 2, 4, 6$ and $\phi^i \rightarrow \phi^i$ for $i = 1, 3, 5$

$$\mathcal{L}_I^{\text{rescaled}} = \frac{\partial \pi g}{2g^3} \text{tr} \left( i \psi^t C \psi + i \psi^\dagger C \psi^* - 4ig^6 \phi^2 [\phi^4, \phi^6] \right)$$

- The rescaled theory admits a **conformal limit** where $g$ is a step function.