Superstring Amplitudes and Effective Interactions

Eric D'Hoker
Mani L. Bhaumik Institute for Theoretical Physics
University of California, Los Angeles

50 Years of the Veneziano Model:

from dual models to strings, M-theory and beyond

Galileo Galilei Institute, Arcetri, Florence

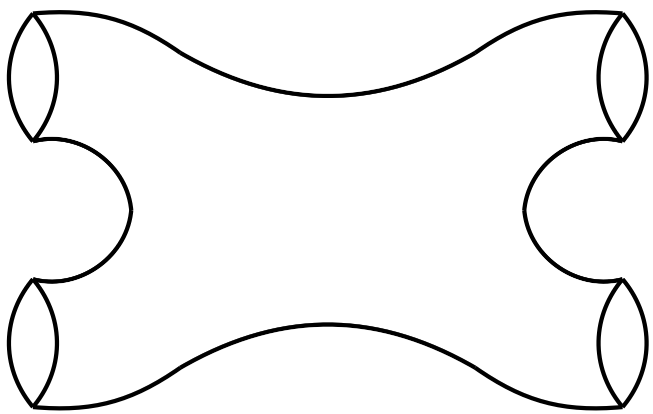
Joining and splitting

Veneziano model 1968

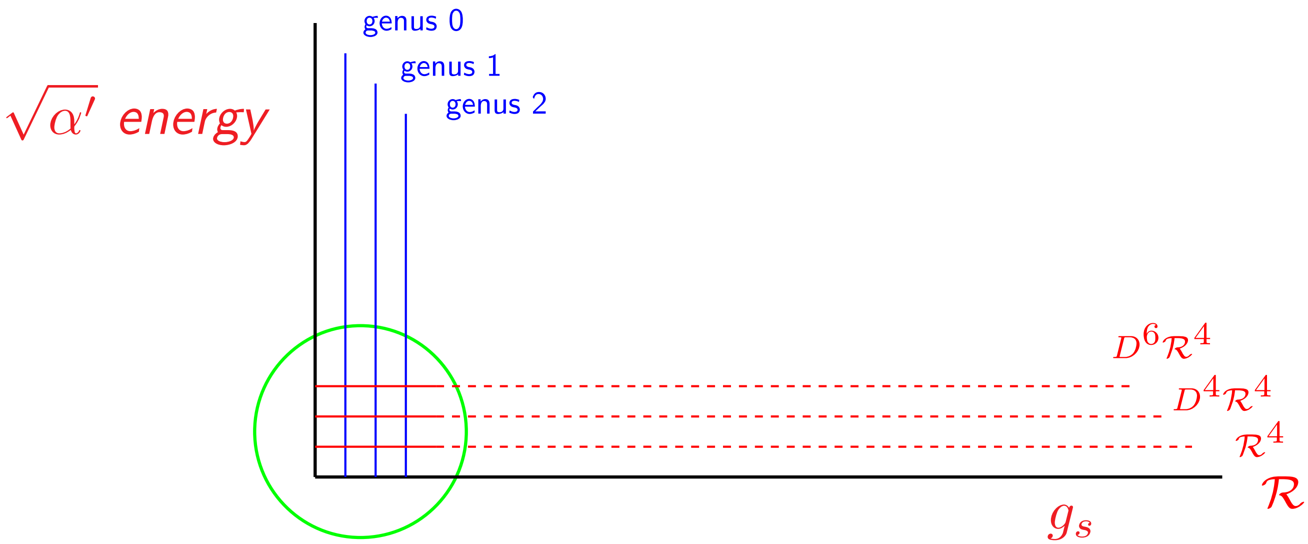
$\downarrow$

strings

(Nambu, Nielsen, Susskind 1969)



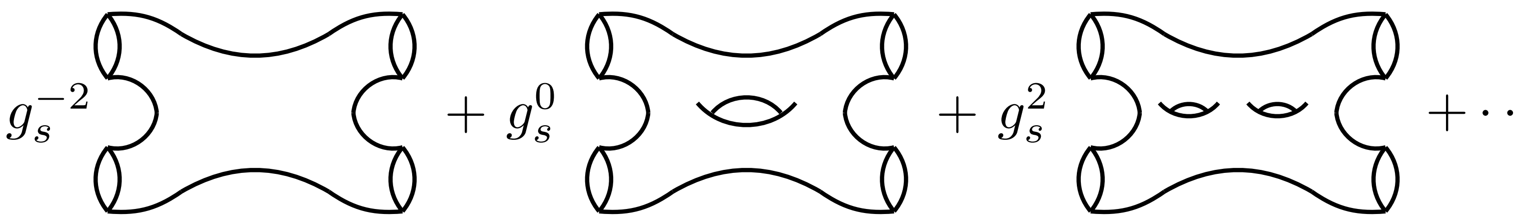
Superstring Perturbation Theory and Supergravity



- **Superstring perturbation theory in powers of the string coupling** g_s
  - holds for weak coupling g_s
  - and for all energies
- **Classical supergravity** "$\mathcal{R}$"
  - leading low energy expansion of string theory
  - holds for all couplings g_s
- **String induced effective interactions** $\mathcal{R}^4, D^4\mathcal{R}^4, D^6\mathcal{R}^4$
  - Evaluated in perturbation theory for $g_s \ll 1$

Closed Superstring Perturbation Theory

- **Theory of fluctuating random surfaces**
  - governed by topological expansion in the genus h weighed by g_s^{2h-2}



- **Closed superstring tree-level four-graviton amplitude**
  - generalization of the Veneziano amplitude $(s_{ij} = -\frac{\alpha'}{4}(k_i + k_j)^2)$

$$\frac{1}{stu} \frac{\Gamma(1-s)\Gamma(1-t)\Gamma(1-u)}{\Gamma(1+s)\Gamma(1+t)\Gamma(1+u)} \times \mathcal{R}^4$$

- **At low energy massive strings produce local effective interactions**
  - string-induced corrections to supergravity; eg. in Type II

$$\frac{1}{stu} + 2\zeta(3) + \zeta(5)(s^2+t^2+u^2) + 2\zeta(3)^2 stu - \frac{1}{2}\zeta(7)(s^2+t^2+u^2)^2 + \cdots$$

massless $\quad \mathcal{R}^4 \quad D^4\mathcal{R}^4 \quad D^6\mathcal{R}^4 \quad D^8\mathcal{R}^4$

Approaches to Superstring Perturbation Theory

- **Goal is to obtain superstring amplitudes at higher genus**
  - Ramond-Neveu-Schwarz formulation of fermionic strings;
  - Gliozzi-Scherk-Olive projection to supersymmetric spectrum;
  - Green-Schwarz space-time supersymmetric formulation;
  - Mandelstam light-cone formulation;
  - String field theory;
  - Topological string theory;
  - Berkovits pure spinor formulation.

- **Different perturbative superstring theories** (in 10 dimensions)
  - Type I open & closed, orientable & non-orientable, D-branes
  - Type IIA,B closed orientable, D-branes
  - Heterotic closed orientable $E_8 \times E_8$, $Spin(32/\mathbb{Z}_2)$

Outline of this talk

RNS formulation, closed orientable superstrings, critical dimension 10

- Formulation of RNS superstrings in terms of super Riemann surfaces
  (Witten 2012-2013)

- Explicit calculations of genus-two amplitudes
  (ED & Phong 2001-2007)

- Perturbative supersymmetry breaking and vacuum energy calculations
  (Atick, Dixon, Sen; Dine, Seiberg, Witten 1987) · · · (Witten; ED, Phong 2013)

- Perturbative evaluation of effective interactions in Type IIB and S-duality
  (Green, Gutperle, Vanhove 1997-2000) · · ·
  (ED, Green, Pioline, R. Russo, Vanhove 2013-present)

Type II worldsheet fields in the RNS formulation

- $M = \mathbb{R}^{10}$ **flat Minkowski space-time with Lorentz group** $SO(1,9)$
  - x^μ scalars on worldsheet Σ, map Σ into M
  - ψ^μ spinors on Σ but Lorentz vector under $SO(1,9)$
    - ⋆ Worldsheet supersymmetry $\Longrightarrow$ Σ is a super Riemann surface
    - ⋆ Two sectors : NS bosons $SO(1,9)$-tensors, R fermions $SO(1,9)$-spinors

- **With Minkowski signature** Σ
  - ψ^μ and $\tilde\psi^\mu$ are *independent* Majorana-Weyl spinors of opposite chirality
  - Locally, Dirac eq solved by $\psi^\mu(\tau-\sigma)$ and $\tilde\psi^\mu(\tau+\sigma)$

- **With Euclidean signature** Σ
  - ψ^μ and $\tilde\psi^\mu$ must be *independent* complex Weyl spinors
  - Locally, Dirac eq solved by $\psi^\mu(z)$ and $\tilde\psi^\mu(\tilde z)$
  - Globally, on a compact Riemann surface of genus h,
    - ⋆ All ψ^μ are sections of a the same spin bundle S (and $\tilde\psi^\mu$ of $\tilde S$)
    - ⋆ 2^{2h} distinct spin structures for S (and 2^{2h} independently for $\tilde S$)

- **GSO projection requires independent summation over spin structures**

Super Riemann surfaces and their moduli

- **Ordinary Riemann surface** (locally $\mathbb{C}$ with coordinate z)
  - complex manifold: holomorphic transition functions $z \to z'(z)$;
  - complex structure = conformal structure J
  - Moduli space $\mathcal{M}_h = \{J\}/\mathrm{Diff}(\Sigma)$ of genus h compact Riemann surfaces

- **Complex super manifold** (locally $\mathbb{C}^{1|1}$ with coordinates $z|\theta$)
  - holó transition functions $z|\theta \to z'(z,\theta)|\theta'(z,\theta)$ generate $\mathcal{N}=2$ super conformal

- **Super Riemann surface** (locally $\mathbb{C}^{1|1}$ with coordinates $z|\theta$)
  - holó transition functions $z|\theta \to z'|\theta'$ rescale $D_\theta = \partial_\theta + \theta\partial_z$
  - Transition functions define $\mathcal{N}=1$ superconformal structure $\mathcal{J}$
  - Globally: $T\Sigma$ has a completely non-integrable subbundle of rank $0|1$

- **Moduli space of compact super Riemann surfaces**: $\mathfrak{M}_h = \{\mathcal{J}\}/\mathrm{Diff}(\Sigma)$

  = equivalence classes of superconformal structures $\mathcal{J}$

$$\dim_{\mathbb{C}} \mathfrak{M}_h = \begin{cases} 0|0 & h=0 \\ 1|0 \text{ or } 1|1 & h=1 \quad \text{even or odd spin structure} \\ 3h-3|2h-2 & h \geq 2 \end{cases}$$

  - odd modulus at $h=1$ odd spin structure is a book keeping device;
  - odd moduli really first appear at genus 2, as curved super spaces.

Superstring worldsheets and moduli spaces

- **Heterotic**
  - Left : RS Σ_L, moduli space $\mathcal{M}_L$ coord resp. $\tilde z$ and $\tilde m^i$
  - Right : SRS Σ_R, moduli space $\mathfrak{M}_R$ coord resp. (z,θ) and (m^i,ζ^α)
  - Worldsheet is a cycle $\Sigma \subset \Sigma_L \times \Sigma_R$ of dim $1|1$

    subject to $\Sigma_{\rm red} = {\rm diag}(\Sigma_{L\,{\rm red}} \times \Sigma_{R\,{\rm red}})$: $\tilde z^* = z + {\rm nilpotent}$
  - Moduli space is a cycle $\Gamma \subset \mathcal{M}_L \times \mathfrak{M}_R$ of dim $3h-3|2h-2$ for $h \geq 2$

    subject to $\Gamma_{\rm red} = {\rm diag}(\mathcal{M}_{L\,{\rm red}} \times \mathfrak{M}_{R\,{\rm red}})$: $(\tilde m^i)^* = m^i + {\rm nilpotent}$

    (reduced space obtained by setting all nilpotent variables to zero)

- **Type II**
  - Left : SRS Σ_L, moduli space $\mathfrak{M}_L$ coord resp. $(\tilde z, \tilde \theta)$ and $(\tilde m^i, \tilde \zeta^\alpha)$
  - Right : SRS Σ_R, moduli space $\mathfrak{M}_R$ coord resp. (z,θ) and (m^i,ζ^α)
  - Worldsheet is a cycle $\Sigma \subset \Sigma_L \times \Sigma_R$ of dim $1|2$
  - Moduli space is cycle $\Gamma \subset \mathfrak{M}_L \times \mathfrak{M}_R$ of dim $3h-3|4h-4$ for $h \geq 2$

    subject to $\tilde z^* = z + {\rm nilpotent}$ and $(\tilde m^i)^* = m^i + {\rm nilpotent}$

- **Super-Stokes theorem ensures independence of the choice of cycles**
  - in amplitudes with BRST invariant vertex operators
  - consistent definition of superstring amplitudes to all genera (Witten 2012)

Worldsheet action for Type II superstrings

- **Worldsheet is $\Sigma \subset \Sigma_L \times \Sigma_R$**
  - Σ_L has superconformal structure $\tilde{\mathcal{J}}$ with local coordinates $\tilde z|\tilde\theta$
  - Σ_R has superconformal structure $\mathcal{J}$ with local coordinates $z|\theta$

- **Superconformal invariant matter action**
  - worldsheet matter field

$$X^\mu(\tilde z, z|\tilde\theta, \theta) = x^\mu(\tilde z, z) + \theta\psi^\mu(\tilde z, z) + \tilde\theta\tilde\psi^\mu(\tilde z, z) + \tilde\theta\theta F^\mu(\tilde z, z)$$

  - Worldsheet action in local coordinates $(D_\theta = \partial_\theta + \theta\partial_z)$

$$I_m[X^\mu, \tilde{\mathcal{J}}, \mathcal{J}] = \int_\Sigma [d\tilde z dz|d\tilde\theta d\theta] \tilde D_{\tilde\theta} X^\mu D_\theta X_\mu$$

  - Superconformal algebra on fields generated by

$$\mathcal{S}_{z\theta} = S_{z\theta} + \theta T_{zz} \qquad S_{z\theta} = \frac{1}{2}\psi^\mu \partial_z x_\mu \qquad T_{zz} = -\frac{1}{2}\partial_z x^\mu \partial_z x_\mu + \frac{1}{2}\psi^\mu \partial_z \psi_\mu$$

$$\tilde{\mathcal{S}}_{\tilde z\tilde\theta} = \tilde S_{\tilde z\tilde\theta} + \tilde\theta \tilde T_{\tilde z\tilde z} \qquad \tilde S_{\tilde z\tilde\theta} = \frac{1}{2}\tilde\psi^\mu \partial_{\tilde z} x_\mu \qquad \tilde T_{\tilde z\tilde z} = -\frac{1}{2}\partial_{\tilde z} x^\mu \partial_{\tilde z} x_\mu + \frac{1}{2}\tilde\psi^\mu \partial_{\tilde z} \tilde\psi_\mu$$

Deformations of superconformal structures

- **Under deformation of $\tilde{\mathcal{J}}$ for Σ_L and $\mathcal{J}$ for Σ_R**

$$\delta I = \int_\Sigma [d\tilde{z}dz|d\tilde{\theta}d\theta] \left(H_{\tilde{\theta}}{}^z \mathcal{S}_{z\theta} + \tilde{H}_\theta{}^{\tilde{z}}\, \tilde{\mathcal{S}}_{\tilde{z}\tilde{\theta}} \right)$$

  - in components by integrating out $\tilde{\theta}, \theta$,

$$\delta I = \int_{\Sigma_{\rm red}} d\tilde{z}dz \left(\mu_{\tilde{z}}{}^z\, T_{zz} + \chi_{\tilde{z}}{}^\theta\, S_{z\theta} + \tilde{\mu}_z{}^{\tilde{z}}\, T_{\tilde{z}\tilde{z}} + \tilde{\chi}_z{}^{\tilde{\theta}}\, \tilde{S}_{\tilde{z}\tilde{\theta}} \right)$$

  - recover Beltrami differentials $\mu, \tilde{\mu}$ and worldsheet gravitino fields $\chi, \tilde{\chi}$

$$H_{\tilde{\theta}}{}^z = \tilde{\theta}(\mu_{\tilde{z}}{}^z + \theta \chi_{\tilde{z}}{}^\theta) \qquad\qquad \tilde{H}_\theta{}^{\tilde{z}} = \theta(\tilde{\mu}_z{}^{\tilde{z}} + \tilde{\theta}\tilde{\chi}_z{}^{\tilde{\theta}})$$

  - Finite deformations of the metric with $\tilde{\mu} = \bar{\mu}$ and $\tilde{\chi} = \bar{\chi}$
    integrate to the standard 2-dim $\mathcal{N} = 1$ supergravity action
    (Brink, Di Vecchia, Howe; Deser, Zumino 1976)

- **Type II superstring perturbation theory requires $\tilde{\mu} \neq \bar{\mu}$ and $\tilde{\chi} \neq \bar{\chi}$**

Type II string amplitude

- **Parametrize deformations $\tilde{H}_\theta{}^{\tilde{z}}, H_{\tilde{\theta}}{}^z$ by slice $\{\tilde{\mathcal{J}}(\tilde{\mathfrak{m}}), \mathcal{J}(\mathfrak{m})\}$ in $\mathfrak{M}_L \times \mathfrak{M}_R$**

$$H_{\tilde{\theta}}{}^z = \tilde{D}_{\tilde{\theta}} V^z + H_A d\mathfrak{m}^A \qquad H_A = \partial \mathcal{J}_{\tilde{\theta}}{}^z / \partial \mathfrak{m}^A \qquad \mathfrak{m}^A = (m^i, \zeta^\alpha)$$

$$\tilde{H}_\theta{}^{\tilde{z}} = D_\theta \tilde{V}^{\tilde{z}} + \tilde{H}_{\tilde{A}} d\tilde{\mathfrak{m}}^{\tilde{A}} \qquad \tilde{H}_{\tilde{A}} = \partial \mathcal{J}_\theta{}^{\tilde{z}} / \partial \tilde{\mathfrak{m}}^{\tilde{A}} \qquad \tilde{\mathfrak{m}}^A = (\tilde{m}^i, \tilde{\zeta}^\alpha)$$

ghost fields $\quad V^z \to C^z = c^z + \theta \gamma^\theta \qquad H_{\tilde{\theta}}{}^z \to B_{z\theta} = \beta_{z\theta} + \theta b_{zz}$

$$V^{\tilde{z}} \to \tilde{C}^{\tilde{z}} = \tilde{c}^{\tilde{z}} + \tilde{\theta}\tilde{\gamma}^{\tilde{\theta}} \qquad H_\theta{}^{\tilde{z}} \to \tilde{B}_{\tilde{z}\tilde{\theta}} = \tilde{\beta}_{\tilde{z}\tilde{\theta}} + \tilde{\theta}\tilde{b}_{\tilde{z}\tilde{z}}$$

  - Super conformal invariant ghost action

$$I_{\rm gh} = \int_\Sigma [d\tilde{z} dz | d\tilde{\theta} d\theta] \Big(B_{z\theta} \tilde{D}_{\tilde{\theta}} C^z + \tilde{B}_{\tilde{z}\tilde{\theta}} D_\theta C^{\tilde{z}} + B_{z\theta} H_A d\mathfrak{m}^A + \tilde{B}_{\tilde{z}\tilde{\theta}} \tilde{H}_{\tilde{A}} d\tilde{\mathfrak{m}}^{\tilde{A}} \Big)$$

- **The integrand for the full amplitude is given by**

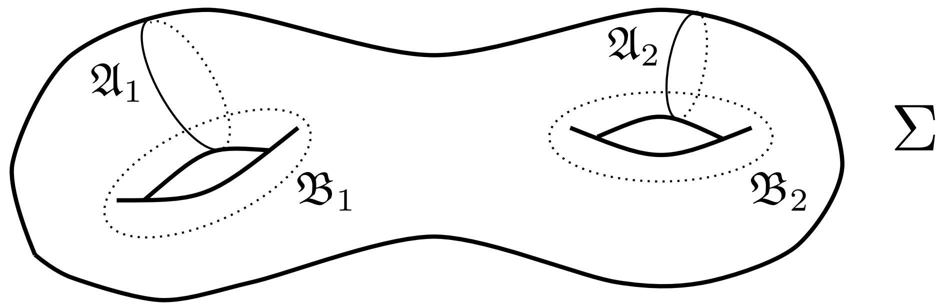
$$\int D(XB\tilde{B}C\tilde{C})\, \mathcal{V}_1 \cdots \mathcal{V}_n \prod_{\tilde{A},A} [d\tilde{\mathfrak{m}}^{\tilde{A}} d\mathfrak{m}^A]\, \delta(\langle \tilde{B}, \tilde{H}_{\tilde{A}} \rangle) \delta(\langle B, H_A \rangle)\, e^{-I_m - I_{\rm gh}}$$

  - $\mathcal{V}_1 \cdots \mathcal{V}_n$ are BRST-invariant vertex operators.
  - Picture Changing Operator formalism (Friedan, Martinec, Shenker 1986)
    - ⋆ may be obtained as singular limit for χ supported at points
    - ⋆ globally regular reformulation via "vertical integration" (Sen, Witten 2016)

Loop momenta

- **Fix a canonical homology basis of cycles** $\mathfrak{A}_I, \mathfrak{B}_I$ **of** $H_1(\Sigma, \mathbb{Z})$ $\quad I = 1, \cdots, h$
  - with canonical intersection pairing

    $\#(\mathfrak{A}_I, \mathfrak{A}_J) = \#(\mathfrak{B}_I, \mathfrak{B}_J) = 0$ and $\#(\mathfrak{A}_I, \mathfrak{A}_J) = \delta_{IJ}$



- h **independent loop momenta** p_I^μ **defined to flow across** $\mathfrak{A}_I$ **cycles**

$$p_I^\mu = \oint_{\mathfrak{A}_I} dz \partial_z x^\mu$$

- At fixed loop momenta, the amplitudes chirally split

Chiral amplitudes

- **Chiral Amplitudes** (ED, Phong 1988)
  - For massless NS bosons with factorized polarization tensor $\varepsilon_i^{\mu\tilde{\mu}} = \varepsilon_i^{\mu} \tilde{\varepsilon}_i^{\tilde{\mu}}$
  - Each chiral amplitude at fixed loop momenta is given by

$$\mathcal{F}_R(\mathcal{J}, \varepsilon_i, k_i, p_I) = \left\langle \mathcal{V}_1 \cdots \mathcal{V}_N \, e^{p_I^\mu \oint_{\mathfrak{B}_I} dz \partial_z x^\mu} \, e^{\int_\Sigma H_{\tilde{\theta}}{}^z S_{z\theta}} \prod_A \delta(\langle B, H_A \rangle) d\mathfrak{m}^A \right\rangle$$

  - Correlation function $\langle \cdots \rangle$ computed with Szegö kernel S_δ and prime for E

$$\langle \psi^\mu(z) \psi^\nu(w) \rangle = -\eta^{\mu\nu} S_\delta(z,w) \qquad \langle x_+^\mu(z) x_+^\nu(w) \rangle = -\eta^{\mu\nu} \ln E(z,w)$$

- **Full Superstring Amplitudes**
  - obtained by pairing left and right and integrating on a cycle $\Gamma \in \mathfrak{M}_L \times \mathfrak{M}_R$

$$\mathcal{A}^{(h)}(\varepsilon_i, \tilde{\varepsilon}_i, k_i) = \int_\Gamma \int_{\mathbb{R}^{10}} dp_I^\mu \, \mathcal{F}_L(\tilde{\mathcal{J}}, \tilde{\varepsilon}_i, k_i, p_I^\mu) \, \mathcal{F}_R(\mathcal{J}, \varepsilon_i, k_i, p_I^\mu)$$

  - cfr "double copy construction" in supergravity calculations (cfr Zvi Bern's talk)

Parametrization of super moduli

- **Superconformal structure $\mathcal{J} \in \mathfrak{M}_h$ specified by transition functions**
  - Concrete calculations use parametrization by gravitino field $\chi_{\tilde{z}}{}^{\theta}$
- **Local parametrization of moduli** (in conformal-invariant theory)
  - Conformal structure J with metric $g = |dz|^2$ in local coordinates $(z, \tilde{z})$
  - deform conformal structure by Beltrami differential to $g' = |dz + \mu d\tilde{z}|^2$
  - realized in CFT by inserting $\int_\Sigma d\tilde{z}dz\, \mu_{\tilde{z}}{}^z\, T_{zz}$ to all orders in μ
- **Local parametrization of supermoduli** (in superconformal-invariant theory)
  - Start with $\Sigma_{\rm red}$ with complex structure given by $J \in \mathfrak{M}_{\rm red}$
  - Deform super conformal structure by inserting T and S

$$\int_{\Sigma_{\rm red}} d\tilde{z}dz\, \left(\mu_{\tilde{z}}{}^z T_{zz} + \chi_{\tilde{z}}{}^{\theta} S_{z\theta}\right)$$

  - χ and μ parametrized by local odd coordinates on $\mathfrak{M}_h$
- **For $h=2$, even spin structures, holó projection $\mathfrak{M}_2 \to \mathcal{M}_2$ exists**
  - via the super period matrix (ED, Phong 2001)
- **For $h \geq 5$ no holó projection $\mathfrak{M}_h \to \mathcal{M}_h$ exists** (Donagi, Witten 2013)

The super period matrix (even spin structures)

- **Start from conformal structure J for $\Sigma_{\rm red}$ with holó 1-forms ω_I**

$$\oint_{\mathfrak{A}_I} \omega_J = \delta_{IJ} \qquad \oint_{\mathfrak{B}_I} \omega_J = \Omega_{IJ} \qquad I,J=1,2$$

- **Deform to superconformal structure $\mathcal{J}$ on Σ with superholó forms $\hat\omega_I$**

$$\oint_{\mathfrak{A}_I} \hat\omega_J = \delta_{IJ} \qquad \oint_{\mathfrak{B}_I} \hat\omega_J = \hat\Omega_{IJ} \qquad I,J=1,2$$

  – Explicit formula for the super period matrix $\hat\Omega$ for even spin structure δ

$$\hat\Omega_{IJ} = \Omega_{IJ} - \frac{i}{8\pi}\int_{\Sigma^2_{\rm red}} \omega_I(z)\chi(z)S_\delta(z,w|\Omega)\chi(w)\omega_J(w) + \int_{\Sigma_{\rm red}} \mu\,\omega_I\omega_J$$

  – $\hat\Omega_{IJ}$ is locally supersymmetric; $\hat\Omega_{IJ}=\hat\Omega_{JI}$; and $\mathrm{Im}\,\hat\Omega>0$
  – Every $\hat\Omega$ corresponds to an ordinary Riemann surface
  – Szegö kernel $S_\delta(z,w|\Omega)$ is non-singular in the interior of $\mathcal{M}_2$

$\Rightarrow$ **Projection using $\hat\Omega$ is holomorphic and natural for genus 2**

Projecting and pairing Chiral Amplitudes

- **Chiral Amplitudes on $\mathfrak{M}_2$**
  - Natural parametrization of $\mathfrak{M}_2$ by $(\hat\Omega_{IJ}, \zeta^\alpha)$ (even spin structure δ)
  - involves measure $d\kappa[\delta](\hat\Omega, \zeta)$ and correlation functions $\mathcal{C}[\delta](\varepsilon_i, k_i, p_I|\hat\Omega, \zeta)$

- **Projection to chiral amplitudes on $\mathcal{M}_2$**
  - by integrating over ζ and summing over δ at fixed $\hat\Omega$

$$\begin{aligned}\mathcal{R}(\varepsilon_i, k_i, p_I|\hat\Omega) &= \sum_\delta \int_\zeta d\kappa[\delta](\hat\Omega, \zeta)\, \mathcal{C}[\delta](\varepsilon_i, k_i, p_I|\hat\Omega, \zeta) \\ \mathcal{L}(\tilde\varepsilon_i, k_i, p_I|\hat\Omega) &= \sum_{\tilde\delta} \int_{\tilde\zeta} d\kappa[\tilde\delta](\hat\Omega, \tilde\zeta)\, \mathcal{C}[\tilde\delta](\tilde\varepsilon_i, k_i, p_I|\hat\Omega, \tilde\zeta)\end{aligned}$$

  - for heterotic, $\mathcal{L}$ is chiral half of bosonic string, has no integral in $\tilde\zeta$
  - phase factors determined by $Sp(4,\mathbb{Z})$ modular invariance

- **Pairing left and right chiral amplitudes, integrating over p_I and $\hat\Omega$**

$$\mathcal{A}^{(2)}(\varepsilon_i, \tilde\varepsilon_i, k_i) = \int_{\mathcal{M}_2} d\hat\Omega \int dp_I^\mu\, \mathcal{R}(\varepsilon_i, k_i, p_I|\hat\Omega)\, \overline{\mathcal{L}(\tilde\varepsilon_i, k_i, p_I|\hat\Omega)}$$

  - Integral over p_I is Gaussian and can be carried out explicitly.

UV-finiteness and one-loop amplitudes

- **Thanks to modular invariance, all string amplitudes are UV-finite**
  - shown for the closed bosonic string (Shapiro 1972)
  - holds for all modular invariant superstrings to all loops (i.e. all genera)
  - all chiral amplitudes have a universal loop momentum factor

$$\mathcal{F}_R(\mathcal{J}, \varepsilon_i, k_i, p_I) = e^{i\pi p_I^\mu \Omega_{IJ} p_J^\mu} \times \cdots$$

  - Modular invariances guarantees $\mathrm{Im}(\Omega)$ bounded from below
    $\Rightarrow$ Gaussian suppression at large loop momenta & UV finiteness

- **One-loop Type II four-graviton amplitude** (Green, Schwarz 1982)

$$\mathcal{A}^{(1)}(\varepsilon_i, k_i) = \mathcal{R}^4 \int_{\mathcal{M}_1} \frac{d^2\tau}{(\mathrm{Im}\tau)^2} \mathcal{B}^{(1)}(s_{ij}|\tau)$$

  - Partial amplitude $\mathcal{B}$ is a modular function in $\tau \in \mathcal{M}_1 = \mathcal{H}_1/SL(2,\mathbb{Z})$

$$\mathcal{B}^{(1)}(s_{ij}|\tau) = \int_{\Sigma^4} \prod_{i=1}^{4} \frac{d^2 z_i}{\mathrm{Im}\tau} \exp\Big(\sum_{i<j} s_{ij}\, G(z_i - z_j|\tau)\Big)$$

  - $G(z|\tau)$ is the scalar Green function on the torus Σ of modulus τ.
  - Analogous formulas for Heterotic string (Gross, Harvey, Martinec, Rohm 1985)

Genus two Riemann surfaces and moduli

- **Siegel Upper half space $\mathcal{H}_h$**

$$\mathcal{H}_h = \Big\{ \Omega = \{\Omega_{IJ}\}_{I,J=1,\cdots,h} \text{ with } \Omega^t = \Omega \text{ and } Y = \mathrm{Im}(\Omega) > 0 \Big\}$$

  – $Sp(2h,\mathbb{R})$ acts by $\Omega \to (A\Omega + B)(C\Omega + D)^{-1}$

$$M^t J M = J \qquad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

  – $\mathcal{H}_h$ has $Sp(2h,\mathbb{R})$-invariant metric ds^2_h and volume form $d\mu_h$

$$ds^2_h = \sum_{I,J,K,L} Y^{-1}_{IJ}\, d\bar\Omega_{JK}\, Y^{-1}_{KL}\, d\Omega_{LI}$$

- **Genus-two compact Riemann surfaces Σ**
  – Canonical homology basis $\mathfrak{A}_I, \mathfrak{B}_I$ with ω_I dual holomorphic (1,0) forms,

$$\oint_{\mathfrak{A}_I} \omega_J = \delta_{IJ} \qquad \oint_{\mathfrak{B}_I} \omega_J = \Omega_{IJ}$$

  – Riemann relations imply $\Omega \in \mathcal{H}_2$; modular group is $Sp(4,\mathbb{Z})$
  – Genus-two moduli space $\mathcal{M}_2 = \mathcal{H}_2/Sp(4,\mathbb{Z})$ (upon removing diagonal Ω)

Genus-two Type II superstring amplitudes

- **Type II four-graviton amplitude** (ED, Phong 2001 – 2005)
  - receives contributions only from even spin structures, as for genus-one
  - use projection by the super period matrix to integral over $\mathcal{M}_2$

$$\mathcal{A}^{(2)}(\varepsilon_i, \tilde{\varepsilon}_I, k_i) = \mathcal{R}^4 \int_{\mathcal{M}_2} d\mu_2 \, \mathcal{B}^{(2)}(s_{ij}|\Omega)$$

  - kinematic factor $\mathcal{R}^4 = \mathcal{K}(\varepsilon_i, k_i)\tilde{\mathcal{K}}(\tilde{\varepsilon}_i, k_i)$ is the same as tree-level and one-loop

$$\mathcal{B}^{(2)}(s_{ij}|\Omega) = \int_{\Sigma^4} \frac{\mathcal{Y} \wedge \bar{\mathcal{Y}}}{(\det \operatorname{Im} \Omega)^2} \exp\Big(\sum_{i<j} s_{ij} \, G(z_i, z_j|\Omega)\Big)$$

  - $G(z_i, z_j)$ is the genus-two scalar Green function;
  - $3\mathcal{Y} = (t-u)\Delta(1,2)\Delta(3,4) + (s-t)\Delta(1,3)\Delta(4,2) + (u-s)\Delta(1,4)\Delta(2,3)$
  - $\Delta(i,j) = \omega_1(z_i) \wedge \omega_2(z_j) - \omega_2(z_i) \wedge \omega_1(z_j)$ is a biholomorphic form
  - reproduced (with fermions) in pure spinor formulation (Berkovits, Mafra 2005)

- **Heterotic four NS boson amplitude** (ED, Phong 2005)
  - obtained by replacing $\tilde{\mathcal{K}}(\tilde{\varepsilon}_i, k_i)\tilde{\mathcal{Y}} \to \mathcal{W}(\tilde{\varepsilon}_i, k_i)$
  - $\mathcal{W}$ correlator depends on the NS boson state (graviton versus gauge).

Singularities in the projection $\overline{\mathfrak{M}}_2 \to \overline{\mathcal{M}}_2$

- **Projection $\mathfrak{M}_2 \to \mathcal{M}_2$ is holó, but integration extends to boundary**
  - are there singularities in the projection $\overline{\mathfrak{M}}_2 \to \overline{\mathcal{M}}_2$?

$$\Omega = \begin{pmatrix} \tau & u \\ u & \sigma \end{pmatrix} \qquad \begin{array}{ll} u \to 0 & \text{separating node} \\ \sigma \to i\infty & \text{non-separating node} \end{array}$$

  - Key ingredient in $\hat{\Omega}$ is the Szegö kernel

$$S_\delta(z, w|\Omega) = \frac{\vartheta[\delta](z - w|\Omega)}{\vartheta[\delta](0|\Omega) E(z, w)}$$

  - As $u \to 0$ we have $\vartheta[\delta](0|\Omega) \to \vartheta[\delta_1](0|\tau)\, \vartheta[\delta_2](0|\tau)$
  - Even $\delta = [\delta_1, \delta_2]$ with δ_1, δ_2 odd produces a singularity in S_δ and $\hat{\Omega}$

- **Physical effects**
  - singularity killed by ψ-zero modes in $\mathbb{R}^{10}$ (space-time susy)
  - contribution when susy is broken by radiative corrections (Witten 2013)
  - Two-loop vacuum energy in Heterotic strings on CY orbifold $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$
    - ⋆ is zero for $E_8 \times E_8 \to E_6 \times E_8$ with unbroken susy
    - ⋆ non-zero for $\mathrm{Spin}(32)/\mathbb{Z}_2 \to SO(26) \times U(1)$ with broken susy

(Atick, Sen 1988; · · ·; ED, Phong 2013; Berkovits, Witten 2014)

Singularities in the projection $\mathfrak{M}_3 \to \mathcal{M}_3$

- **Some basic structure theorems**
  - A hyper-elliptic surface is a branched double cover of the sphere S^2;
  - All genus 1 and all genus 2 surfaces are hyper-elliptic;
  - Hyper-elliptic surfaces form a co-dim 1 sub-variety in the interior of $\mathcal{M}_3$
    (referred to as the hyper-elliptic divisor)

- **The genus-three period matrix (for even spin structure)**

$$\hat{\Omega}_{IJ} = \Omega_{IJ} - \frac{i}{8\pi} \int\!\!\int \omega_I(z)\chi(z)S_\delta(z,w|\Omega)\chi(w)\omega_J(w) + \mathcal{O}(\chi^4)$$

  - For Ω on the hyper-elliptic divisor of $\mathfrak{M}_3$
    there always exists an even spin structure δ such that $\vartheta[\delta](0|\Omega) = 0$
  - the presence of the extra Dirac zero modes kills effects of this singularity
  - A different even δ does produce a *subtle singularity* in $\hat{\Omega}$ (Witten 2015)
  - Rules out earlier beautiful proposals for the genus 3 superstring measure
    (Cacciatori, Dalla Piazza, van Geemen 2008)

Effective interactions from Type IIB superstrings

- $SL(2,\mathbb{Z})$**-duality symmetry of Type IIB superstrings**
  - requires effective interactions to be $SL(2,\mathbb{Z})$-invariant;
  - Einstein frame $\mathcal{R}^4$ invariant; axion-dilaton $\rho = \chi + ie^{-\Phi}$ by Möbius;
  - effective interactions via four-graviton amplitude in Type IIB,

$$\mathcal{E}_0(\rho)\mathcal{R}^4 + \mathcal{E}_4(\rho)D^4\mathcal{R}^4 + \mathcal{E}_6(\rho)D^6\mathcal{R}^4 + \mathcal{E}_8(\rho)D^8\mathcal{R}^4 + \cdots$$

  - $\mathcal{E}_p(\rho)$ is a real-valued $SL(2,\mathbb{Z})$-invariant of ρ, or *modular function*

- **Non-holomorphic Eisenstein series**
  - The functions $E_s(\rho)$ are $SL(2,\mathbb{Z})$-invariant $(\rho = \rho_1 + i\rho_2, \rho_1, \rho_2 \in \mathbb{R})$

$$\Delta_\rho E_s(\rho) = s(1-s)E_s \qquad\qquad \Delta = 4\rho_2^2 \partial_\rho \partial_{\bar\rho}$$

  - For $\mathrm{Re}(s) > 1$ they are given by a Kronecker-Eisenstein sum

$$E_s(\rho) = \sum_{m,n\in\mathbb{Z}}{}' \frac{\rho_2^s}{\pi^s |m+\rho n|^{2s}}$$

  - with asymptotic expansion for $\rho_2 \to \infty$ = weak string coupling

$$E_s(\rho) = 2\zeta(2s)\frac{\rho_2^s}{\pi^s} + \frac{2\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)\pi^{s-\frac{1}{2}}\rho_2^{s-1}} + \mathcal{O}(e^{-2\pi\rho_2})$$

Effective interactions (cont'd)

- **Space-time supersymmetry and S-duality**
  - D-instantons and space-time susy (Green, Gutperle, Vanhove; Green, Sethi 1997)
  - M-theory perturbation theory on torus (Green, Kwon, Vanhove 1999; GV 2005)

$$\frac{1}{stu} + 2\zeta(3) + \zeta(5)(s^2+t^2+u^2) + 2\zeta(3)^2 stu - \frac{1}{2}\zeta(7)(s^2+t^2+u^2)^2 + \cdots$$

$\mathcal{R}^4$	$D^4\mathcal{R}^4$	$D^6\mathcal{R}^4$	$D^8\mathcal{R}^4$
$\mathcal{E}_0 \approx E_{\frac{3}{2}}$	$\mathcal{E}_4 \approx E_{\frac{5}{2}}$	$(\Delta - 12)\mathcal{E}_6 \approx E_{\frac{3}{2}}^2$	??
$\frac{1}{2}$BPS	$\frac{1}{4}$BPS	$\frac{1}{8}$BPS	unprotected

  - $\mathcal{E}_0$ only tree-level $\approx e^{-2\Phi}$ and one-loop $\approx e^{0\Phi}$ (Green, J. Russo, Vanhove 2008)
  - $\mathcal{E}_4$ only tree-level and two-loop $\approx e^{2\Phi}$ (ED, Gutperle, Phong 2005)
  - $\mathcal{E}_6$ only tree-level and two-loops (ED, Green, Pioline, R. Russo 2014) and three-loops $\approx e^{4\Phi}$ (Gomez, Mafra 2015)

- **Non-renormalization theorems**: no perturbative corrections
  - for $\mathcal{E}_0$ for $h \geq 2$, for $\mathcal{E}_4$ for $h \geq 3$, for $\mathcal{E}_6$ for $h \geq 4$

Low energy expansion of the genus-one string integrand

- **Type II genus-one four-graviton amplitude is an integral over $\mathcal{M}_1$ of**

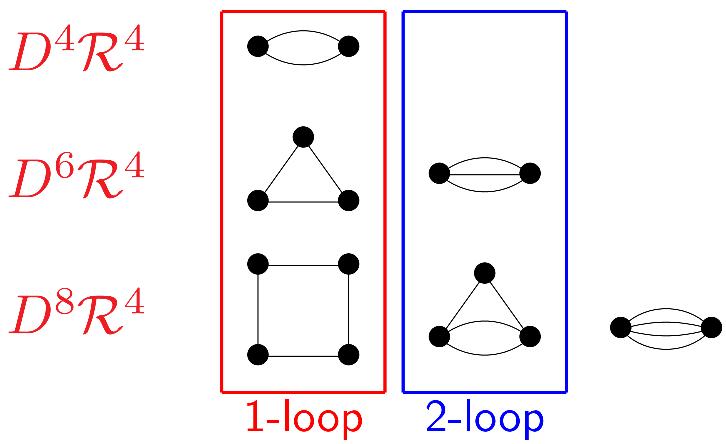
$$\mathcal{B}^{(1)}(s_{ij}|\tau) = \prod_{i=1}^{4} \int_\Sigma \frac{d^2 z_i}{\tau_2} \exp\Big\{ \sum_{i<j} s_{ij}\, G(z_i - z_j|\tau) \Big\}$$

  – The scalar Green function G is given by a Fourier sum

$$G(\alpha + \beta\tau|\tau) = \sum_{m,n\in\mathbb{Z}}{}' \frac{\tau_2}{\pi} \frac{e^{2\pi i(m\beta - n\alpha)}}{|m+\tau n|^2}$$

  – For fixed τ the expansion of $\mathcal{B}^{(1)}$ in powers of s_{ij} converges for $|s_{ij}| < 1$

- **Graphical expansion of $\mathcal{B}^{(1)}(s_{ij}|\tau) \Longrightarrow$ Modular Graph Functions of τ**



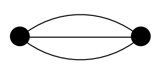
Genus-one modular graph functions

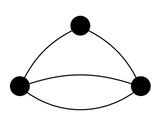
- **One-loop graphs with w Green functions**
  - evaluate to non-holomorphic Eisenstein series $E_w(\tau)$

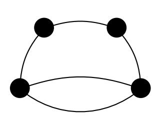
- **Two-loop graphs evaluate to modular graph functions**
  - given by multiple Kronecker-Eisenstein sums over $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$

$$C_{w_1,w_2,w_3}(\tau) = \sum_{p_1,p_2,p_3 \in \Lambda'_\tau} \delta\left(p_1+p_2+p_3\right) \prod_{r=1}^{3} \left(\frac{\tau_2}{\pi |p_r|^2} \right)^{w_r}$$

  - contribute to $D^{2w}\mathcal{R}^4$ with the *weight* given by $w = w_1 + w_2 + w_3$
  - satisfy (inhomogeneous) Laplace-eigenvalue equations

$w=3$ $\quad C_{1,1,1} =$  $\quad (\Delta - 0)C_{1,1,1} = 6E_3$

$w=4$ $\quad C_{2,1,1} =$  $\quad (\Delta - 2)C_{2,1,1} = 9E_4 - E_2^2$

$w=5$ $\quad C_{3,1,1} =$  $\quad (\Delta - 6)C_{3,1,1} = 3C_{2,2,1} + 16E_5 - 4E_2E_3$

Low energy expansion of the genus-two string integrand

- **Recall genus-two four-graviton amplitude is an integral over $\mathcal{M}_2$ of**

$$\mathcal{B}^{(2)}(s_{ij}|\Omega) = \int_{\Sigma^4} \frac{\mathcal{Y} \wedge \bar{\mathcal{Y}}}{(\det \mathrm{Im}\, \Omega)^2} \exp \sum_{i<j} s_{ij}\, G(z_i, z_j)$$

  – $\mathcal{Y} = (s-t)\Delta(z_1, z_3)\Delta(z_4, z_2) + 2$ permutations.

- **Genus-two contributions to local effective interactions**

$$\mathcal{B}^{(2)}(s_{ij}|\Omega) = 0 + 32(s^2+t^2+u^2) + 192\, stu\, \varphi(\Omega) + (s^2+t^2+u^2)^2 \mathcal{B}_8^{(2)}(\Omega) + \cdots$$

$$\mathcal{R}^4 \qquad D^4\mathcal{R}^4 \qquad D^6\mathcal{R}^4 \qquad D^8\mathcal{R}^4$$

  – $\varphi(\Omega)$ = Kawazumi-Zhang invariant (Kawazumi; Zhang 2008)
    ⋆ satisfies differential equation on $\mathcal{M}_2$ (ED, Green, Pioline, R. Russo 2014)

$$(\Delta - 5)\varphi = -2\pi\delta_{SN}$$

    ⋆ Δ is the Laplace-Beltrami operator on $\mathcal{M}_2$ with Siegel metric;
    ⋆ δ_{SN} has support on separating node (into two genus-one surfaces)

  – $\mathcal{B}_8^{(2)}(\Omega)$ satisfies remarkably systematic asymptotics (ED, Green, Pioline 2017)
    ⋆ system of differential equations on $\mathcal{M}_2$ (ED, Green, Pioline, in progress)

Outlook

- **Some additional developments**
  - Clarification of super Riemann surfaces with R-punctures (Witten 2012)
  - There exists a super-period matrix for R-punctures (Witten; ED, Phong 2015)
  - New relations between open and closed string amplitudes (Schlotterer's talk)

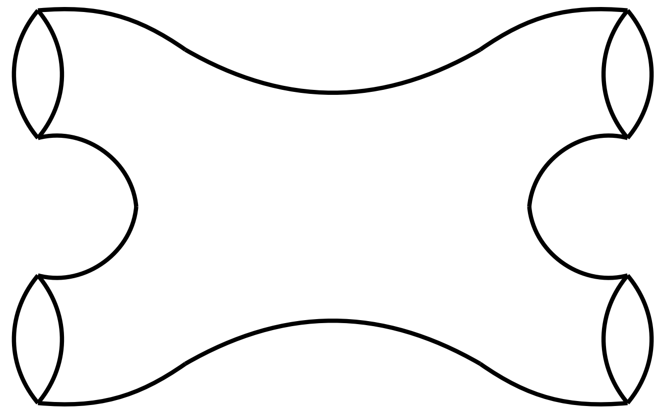
- **Some outstanding issues**
  - Systematic structure of low energy effective interactions
    in terms of properties of modular graph functions w/ Green, Pioline
    calculation without requiring subtleties of supermoduli space w/ Green
    UV divergences in supergravity and effective interactions

  - Ambi-twistor strings ? w/ Casali, Tourkine

  - string perturbation theory on curved spaces with RR flux, e.g. $AdS_5 \times S^5$

Veneziano model 1968

$\downarrow$



Thank You