Lectures on Superstring Amplitudes

Part 1: Bosonic String

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Outline of lectures

• Lecture 1
  Bosonic strings and conformal field theory

• Lecture 2
  Superstring amplitudes

• Lecture 3
  Low energy effective interactions
Strings

- A string is a 1-dimensional object
  - open string = topology of an interval;
  - closed string = topology of a circle;
  - physical size Planck length $\ell_P \approx 10^{-33}\text{cm} \approx 10^{-19} \times$ size of the proton.

- Ultimate goal: unified theory of particle physics and gravity
  - elementary particles correspond to strings and their excited states;
  - consistently with quantum mechanics and general relativity;
  - remarkably unique structure.

- Immediate goal: relating string amplitudes and field theory amplitudes
  - at distance scales larger than the Planck length (low energy)
    a string effectively behaves as a point particle
  - string theory exhibits powerful structure of amplitudes
String Topology

- **Consistent interacting string theories**
  - only closed strings (Type IIA,B and heterotic)
  - closed and open strings (Type I)
  - Type II theories have open strings in the presence of D-branes

- **Strings live in a physical space-time** $M$
  - $M$ may be a manifold or an orbifold (with mild isolated singularities)
  - superstring theory predicts 10-dim
  - but space-time visible to us is 4-dim. $\Rightarrow$ requires “compactification”

- **Under time-evolution strings sweep out a 2-dim. surface**

  closed strings

  \[
  \begin{align*}
  \text{time-evolution} & \quad \text{(freely propagating)} \\
  \end{align*}
  \]

  basic interaction
  \[
  \begin{align*}
  \text{(purely topological)} \\
  \end{align*}
  \]
Perturbative String Amplitudes

- **Quantum probability “scattering” amplitudes**
  - Feynman functional integral/sum over all surfaces with given boundary components for initial and final strings

- **Closed oriented string perturbation theory**
  - The only remaining topological characterization is the genus $h \geq 0$
    - Probability amplitude includes sum over all genera
    - Weighed by a factor $g_s^{2h-2}$ where $g_s$ is the “string coupling”

- genus $h = \text{number of “loops”}$

$$g_s^{-2} + g_s^0 + g_s^2 + \cdots$$
Structure of string amplitudes

- **Perturbative part of string amplitude decomposes into a sum over topologies**

\[
A_{\text{perturbative}} = \sum_{h=0}^{\infty} g_s^{2h-2} \times A^{(h)}
\]

- \(A^{(h)}\) is the amplitude at genus \(h\)

- The perturbative expansion in \(g_s\) is asymptotic but not convergent (just as in field theory)

- **Non-perturbative part** (not considered here)
  - instantons \(\approx e^{-1/g_s^2}\)
  - D-branes contribute \(\approx e^{-1/g_s}\).
String Data (closed oriented bosonic strings)

- **Assume fixed space-time \( M \), with fixed metric \( G \)
  - Physical space-time has Minkowski signature metric \( G \)
  - Starting point for string theory is often a Riemannian metric
    (if needed to be analytically continued to Minkowski signature)

- **The 2-dimensional worldsheet \( \Sigma \) is mapped into space-time \( M \)
  - The space of all such maps \( x : \Sigma \rightarrow M \) is denoted \( \text{Map}(\Sigma) \).

- **Riemannian metric \( G \) induces a Riemannian metric \( x^*(G) \) on \( \Sigma \)
  - Hence \( \Sigma \) is a Riemann surface (i.e. complex manifold with holó transition functions)

- **Polyakov formulation invokes an independent metric
  - Riemannian metric \( g \) on \( \Sigma \)
  - Denote the \( \infty \)-dim. Riemannian manifold of such metrics by \( \text{Met}(\Sigma) \)
  - String amplitude at fixed genus \( h \) obtained by weighed sum over \( g, x \)

\[
A^{(h)} = \int_{\text{Met}(\Sigma)} Dg \int_{\text{Map}(\Sigma)} Dx \ e^{-I_G[x,g]}
\]
The worldsheet action $I_G$ and the measure $Dx$

- **Basic Criteria**
  - Intrinsic = invariant under “reparametrizations” $\text{Diff}(\Sigma)$ of $\Sigma$
  - lead to a well-defined QFT (renormalizable)

- e.g. **Non-linear sigma model action** with Riemannian metric $G$

$$I_G[x, g] = \frac{1}{\alpha'} \int_{\Sigma} d^2\xi \sqrt{g} g^{mn} \partial_m x^\mu \partial_n x^\nu G_{\mu\nu}(x)$$

$m, n = 1, 2$ \hspace{1cm} worldsheet indices
$\mu, \nu = 1, \cdots, D$ \hspace{1cm} space-time Einstein indices

- **The measure is governed by the $L^2$-norm**

$$\|\delta x\|_G^2 = \int_{\Sigma} d^2\xi \sqrt{g} \delta x^\mu \delta x^\nu G_{\mu\nu}(x)$$

  - manifestly intrinsic
  - renormalizable in a generalized sense (the metric $G$ is renormalized)
**Weyl(Σ)-invariance**

- **Weyl transformations:** $g_{mn} \rightarrow e^{2\sigma} g_{mn}$ leaving $x^\mu$ and $G$ unchanged

- **The classical action $I_G$ is Weyl-invariant for any metric $G$**
  - but the measure $Dx$ is not Weyl-invariant
  - which gives rise to a “Weyl-anomaly”
    - = symmetry of classical action not preserved by quantization

- **The action $I_G$ defines a conformal quantum field theory**
  $$e^{-W_G[g]} = \int_{\text{Map}(\Sigma)} Dx \ e^{-I_G[x,g]}$$
  - provided $W_G$ is $\text{Diff}(\Sigma)$-invariant
  - obeys the following Ward identity under Weyl transformations
    $$\delta W_G[g] = \frac{c}{24} \int_\Sigma d^2\xi \sqrt{g} R_g \delta \sigma$$
    - where $R_g$ is the scalar curvature of the metric $g$ on the surface $\Sigma$

- **The measure $Dg$ is not Weyl-invariant, but the combined amplitude**
  - is Weyl invariant for central charge $c = 26 = \dim(M)$
  - later we shall see for the superstring $D = 10 = \dim(M)$
Conformal Field Theory

- Stress tensor encodes response of field theory to change in metric
  \[ T^c_{mn} = \frac{\delta W_G[g]}{\sqrt{g} \delta g^{mn}} \quad \text{subject to} \quad T^c_{mn} = T^c_{nm} \]
  - Diff(\Sigma)-invariance requires “a conserved stress tensor” \( \nabla^m T^c_{mn} = 0 \)
  - Weyl anomaly requires \( g^{mn} T^c_{mn} = -\frac{c}{12} R_g \)

- Traceless stress tensor \( T_{mn} \) obtained by adding a local counter-term
  - In local complex coordinates \((z, \bar{z})\) we have \( T_{zz} = T_{z\bar{z}} = 0 \) and
  \[ T_{zz} = T^c_{zz} + \frac{c}{6} \left( 2 \partial_z \Gamma^z_{zz} - (\Gamma^z_{zz})^2 \right) \quad \Gamma^z_{zz} = \partial_z \ln g_{z\bar{z}} \]
  - Successive derivatives of \( W \) in \( g_{mn} \) give correlators of \( T_{mn} \)
  - Their singular part is governed by the OPE and the Ward identities
  \[ T_{zz} T_{ww} = \frac{c/2}{(z - w)^4} + \frac{2 T_{ww}}{(z - w)^2} + \frac{\partial_w T_{ww}}{z - w} + \text{regular} \]
  - The mode expansion \( T_{zz} = \sum_m z^{-2-m} L_m \) gives the Virasoro algebra
  \[ [L_m, L_n] = (m - n) L_{m+n} + \frac{c}{12} m(m^2 - 1) \delta_{m+n,0} \]
Negative norm states

- Consider flat Minkowski $M = \mathbb{R}^{26}$ with metric $\eta = \text{diag}(- + \cdots +)$
  - Maps $x : \Sigma \to M$ satisfy Laplace equation $\partial_{z} \partial_{\bar{z}} x^\mu = 0$ for $\mu = 1, \ldots, 26$
  - Concentrate on holomorphic field
    \[
    \partial_{z} x^\mu = \sum_{m \in \mathbb{Z}} x^\mu_m z^{-m-1}
    \]
    \[
    [x^\mu_m, x^n_n] = m \delta_{m+n,0} \eta^{\mu \nu}
    \]
    \[
    (x^\mu_n)^\dagger = x^\mu_{-n}
    \]
  - Similarly anti-holomorphic field $\partial_{\bar{z}} x^\mu$ produces modes $\bar{x}^\mu$

- Single string ground state $|0, k\rangle$ labeled by its momentum $k$ satisfies
  \[
  x^\mu_0 |0, k\rangle = k^\mu |0, k\rangle \quad x^\mu_m |0, k\rangle = 0 \text{ for } m > 0
  \]
  - Fock space (holo sector) generated by linear combinations of
    \[
    x^\mu_{m_1} \cdots x^\mu_{m_p} |0, k\rangle \quad m_1, \ldots, m_p < 0
    \]
  - Lowest excited state $\varepsilon^\mu(k) x^\mu_{-1} |0, k\rangle$ has norm
    \[
    \|\varepsilon^\mu(k) x^\mu_{-1} |0, k\rangle\|^2 = \varepsilon^\mu(k) \varepsilon^\nu(k) \eta^{\mu \nu} \| |0, k\rangle\|^2
    \]
  - Component $\varepsilon^\mu = \delta^\mu,0$ produces negative norm state (assuming $\| |0, k\rangle\|^2 > 0$)
    \[
    \Rightarrow \text{ inconsistent with quantum mechanical probability interpretation}
    \]
Eliminating negative norm states – conformal symmetry

• Conformal symmetry guarantees the existence of Virasoro algebra

\[[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n,0}\]

– for the bosonic string $c = 26$ and

\[
L_m = \sum_{n \in \mathbb{Z}} \frac{1}{2} x_{m-n} \cdot x_n \quad L_0 = \frac{1}{2} x_0^2 + \sum_{n \in \mathbb{N}} x_{-n} \cdot x_n
\]

• A state $|\psi\rangle$ is “physical” if $(L_0 - 1)|\psi\rangle = L_m|\psi\rangle = 0$ for $m \in \mathbb{N}$

– Eliminates all negative norm states;
– Decouples all null states produced by gauge transformations;

– e.g. on states $|\psi\rangle = \varepsilon(k) \cdot x_{-1}|0, k\rangle$

  $\star$ $L_1$ constraint imposes $k \cdot \varepsilon(k) = 0$
  $\star$ $L_0$ constraint imposes $k^2 = 0$
  $\star$ $L_m$ constraints are automatic for $m \geq 2$ for this particular state

– the state $|0, k\rangle$ itself is a tachyon (to be absent in the superstring)

$\Rightarrow$ Negative norm and null states eliminated by conformal symmetry
Conformal symmetry in curved space-times

- **Condition for Weyl-invariance on the metric** $G$
  - Infinitesimal Weyl variation for arbitrary $G$ to one-loop order in $\alpha'$

$$
\delta W_G[g] = \int_{\Sigma} d^2 \xi \sqrt{g} g^{mn} \partial_m x^\mu \partial_n x^\nu R_{\mu\nu}(x) \delta \sigma + \cdots + \mathcal{O}(\alpha')
$$

where $R_{\mu\nu}$ is the Ricci tensor of the metric $G_{\mu\nu}$

- Thus, to leading order in $\alpha'$ conformal invariance requires $R_{\mu\nu} = 0$
Conformal symmetry in curved space-times

- **Condition for Weyl-invariance on the metric** $G$
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$$\delta W_G[g] = \int_{\Sigma} d^2 \xi \sqrt{g} \ g^{mn} \partial_m x^\mu \partial_n x^\nu R_{\mu\nu}(x) \delta \sigma + \cdots + O(\alpha')$$

where $R_{\mu\nu}$ is the Ricci tensor of the metric $G_{\mu\nu}$
  - Thus, to leading order in $\alpha'$ conformal invariance requires $R_{\mu\nu} = 0$
Vertex operators

• Small fluctuations in the metric are gravitons
  – A string couples to $N$ gravitons in flat space by slightly perturbing the metric

$$G_{\mu\nu}(x) = \eta_{\mu\nu} + \sum_{i=1}^{N} \varepsilon_{i\mu\nu}(k_i) e^{ik_i\mu x^\mu} + O(\varepsilon^2)$$

  – conformal invariance requires $G$ to satisfy the linearized Einstein equations

$$k_i^2 = 0 \quad k_i^{\mu} \varepsilon_{i\mu\nu}(k_i) = 0 \quad \text{for } i = 1, \ldots, n$$

• Vertex operator formulation is obtained by expanding in powers of $\varepsilon_i$

$$A = \sum_{h=0}^{\infty} g_s^{2h-2} \int_{\text{Met}(\Sigma)} Dg \int_{\text{Map}(\Sigma)} Dx \mathcal{V}_1[x, g] \cdots \mathcal{V}_N[x, g] e^{-I_{\eta}[x, g]}$$

  – where the vertex operator for an on-shell physical graviton is given by

$$\mathcal{V}_i[x, g] = \varepsilon_{i\mu\nu}(k_i) \int_{\Sigma} d^2\xi \sqrt{g} g^{mn} \partial_m x^\mu \partial_n x^\nu e^{ik_\mu x^\mu}$$

  – On-shell conditions $k_i^2 = k_i \cdot \varepsilon_i = 0$ guarantee conformal invariance
**Diff(Σ) × Weyl(Σ) and Moduli space**

- **Fix topology of Σ**
  - Diff(Σ) re-parametrizes $\xi^m$ on Σ by vector field $\delta \xi^m = -\delta v^m$
  - Weyl (Σ) $\delta g_{mn} = \nabla_m \delta v_n + \nabla_n \delta v_m$
  - Weyl (Σ) $\delta g_{mn} = 2\delta \sigma g_{mn}$ with $\delta \sigma$ an arbitrary real function of Σ

- **Orbits of Diff(Σ) × Weyl(Σ) acting on the space Met(Σ)**

  ![Diagram of orbits]

  \[\text{Met}(\Sigma)/\text{Diff}(\Sigma) \times \text{Weyl}(\Sigma) = \mathcal{M}_h\]

- **Moduli space $\mathcal{M}_h$ of compact Riemann surfaces of genus $h$ (no boundaries)**
  = space of conformal structures (= space of complex structures)

  \[\dim_{\mathbb{C}} \mathcal{M}_h = \begin{cases} 
  0 & h = 0 \\
  1 & h = 1 \\
  3h - 3 & h \geq 2
\end{cases}\]
Some trivial moduli spaces

- **Given an infinitesimal** $\delta g_{mn}$ **can one solve for** $\delta \sigma$ **and** $\delta v_m$ **?**

  $$\delta g_{mn} = 2\delta \sigma g_{mn} + \nabla_m \delta v_n + \nabla_n \delta v_m$$

- Eliminate the trace part by choosing $\delta \sigma = g^{mn} \delta g_{mn} + \nabla_m \delta v^m$

- In local complex coordinates $(z, \bar{z})$, remaining eqs for traceless part

  $$\delta g_{zz} = \nabla_z v_z \quad \delta g_{\bar{z}\bar{z}} = \nabla_{\bar{z}} v_{\bar{z}}$$

- Integrability automatic since $\nabla_z$ and $\nabla_{\bar{z}}$ act on different functions
  $\Rightarrow$ locally, or in any simply connected set, you can always solve

- **The sphere $S^2$ has no moduli** (compact)
  - Its stereographic projection onto $\mathbb{C}$ admits a globally conformally flat metric

  $$ds^2 = \frac{|dz|^2}{(1 + |z|^2)^2}$$

- **The Poincaré upper half plane $\mathcal{H}$ has no moduli** (non-compact)

  $$ds^2 = \frac{|dz|^2}{(\text{Im } z)^2} \quad \text{Im } z > 0$$
Moduli deformations of the torus

- The torus may be viewed as the product of two circles $A$ and $B$
  - The ratio of their lengths and relative angle provide two real moduli
  - equivalently represented by parallelogram in $\mathbb{C}$ with sides pairwise identified

$A \quad B$

$\Sigma$

- The complex number $\tau$ contains the information of relative lengths and angle

- Constant metric deformations equivalently provide a complex modulus
  - translation invariance on the circles induces translation invariance on the torus
  - by translation invariance, metric is constant on $\Sigma$
  - constant trace-part of $\delta g_{mn}$ eliminated by constant $\sigma$
  - but constant $\delta g_{zz} = \partial_z v_z$ has no periodic solutions $v_z$
    $\Rightarrow$ constant $\delta g_{zz}$ provides the deformation of the complex modulus of the torus.
Moduli space of the torus

• Oriented Riemann surfaces: cycles $\mathcal{A}$ and $\mathcal{B}$ ordered
  – equivalently choose orientation $\tau \in \mathcal{H}_1 = \{ \tau \in \mathbb{C}, \text{Im}(\tau) > 0 \}$

• Space of inequivalent tori = space of inequivalent lattices $\Lambda_\tau = \mathbb{Z} \oplus \tau \mathbb{Z}$
  – but different values of $\tau$ may give the same lattice

\[
\begin{align*}
\omega_1' &= a \omega_1 + b \omega_2 \\
\omega_2' &= c \omega_1 + d \omega_2 \\
\tau &= \omega_1 / \omega_2 \\
\tau' &= (a \tau + b) / (c \tau + d)
\end{align*}
\]

– identical lattices requires $\Lambda_{\tau'} \subset \Lambda_\tau$ and $\Lambda_\tau \subset \Lambda_{\tau'}$
– so that $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$ and \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2, \mathbb{Z}) \)
– generated by $\tau \to \tau + 1$ and $\tau \to -\tau^{-1}$

• Moduli space of tori = space of inequivalent lattices $= \mathcal{H}_1 / SL(2, \mathbb{Z})$
  – standard fundamental domain

\[
\mathcal{H}_1 / SL(2, \mathbb{Z}) \equiv \left\{ \tau \in \mathcal{H}_1, |\tau| \geq 1, |\text{Re}(\tau)| \leq \frac{1}{2} \right\}
\]
Decomposing the measure $Dg$

- At any point $g \in \text{Met}(\Sigma)$ the measure $Dg$ factors

$$Dg = Z_g \times D\sigma \times Dv \times d\mu_{M_h}$$

Jacobian Weyl Diff$_0$ $M_h$

- infinitesimal Weyl $\delta g_{mn} = \delta \sigma g_{mn}$
- infinitesimal Diff$_0$ $\delta g_{mn} = \nabla_m \delta v_n + \nabla_n \delta v_m$
- infinitesimal moduli deformations $\delta g_{mn}$

Goal

- compute $Z_g$
- formulate $Z_g$ in terms of ghosts
- omit volume factors $D\sigma Dv$ of the group $\text{Diff}^+(\Sigma) \ltimes \text{Weyl}(\Sigma)$

To decompose $Dg$ we study tensor spaces (alias line bundles) on $\Sigma$
Tensor Spaces - Line Bundles on $\Sigma$

- A one-form $\phi = \phi_z dz + \phi_{\bar{z}} d\bar{z}$ on $\Sigma$ decomposes into $K \oplus \bar{K}$
  
  $K = \{ \phi_z dz \}$ is the (space of sections of the) canonical bundle on $\Sigma$
  
  for $m \in \mathbb{Z}$ define $K^m = \{ \phi_z \cdots dz^m \}$ and $\bar{K}^m = \{ \phi_{\bar{z}} \cdots d\bar{z}^m \} \approx K^{-m}$

- $L^2$ inner product for $\phi_1, \phi_2 \in K^m$
  
  $$(\phi_1, \phi_2) = \int_\Sigma d\bar{z}dz \sqrt{g}(g_{z\bar{z}})^{-m} \phi_1^* \phi_2$$

  The spaces $K^m$ and $K^n$ with $m \neq n$ are mutually orthogonal

- Covariant derivative on $\phi \in K^m$ decomposes $\nabla\phi = \nabla_z^{(m)} \phi + \nabla_{\bar{z}}^{(m)} \phi$
  
  $\nabla_z^{(m)} : K^m \rightarrow K^{m+1}$ mutual adjoint operators $(\nabla_z^{(m)})^\dagger = -\nabla_{\bar{z}}^{(m+1)}$

  $\nabla_{\bar{z}}^{(m)} : K^m \rightarrow K^{m-1}$ with $\nabla_{\bar{z}}^{(m)} = g_{z\bar{z}} \nabla_{\bar{z}}^{(m)}$

- Riemann-Roch and Vanishing Theorems
  
  $\dim_{\mathbb{C}} \text{Ker} \nabla_{\bar{z}}^{(m)} - \dim_{\mathbb{C}} \text{Ker} \nabla_{z}^{(1-m)} = (2m - 1)(h - 1)$

  $\text{Ker}\nabla_{\bar{z}}^{(m)} = 0$ for $h \geq 2$ and $m \leq -1$ (no holó vector fields for $h \geq 2$)

  $\text{Ker}\nabla_{\bar{z}}^{(m)} = 0$ for $h = 0$ and $m \geq 1$ (no holó forms on the sphere)
Decomposing the tangent space to \( \text{Met}(\Sigma) \)

- **Orthogonal decomposition of** \( T_g(\text{Met}(\Sigma)) \)

\[
T_g(\text{Met}(\Sigma)) = \{ \delta \sigma \ g_{z \bar{z}} \} \oplus \{ \delta g_{zz} = g_{z \bar{z}} \delta \eta_{\bar{z}} \} \oplus \{ \delta g_{\bar{z} \bar{z}} = g_{z \bar{z}} \delta \eta_{z \bar{z}} \}
\]

\[
\delta \sigma \in K^0 \quad \delta \eta_{z \bar{z}} \in K \otimes \bar{K}^{-1} \quad \delta \eta_{z \bar{z}} \in \bar{K} \otimes K^{-1}
\]

- **\( \text{Diff}_0 \) acts by** \( \delta \eta_{z \bar{z}} = \nabla^{(1)}_{\bar{z}} \delta v_{\bar{z}} \)

  - For \( h \geq 1 \), the range of the operator \( \nabla^{(1)}_{\bar{z}} \) is NOT all of \( K \otimes \bar{K}^{-1} \)
  - The orthogonal complement of the range of \( \nabla^{(1)}_{\bar{z}} \) is given by

\[
\text{Range} \nabla^{(1)}_{\bar{z}} \oplus \ker(\nabla^{(1)}_{\bar{z}})^\dagger = K \otimes \bar{K}^{-1} \approx K^2
\]

- **Holomorphic quadratic differentials** \( \phi^j \in \ker \nabla^{(2)}_{\bar{z}} \approx \ker(\nabla^{(1)}_{\bar{z}})^\dagger \)

  - Hence we may identify \( \ker \nabla^{(2)}_{\bar{z}} = T^*_{(1,0)}(\mathcal{M}_h) \)
  - One-forms \( \delta m^j \in T^*_{(1,0)}(\mathcal{M}_h) \) given by linear forms on \( \bar{K} \otimes K^{-1} \)

\[
\delta m^j = (\delta \eta, \phi^j) = \int_{\Sigma} d\bar{z}dz \delta \eta_{z \bar{z}} \phi^j_{z \bar{z}}
\]

  - Weyl-invariant pairing and vanishes on \( \delta \eta \in \text{Range} \nabla^{(1)}_{\bar{z}} \)
  - Riemann-Roch and Vanishing give \( \text{dim}_\mathbb{C} \mathcal{M}_h = 3h - 3 \) for \( h \geq 2 \)
Decomposing the measure $Dg$ (cont’d)

• Parametrize $\mathcal{M}_h$ by a slice in $\text{Met}(\Sigma)$ transverse to $\text{Weyl} \ltimes \text{Diff}_0$

  orbits of $\text{Diff}(\Sigma) \ltimes \text{Weyl}(\Sigma)$

  $m^j, \tilde{m}^j$ local coordinates on $\mathcal{M}_h$

• Carry out a change of integration variables

\[
T_g(\text{Met}(\Sigma)) = \{\delta \sigma g_{zz}\} \oplus \{\delta \eta_z \bar{z}\} \oplus \{\delta \eta_{\bar{z}} z\}
\]

– Orthogonality implies that the measure factorizes $Dg = D\sigma D\eta D\bar{\eta}$

– The change of variables is given by (repeated indices $j$ are summed)

\[
\delta \eta_{\bar{z}} z' = \nabla_{\bar{z}}^{(-1)} \delta v z + (\mu_j)_z \bar{z} \delta m^j
\]

\[
(\mu_j)_z \bar{z} = g^{z\bar{z}} \frac{\partial g_{z\bar{z}}}{\partial m^j}
\]

\[
\delta \eta_z \bar{z}' = \nabla_{z}^{(1)} \delta v \bar{z} + (\tilde{\mu}_j)_z \bar{z} \delta \tilde{m}^j
\]

\[
(\tilde{\mu}_j)_z \bar{z} = g^{z\bar{z}} \frac{\partial g_{z\bar{z}}}{\partial \tilde{m}^j}
\]
Ghosts

- **Use standard rules to introduce ghosts for the determinant**
  - gauge transformations \((\delta v^z, \delta v^{\tilde{z}}) \rightarrow (c^z, c^{\tilde{z}})\) Grassmann-odd ghosts
  - conjugate \((\delta \eta^z_{\tilde{z}}, \delta \eta^{\tilde{z}}z) \rightarrow (b_{zz}, \tilde{b}_{\tilde{z}z})\) Grassmann-odd anti-ghosts
  - extended ghost action
    \[
    \int \Sigma d^2 z \left[ b_{zz} (\partial_z c^z + \mu_j \delta m^j) + \tilde{b}_{\tilde{z}z} (\partial_{\tilde{z}} c^{\tilde{z}} + \tilde{\mu}_j \delta \tilde{m}^j) \right]
    \]
  - Here \(\delta m^j, \delta \tilde{m}^j\) are differential one-forms which are Grassmann odd

- **Integrating out** \(\delta m^j, \delta \tilde{m}^j\) **gives the standard ghost representation**
  \[
  \int D(x^\mu, b, \tilde{b}, c, \tilde{c}) \mathcal{V}_1 \cdots \mathcal{V}_N e^{-I_G - I_{gh}} \prod_j \delta(\langle b, \mu_j \rangle) \delta(\langle \tilde{b}, \tilde{\mu}_j \rangle) dm^j d\tilde{m}^j
  \]
  - where \(I_{gh}\) is the standard ghost action
    \[
    I_{gh} = \int \Sigma d^2 z \left[ b_{zz} \partial_z c^z + \tilde{b}_{\tilde{z}z} \partial_{\tilde{z}} c^{\tilde{z}} \right]
    \]
  - gauge fixed formulation has BRST invariance
  - for the sphere and the torus, quotient out by conformal automorphisms
Bosonic string has tachyon and no fermions: unphysical

• Warm-up: tree-level tachyon scattering amplitude
  – tachyon vertex operator \( V(k_i) = \int_\Sigma d^2z_i \sqrt{g(z_i)} : e^{ik_i \cdot x(z_i)} : \)
  – scalar Green function on the sphere with metric \( |dz|^2/(1 + |z|^2)^2 \)
  \[
  \langle x^\mu(z)x^\nu(w) \rangle = \eta^{\mu\nu}G(z, w) \\
  G(z, w) = -\ln \frac{|z - w|^2}{(1 + |z|^2)(1 + |w|^2)}
  \]

• Sphere has no moduli, ghost and scalar partition functions are constant
  \[
  \langle \prod_{i=1}^N d^2z_i \sqrt{g(z_i)} : e^{ik_i \cdot x(z_i)} : \rangle = \prod_{i=1}^N d^2z_i \prod_{i<j}^N |z_i - z_j|^\alpha' k_i \cdot k_j
  \]
  – Integrand invariant under \( z_i \to (\alpha z_i + \beta)/(\gamma z_i + \delta) \) with \( \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \in SL(2, \mathbb{C}) \)
  – Factor out volume of \( SL(2, \mathbb{C}) \) by fixing \( z_N = \infty, z_{N-1} = 1, z_{N-2} = 0 \)

• The 4-tachyon amplitude with \( s_{ij} = -\alpha'(k_i + k_j)^2/4 \)
  \[
  \frac{1}{g_s^2} \int \Sigma d^2z |z|^{\alpha' k_1 \cdot k_2} |z - 1|^{\alpha' k_1 \cdot k_3} = \frac{\Gamma(-1 - s)\Gamma(-1 - t)\Gamma(-1 - u)}{g_s^2 \Gamma(2 + s)\Gamma(2 + t)\Gamma(2 + u)}
  \]
  – Tachyon poles at \( s, t, u = -1 \)
Kawai-Lewellen-Tye (KLT) relations

- **Tree-level closed string amplitudes are bilinears in open string amplitudes**
  - Closed string amplitudes on the sphere, vertex operators in interior
  - Open string amplitude on upper half plane, vertex operators on boundary
  - Consider open and closed string 4-tachyon amplitudes

\[ A^{(0)}_{\text{open}}(s, t) = \int_0^1 d\xi |\xi|^k_1 \cdot k_2 |1 - \xi|^{k_2 \cdot k_3} \]
\[ A^{(0)}_{\text{closed}}(s, t, u) = \int_{S^2} d^2 z |z|^{2k_1 \cdot k_2} |1 - z|^{2k_2 \cdot k_3} \]

- Parametrize \( z = \alpha + i\beta \) then \( z \)-integrand is analytic function of \( \beta \) with branch points at \( \beta = \pm i\alpha \) and \( \beta = \pm i(1 - \alpha) \)
- Deform \( \beta \)-contour from real to imaginary axis, but pick up phases
\[ \int_{S^2} d^2 z |z|^{2k_1 \cdot k_2} |1 - z|^{2k_2 \cdot k_3} = \sin(\pi k_2 \cdot k_3) \int_0^1 d\xi |\xi|^{k_1 \cdot k_2} |1 - \xi|^{k_2 \cdot k_3} \int_1^\infty d\eta |\eta|^{k_1 \cdot k_2} |1 - \eta|^{k_2 \cdot k_3} \]

- Converting the second integral back to \( A^{(0)}_{\text{open}} \), we obtain the KLT relation
\[ A^{(0)}_{\text{closed}}(s, t, u) = \sin(\pi k_2 \cdot k_3) A^{(0)}_{\text{open}}(s, t) A^{(0)}_{\text{open}}(t, u) \]

- Does the worldsheet secretly have a Minkowski signature structure?
- No generalization known to loop level
Lectures on Superstring Amplitudes

Part 2: Superstrings

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Center for Quantum Mathematics and Physics - 2018
Amplitudes 2018 Summer School
Superstring Perturbation Theory

- **Theory of fluctuating random surfaces** (closed strings shown)
  
  - governed by topological expansion in the genus $h$ weighed by $g_s^{2h-2}$

\[ g_s^{-2} + g_s^0 + g_s^2 + \cdots \]

- **Bosonic string**
  
  - unstable with closed string tachyon
  
  - Nature has fermions!

- **Superstrings generalize bosonic string**
  
  - they have fermions
  
  - no tachyon
  
  - supersymmetry
**Approaches to Superstring Perturbation Theory**

- **Goal is to obtain superstring amplitudes at all genera**
  - Ramond-Neveu-Schwarz formulation of fermionic strings; w/ Gliozzi-Scherk-Olive projection to supersymmetric spectrum;
  - Green-Schwarz space-time supersymmetric formulation;
  - Mandelstam light-cone formulation;
  - String field theory;
  - Topological string theory;
  - Berkovits pure spinor formulation.

- **Different perturbative superstring theories** (in 10 dimensions)
  - Type I open & closed, orientable & non-orientable, D-branes
  - Type IIA,B closed orientable, D-branes
  - Heterotic closed orientable $E_8 \times E_8$, $Spin(32/\mathbb{Z}_2)$

- Here: **RNS formulation, closed orientable superstrings, dimension 10**
Genus-zero four-graviton superstring amplitude

- Kinematics of the four-graviton amplitude
  - momenta of gravitons $k_i^\mu$ are conserved $\sum_i k_i^\mu = 0$
  - choose basis of factorized polarization tensors $\varepsilon_i^{\mu\nu} = \varepsilon_i^\mu \tilde{\varepsilon}_i^\nu$
  - masslessness $k_i^2 = 0$ and transversality $k_i^\mu \varepsilon_i^\mu = k_i^\mu \tilde{\varepsilon}_i^\mu = 0$ for $i = 1, 2, 3, 4$
  - kinematic invariants $s = s_{12} = s_{34}, t = s_{14} = s_{23}, u = s_{13} = s_{24}$
  
  $$s_{ij} = -\alpha'(k_i + k_j)^2/4$$

- Tree-level four-graviton amplitude is given by

$$A^{(0)}(\varepsilon_i, \tilde{\varepsilon}_i, k_i) = \frac{1}{g_s^2} \times K\tilde{K} \times \frac{1}{stu} \frac{\Gamma(1-s)\Gamma(1-t)\Gamma(1-u)}{\Gamma(1+s)\Gamma(1+t)\Gamma(1+u)}$$

  - Kinematical factor $K$ given in terms of $f_i^{\mu\nu} = k_i^\mu \varepsilon_i^\nu - k_i^\nu \varepsilon_i^\mu$ by

  $$K = (f_1f_2)(f_3f_4) + (f_1f_3)(f_2f_4) + (f_1f_4)(f_2f_3) - 4(f_1f_2f_3f_4) - 4(f_1f_2f_4f_3) - 4(f_1f_3f_2f_4)$$

  - for $\tilde{K}$ replace $\varepsilon_i$ by $\tilde{\varepsilon}_i$

  - Equivalently, $K \times \tilde{K} = R^4$ with $R$ the linearized Weyl tensor

  - String duality: symmetric in $s, t, u$

  - Poles in each channel, at $s, t, u = 0, 1, 2, \cdots$
Genus-one four-graviton superstring amplitude

- **Type II four-graviton amplitude to one-loop order** (Green, Schwarz 1982)

\[
A^{(1)}(\varepsilon_i, \bar{\varepsilon}_i, k_i) = \mathcal{R}^4 \int_{\mathcal{M}_1} \frac{d^2 \tau}{(\text{Im} \, \tau)^2} B^{(1)}(s_{ij}|\tau)
\]

- Partial amplitude \(B^{(1)}\) is a modular function in \(\tau \in \mathcal{M}_1 = \mathcal{H}_1/SL(2, \mathbb{Z})\)

\[
B^{(1)}(s_{ij}|\tau) = \int_{\Sigma} \prod_{i=1}^4 \frac{d^2 z_i}{\text{Im} \, \tau} \exp \left( \sum_{i<j} s_{ij} G(z_i - z_j|\tau) \right)
\]

- \(G(z|\tau)\) is the scalar Green function on the torus \(\Sigma\) of modulus \(\tau\).
- Analogous formulas for Heterotic strings and more external states.

- **Singularity structure**
  - For fixed \(\tau\) integrations over \(\Sigma\) produce poles in \(B^{(1)}\) at positive integers \(s_{ij}\).
  - The integral over \(\tau\) converges absolutely only for \(\text{Re}(s_{ij}) = 0\).
  - Analytic continuation to \(s_{ij} \in \mathbb{C}\) via decomposition of \(\mathcal{M}_1\).
  - Branch cuts in \(s_{ij}\) starting at integers \(\geq 0\) are produced by \(\tau \rightarrow i\infty\) region.
Loop momenta

- Loop momenta may be exposed
  - Choose a canonical basis of homology cycles $\mathcal{A}, \mathcal{B}$.
  - Choose loop momentum $p$ flowing through the cycle $\mathcal{A}$,

$$
\int_{\mathcal{M}_1} \frac{d^2 \tau}{(\operatorname{Im} \tau)^2} \mathcal{B}^{(1)} (s_{ij} | \tau) = \int_{\mathbb{R}^{10}} d^{10} p \int_{\mathcal{M}_1} \int_{\Sigma^4} \left| \mathcal{F}(z_i, k_i, p | \tau) \right|^2
$$

- Chiral amplitude $\mathcal{F}$ is locally holomorphic in $\tau$ and $z_i$

$$
\mathcal{F}(z_i, k_i, p | \tau) = e^{i \pi p^2 + 2i \pi p} \prod_{i<j} \vartheta_1 (z_i - z_j | \tau)^{-s_{ij}} \, d\tau \prod_{i=1}^{4} dz_i
$$

  - at the cost of non-trivial monodromy

$$
\mathcal{F}(z_i + \delta_i, \ell \mathcal{A}, k_i, p | \tau) = e^{2i \pi k \ell \cdot p} \mathcal{F}(z_i, k_i, p | \tau)
$$

$$
\mathcal{F}(z_i + \delta_i, \ell \mathcal{B}, k_i, p | \tau) = \mathcal{F}(z_i, k_i, p + k_\ell | \tau)
$$

  - Modular invariance of $\mathcal{A}^{(1)}$ guarantees independence of choices.
  - Hermitian pairing of $\mathcal{F}$ and $\bar{\mathcal{F}}$ is familiar from 2-d CFT where loop momentum $p$ labels conformal blocks of 10 copies of $c = 1$. 
**RNS formulation of superstrings**

- $M = \mathbb{R}^{10}$ flat Minkowski space-time with Lorentz group $SO(1,9)$
  - $x^\mu$ scalars on worldsheet $\Sigma$, map $\Sigma$ into $M$
  - $\psi^\mu$ spinors on $\Sigma$ but Lorentz vector under $SO(1,9)$
    * Worldsheet supersymmetry $\Rightarrow \Sigma$ is a super Riemann surface
    * Two sectors: NS bosons $SO(1,9)$-tensors
      R fermions $SO(1,9)$-spinors

- With Minkowski signature $\Sigma$
  - $\psi^\mu$ and $\tilde{\psi}^\mu$ are *independent* Majorana-Weyl spinors of opposite chirality

- With Euclidean signature $\Sigma$
  - $\psi^\mu$ and $\tilde{\psi}^\mu$ must be *independent* complex Weyl spinors
  - Globally, on a compact Riemann surface of genus $h$,
    * All $\psi^\mu$ are sections of a the same spin bundle $S$ (and $\tilde{\psi}^\mu$ of $\tilde{S}$)
    * $2^{2h}$ distinct spin structures for $S$ (and $2^{2h}$ independently for $\tilde{S}$)

- GSO projection requires independent summation over spin structures
Quantization of worldsheet spinor fields

• Illustrate
  – Ramond and Neveu-Schwarz sectors
  – independence of chiralities

• Dirac action and equation for flat $M = \mathbb{R}^{10}$ with metric $\eta$
  – All components of $\psi_{+}^\mu$ are sections of the same spin bundle $S$
  – Complex structure $J$ with local complex coordinates $(z, \bar{z})$
  – Dirac action,

$$I_{\psi}[\psi, J] = \frac{1}{2\pi} \int_{\Sigma} d\bar{z}dz \psi_{+}^\mu \partial_{\bar{z}}\psi_{+}^\nu \eta_{\mu\nu}$$

  – Dirac equation $\partial_{\bar{z}}\psi_{+}^\mu = 0$ has locally holomorphic solutions,
  – but products of operators produce singularities

$$\psi_{+}^\mu(z)\psi_{+}^\nu(w) = \frac{\eta_{\mu\nu}}{z-w} + \text{regular}$$

  – each component $\psi^\mu$ generates a CFT with central charge $c = \frac{1}{2}$. 
Quantization of worldsheet spinor fields (cont’d)

• Quantization on flat cylinder or conformal equivalent flat annulus
  – cylinder \( w = \tau + i\sigma \) with identification \( \sigma \approx \sigma + 2\pi \)
  – annulus centered at \( z = 0 \), conformally mapped by \( z = e^w \)
  – one-forms related by \( dz = e^w \, dw \), spinors by \( (dz)^{\frac{1}{2}} = e^{w/2} \, (dw)^{\frac{1}{2}} \)
  – fields related by conformal transformation \( \psi_{cyl}(z) = e^{w/2} \psi_{ann}(w) \)

• Two possible spin structures

  \( \text{NS} \quad \psi^\mu_{cyl}(\tau, \sigma + 2\pi) = -\psi^\mu_{cyl}(\tau, \sigma) \) or \( \psi^\mu_{ann}(e^{2\pi i} \, z) = +\psi^\mu_{ann}(z) \)
  \( \text{R} \quad \psi^\mu_{cyl}(\tau, \sigma + 2\pi) = +\psi^\mu_{cyl}(\tau, \sigma) \) or \( \psi^\mu_{ann}(e^{2\pi i} \, z) = -\psi^\mu_{ann}(z) \)

• Free field quantization in annulus representation

  \( \text{NS} \quad \psi^\mu(z) = \sum_{r \in \frac{1}{2} + \mathbb{Z}} b^\mu_r \, z^{-\frac{1}{2} - r} \quad \{b^\mu_r, b^\nu_s\} = \eta^{\mu \nu} \delta_{r+s,0} \)
  \( \text{R} \quad \psi^\mu(z) = \sum_{n \in \mathbb{Z}} d^\mu_n \, z^{-\frac{1}{2} - n} \quad \{d^\mu_m, d^\nu_n\} = \eta^{\mu \nu} \delta_{m+n,0} \)
Quantization of worldsheet spinor fields (cont’d)

• Lorentz generators of $SO(1, 9)$:

$$[J^{\mu\nu}, \psi^\kappa(z)] = \eta^{\nu\kappa} \psi^\mu(z) - \eta^{\mu\kappa} \psi^\nu(z)$$

$$J^{\mu\nu}_{NS} = \sum_{r \in \mathbb{N}-\frac{1}{2}} (b^\mu_{-r} b^\nu_r - b^\nu_{-r} b^\mu_r)$$

$$J^{\mu\nu}_R = \frac{1}{2}[d^\mu_0, d^\nu_0] + \sum_{n \in \mathbb{N}} (d^\mu_{-n} d^\nu_n - d^\nu_{-n} d^\mu_n)$$

• Fock space construction produces two sectors

★ NS ground state defined by $b^\mu_r |0; NS\rangle = 0$ for all $r > 0$
  – $|0; NS\rangle$ is unique and in trivial representation of $SO(1, 9)$
  – Fock space = linear combinations of $b^{\mu_1}_{-r_1} \cdots b^{\mu_p}_{-r_p} |0; NS\rangle$, $r_i > 0$
  – All states in tensor reps of $SO(1, 9)$ are space-time bosons.

★ R ground state defined by $d^\mu_n |0, \alpha; R\rangle = 0$ for all $n > 0$
  – $|0, \alpha; R\rangle$ is degenerate and in spinor rep. of $SO(1, 9)$, states labelled by $\alpha$
  – Fock space = linear combinations of $d^{\mu_1}_{-n_1} \cdots d^{\mu_p}_{-n_p} |0, \alpha; R\rangle$, $n_i > 0$
  – All states in spinor reps of $SO(1, 9)$ are space-time fermions.
**Summation over spin structures**

- **Theory with bosons and fermions requires both NS and R sectors**
  - to include both, one must sum over two spin structures of the annulus

- **Type II spin structures of** $\psi^\pm$ **are independent of one another**
  - space-time fermions are in the $R \otimes NS$ and $NS \otimes R$ sectors
    - which could never arise if spin structures for opposite chiralities coincided

- **On the torus, viewed as cylinder + identification**
  - spin structures along cycle of cylinder produce R and NS sectors
  - sum over spin structures along conjugate cycle produces GSO-projection
    - reduces to half the states in both R and NS sectors
    - R-sector: space-time spinor of definite chirality
    - NS-sector: eliminates the tachyon
  $\Rightarrow$ sum over *all* spin structures
Summation over spin structures (cont’d)

- **Fix a canonical homology basis of cycles** $\mathcal{A}_I, \mathcal{B}_I$ of $H_1(\Sigma, \mathbb{Z}) \quad I = 1, \ldots, h$
  - with canonical intersection pairing
    $\#(\mathcal{A}_I, \mathcal{A}_J) = \#(\mathcal{B}_I, \mathcal{B}_J) = 0$ and $\#(\mathcal{A}_I, \mathcal{A}_J) = \delta_{IJ}$

- **Transformations which maps one canonical basis into another**
  - linear with integer coefficients
  - preserve the intersection matrix: $Sp(2h, \mathbb{Z})$

- **On Riemann surface of higher genus $h$ sum over all spin structures**
  - along $\mathcal{A}$-cycles produces R and NS sectors
  - along $\mathcal{B}$-cycles produces GSO-projection
  - mapped into one another by $Sp(2h, \mathbb{Z}_2)$
Super Riemann surfaces

- **Ordinary Riemann surface** (locally $\mathbb{C}$ with coordinate $z$)
  - complex manifold: holomorphic transition functions $z \to z'(z)$;
  - complex structure $= \text{conformal structure } J$
  - Moduli space $\mathcal{M}_h = \{ J \}/\text{Diff}(\Sigma)$ of genus $h$ compact Riemann surfaces

- **Complex super manifold** (locally $\mathbb{C}^{1|1}$ with coordinates $z|\theta$)
  - holó transition functions $z|\theta \to z'(z, \theta)|\theta'(z, \theta)$ generate $\mathcal{N} = 2$ super conformal

- **Super Riemann surface** (locally $\mathbb{C}^{1|1}$ with coordinates $z|\theta$)
  - holó transition functions $z|\theta \to z'|\theta'$ rescale $D_\theta = \partial_\theta + \theta \partial_z$
  - Transition functions define $\mathcal{N} = 1$ superconformal structure $J$
  - Globally: $T\Sigma$ has a completely non-integrable subbundle of rank $0|1$

- **Moduli space of compact super Riemann surfaces**: $\mathcal{M}_h = \{ J \}/\text{Diff}(\Sigma)$
  - equivalence classes of superconformal structures $J$
  \[
  \dim_{\mathbb{C}} \mathcal{M}_h = \begin{cases} 
  0|0 & h = 0 \\
  1|0 \text{ or } 1|1 & h = 1 \text{ even or odd spin structure} \\
  3h - 3|2h - 2 & h \geq 2
  \end{cases}
  \]

  - odd modulus at $h = 1$ odd spin structure is a book keeping device;
  - odd moduli really first appear at genus 2, as curved super spaces.
Superstring worldsheets and moduli spaces

• Heterotic
  – Left : RS $\Sigma_L$, moduli space $\mathcal{M}_L$ coord resp. $\tilde{z}$ and $\tilde{m}^i$
  – Right : SRS $\Sigma_R$, moduli space $\mathcal{M}_R$ coord resp. $(z, \theta)$ and $(m^i, \zeta^\alpha)$
  – Worldsheet is a cycle $\Sigma \subset \Sigma_L \times \Sigma_R$ of dim $1|1$
    subject to $\Sigma_{\text{red}} = \text{diag}(\Sigma_{L\text{red}} \times \Sigma_{R\text{red}}) : \tilde{z}^* = z + \text{nilpotent}$
  – Moduli space is a cycle $\Gamma \subset \mathcal{M}_L \times \mathcal{M}_R$ of dim $3h - 3|2h - 2$ for $h \geq 2$
    subject to $\Gamma_{\text{red}} = \text{diag}(\mathcal{M}_{L\text{red}} \times \mathcal{M}_{R\text{red}}) : (\tilde{m}^i)^* = m^i + \text{nilpotent}$
    (reduced space obtained by setting all nilpotent variables to zero)

• Type II
  – Left : SRS $\Sigma_L$, moduli space $\mathcal{M}_L$ coord resp. $(\tilde{z}, \tilde{\theta})$ and $(\tilde{m}^i, \tilde{\zeta}^\alpha)$
  – Right : SRS $\Sigma_R$, moduli space $\mathcal{M}_R$ coord resp. $(z, \theta)$ and $(m^i, \zeta^\alpha)$
  – Worldsheet is a cycle $\Sigma \subset \Sigma_L \times \Sigma_R$ of dim $1|2$
  – Moduli space is cycle $\Gamma \subset \mathcal{M}_L \times \mathcal{M}_R$ of dim $3h - 3|4h - 4$ for $h \geq 2$
    subject to $\tilde{z}^* = z + \text{nilpotent}$ and $(\tilde{m}^i)^* = m^i + \text{nilpotent}$

• Super-Stokes theorem ensures independence of the choice of cycles
  – in amplitudes with BRST invariant vertex operators
  – consistent definition of superstring amplitudes to all genera (Witten 2012)
Worldsheet action for Type II superstrings

- **Worldsheet is** $\Sigma \subset \Sigma_L \times \Sigma_R$
  - $\Sigma_L$ has superconformal structure $\tilde{J}$ with local coordinates $\tilde{z}|\tilde{\theta}$
  - $\Sigma_R$ has superconformal structure $J$ with local coordinates $z|\theta$

- **Superconformal invariant matter action**
  - worldsheet matter field
    
    $$X^\mu(\tilde{z}, z|\tilde{\theta}, \theta) = x^\mu(\tilde{z}, z) + \theta \psi^\mu(\tilde{z}, z) + \tilde{\theta} \tilde{\psi}^\mu(\tilde{z}, z) + \tilde{\theta} \theta F^\mu(\tilde{z}, z)$$

  - Worldsheet action in local coordinates ($D_\theta = \partial_\theta + \theta \partial_z$)
    
    $$I_m[X^\mu, \tilde{J}, J] = \int_{\Sigma} [d\tilde{z}dz|d\tilde{\theta}d\theta] \tilde{D}_{\tilde{\theta}}X^\mu D_\theta X_\mu$$

  - Superconformal algebra on fields generated by

    $S_{z\theta} = S_{z\theta} + \theta T_{zz}$
    
    $S_{z\theta} = \frac{1}{2} \psi^\mu \partial_z x_\mu$
    
    $T_{zz} = -\frac{1}{2} \partial_z x^\mu \partial_z x_\mu + \frac{1}{2} \psi^\mu \partial_z \psi_\mu$

    $\tilde{S}_{\tilde{z}\tilde{\theta}} = \tilde{S}_{\tilde{z}\tilde{\theta}} + \tilde{\theta} \tilde{T}_{\tilde{z}\tilde{z}}$
    
    $\tilde{S}_{\tilde{z}\tilde{\theta}} = \frac{1}{2} \tilde{\psi}^\mu \partial_{\tilde{z}} x_\mu$
    
    $\tilde{T}_{\tilde{z}\tilde{z}} = -\frac{1}{2} \partial_{\tilde{z}} x^\mu \partial_{\tilde{z}} x_\mu + \frac{1}{2} \tilde{\psi}^\mu \partial_{\tilde{z}} \tilde{\psi}_\mu$
Deformations of superconformal structures

- Under deformation of $\tilde{J}$ for $\Sigma_L$ and $J$ for $\Sigma_R$

$$\delta I = \int_{\Sigma} [d\tilde{z}dz|d\tilde{\theta}d\theta] \left( H_{\tilde{\theta}}\tilde{z} S_{\tilde{z}\theta} + \tilde{H}_{\theta}\tilde{\tilde{z}} \tilde{S}_{\tilde{z}\tilde{\theta}} \right)$$

- in components by integrating out $\tilde{\theta}, \theta$,

$$\delta I = \int_{\Sigma_{\text{red}}} d\tilde{z}dz \left( \mu_{\tilde{z}} \tilde{T}_{zz} + \chi_{\tilde{z}} \theta S_{\tilde{z}\theta} + \tilde{\mu}_{\tilde{z}} \tilde{T}_{\tilde{z}\tilde{z}} + \tilde{\chi}_{\tilde{z}} \tilde{\theta} \tilde{S}_{\tilde{z}\tilde{\theta}} \right)$$

- recover Beltrami differentials $\mu, \tilde{\mu}$ and worldsheet gravitino fields $\chi, \tilde{\chi}$

$$H_{\tilde{\theta}}\tilde{z} = \tilde{\theta}(\mu_{\tilde{z}} \tilde{z} + \theta \chi_{\tilde{z}} \theta) \quad \tilde{H}_{\theta}\tilde{\tilde{z}} = \theta(\tilde{\mu}_{\tilde{z}} \tilde{\tilde{z}} + \tilde{\theta}\tilde{\chi}_{\tilde{z}} \tilde{\theta})$$

- Finite deformations of the metric with $\tilde{\mu} = \tilde{\mu}$ and $\tilde{\chi} = \tilde{\chi}$

integrate to the standard 2-dim $\mathcal{N} = 1$ supergravity action

(Brink, Di Vecchia, Howe; Deser, Zumino 1976)

- Type II superstring perturbation theory requires $\tilde{\mu} \neq \tilde{\mu}$ and $\tilde{\chi} \neq \tilde{\chi}$
Type II string amplitude

- Parametrize deformations \( \tilde{H}_\theta \tilde{z}, H_\theta \tilde{z} \) by slice \( \{ \tilde{J}(\tilde{m}), J(m) \} \) in \( \mathcal{M}_L \times \mathcal{M}_R \)

\[
\begin{align*}
H_\theta \tilde{z} &= \tilde{D}_\theta V \tilde{z} + H_A \delta m^A \\
\tilde{H}_\theta \tilde{z} &= D_\theta \tilde{V} \tilde{z} + \tilde{H}_A \delta \tilde{m} \tilde{A}
\end{align*}
\]

\( m^A = (m^i, \zeta^\alpha) \)

- Super conformal invariant ghost action

\[
I_{gh} = \int_{\Sigma} [d\tilde{z}d\tilde{z}d\theta d\bar{\theta}]
\left( B_{z\theta} \tilde{D}_{\tilde{\theta}} C \tilde{z} + \tilde{B}_{\tilde{z}\tilde{\theta}} D_{\theta} C \tilde{z} + B_{z\theta} H_A \delta m^A + \tilde{B}_{\tilde{z}\tilde{\theta}} \tilde{H}_A \delta \tilde{m} \tilde{A} \right)
\]

- The integrand for the full amplitude is given by

\[
\int D(X B \tilde{B} C \tilde{C}) \mathcal{V}_1 \cdots \mathcal{V}_n \prod_{\tilde{A}, A} [d\tilde{m}^{\tilde{A}} dm^A] \delta(\langle \tilde{B}, \tilde{H}_A \rangle) \delta(\langle B, H_A \rangle) e^{-I_m - I_{gh}}
\]

- \( \mathcal{V}_1 \cdots \mathcal{V}_n \) are BRST-invariant vertex operators.
- Picture Changing Operator formalism \( \text{(Friedan, Martinec, Shenker 1986)} \)
  * may be obtained as singular limit for \( \chi \) supported at points
  * globally regular reformulation via “vertical integration” \( \text{(Sen, Witten 2016)} \)
Loop momenta and Chiral amplitudes

- $h$ independent loop momenta $p^\mu_I$ defined to flow across $\mathcal{A}_I$ cycles

$$p^\mu_I = \oint_{\mathcal{A}_I} dz \partial_z x^\mu$$

- Chiral Amplitudes (ED, Phong 1988)
  - Massless NS bosons with factorized polarization tensor $\tilde{\varepsilon}^\mu_{i} = \varepsilon^\mu_i \tilde{\varepsilon}_i$
  - Chiral amplitude at fixed loop momenta is given by

$$\mathcal{F}_R(\mathcal{J}, \varepsilon_i, k_i, p_I) = \left\langle \mathcal{V}_1 \cdots \mathcal{V}_N e^{\frac{p^\mu_I}{\delta} \oint_{\mathcal{B}_I} dz \partial_z x^\mu} e^{\int \Sigma H_{\tilde{z}}^z S_{z\tilde{z}}} \prod_A \delta(\langle B, H_A \rangle) \, dm^A \right\rangle$$

- Correlation functions $\langle \cdots \rangle$ computed with chiral Green functions

- Full Superstring Amplitudes
  - obtained by pairing left and right and integrating over $\Gamma \in \mathcal{M}_L \times \mathcal{M}_R$

$$\mathcal{A}^{(h)}(\varepsilon_i, \tilde{\varepsilon}_i, k_i) = \int_{\mathbb{R}^{10}} dp^\mu_I \int_{\Gamma} \mathcal{F}_L(\tilde{\mathcal{J}}, \tilde{\varepsilon}_i, k_i, p^\mu_I) \mathcal{F}_R(\mathcal{J}, \varepsilon_i, k_i, p^\mu_I)$$

- integration over vertex operator insertion points included in integration over $\Gamma$
- cfr “double copy construction” in supergravity calculations
Parametrization of super moduli

- **Superconformal structure** $\mathcal{J} \in \mathcal{M}_h$ specified by transition functions
  - Concrete calculations use parametrization by gravitino field $\chi \tilde{z}^\theta$

- **Local parametrization of moduli** (in conformal-invariant theory)
  - Conformal structure $J$ with metric $g = |dz|^2$ in local coordinates $(z, \tilde{z})$
  - deform conformal structure by Beltrami differential to $g' = |dz + \mu d\tilde{z}|^2$
  - realized in CFT by inserting $\int_{\Sigma} d\tilde{z}dz \mu \tilde{z}^z T_{zz}$ to all orders in $\mu$

- **Local parametrization of supermoduli** (in superconformal-invariant theory)
  - Start with $\Sigma_{\text{red}}$ with complex structure given by $J \in \mathcal{M}_{\text{red}}$
  - Deform super conformal structure by inserting $T$ and $S$

\[ \int_{\Sigma_{\text{red}}} d\tilde{z}dz \left( \mu \tilde{z}^z T_{zz} + \chi \tilde{z}^\theta S_{z\theta} \right) \]

  - $\chi$ and $\mu$ parametrized by local odd coordinates on $\mathcal{M}_h$

- **For $h = 2$, even spin structures, holó projection $\mathcal{M}_2 \rightarrow \mathcal{M}_2$ exists**
  - via the super period matrix (ED, Phong 2001)

- **For $h \geq 5$ no holó projection $\mathcal{M}_h \rightarrow \mathcal{M}_h$ exists** (Donagi, Witten 2013)
The super period matrix \((\text{even spin structures})\)

- Start from conformal structure \(J\) for \(\Sigma_{\text{red}}\) with holó 1-forms \(\omega_I\)
  \[
  \oint_{\partial I} \omega_J = \delta_{IJ} \quad \oint_{\Sigma_B} \omega_J = \Omega_{IJ} \quad I, J = 1, 2
  \]

- Deform to superconformal structure \(J\) on \(\Sigma\) with superholó forms \(\hat{\omega}_I\)
  \[
  \oint_{\partial I} \hat{\omega}_J = \delta_{IJ} \quad \oint_{\Sigma_B} \hat{\omega}_J = \hat{\Omega}_{IJ} \quad I, J = 1, 2
  \]

- Explicit formula for the super period matrix \(\hat{\Omega}\) for even spin structure \(\delta\)
  \[
  \hat{\Omega}_{IJ} = \Omega_{IJ} - \frac{i}{8\pi} \int_{\Sigma_{\text{red}}} \omega_I(z) \chi(z) S_{\delta}(z, w|\Omega) \chi(w) \omega_J(w) + \int_{\Sigma_{\text{red}}} \mu \omega_I \omega_J
  \]

- \(\hat{\Omega}_{IJ}\) is locally supersymmetric; \(\hat{\Omega}_{IJ} = \hat{\Omega}_{JI}\); and \(\text{Im} \hat{\Omega} > 0\)
- Every \(\hat{\Omega}\) corresponds to an ordinary Riemann surface
- Szegö kernel \(S_{\delta}(z, w|\Omega)\) is non-singular in the interior of \(\mathcal{M}_2\)

\(\Rightarrow\) Projection using \(\hat{\Omega}\) is holomorphic and natural for genus 2
Projecting and pairing Chiral Amplitudes

**Chiral Amplitudes on $\mathcal{M}_2$**
- Natural parametrization of $\mathcal{M}_2$ by $(\hat{\Omega}_{IJ}, \zeta^\alpha)$ (even spin structure $\delta$)
- involves measure $d\kappa[\delta](\hat{\Omega}, \zeta)$ and correlation functions $C[\delta](\varepsilon_i, k_i, p_I|\hat{\Omega}, \zeta)$

**Projection to chiral amplitudes on $\mathcal{M}_2$**
- by integrating over $\zeta$ and summing over $\delta$ at fixed $\hat{\Omega}$
\[
\mathcal{R}(\varepsilon_i, k_i, p_I|\hat{\Omega}) = \sum_\delta \int_\zeta d\kappa[\delta](\hat{\Omega}, \zeta) C[\delta](\varepsilon_i, k_i, p_I|\hat{\Omega}, \zeta)
\]
\[
\mathcal{L}(\bar{\varepsilon}_i, k_i, p_I|\hat{\Omega}) = \sum_\tilde{\delta} \int_{\tilde{\zeta}} d\kappa[\tilde{\delta}](\hat{\Omega}, \tilde{\zeta}) C[\tilde{\delta}](\bar{\varepsilon}_i, k_i, p_I|\hat{\Omega}, \tilde{\zeta})
\]
- for heterotic, $\mathcal{L}$ is chiral half of bosonic string, has no integral in $\tilde{\zeta}$
- phase factors determined by $Sp(4, \mathbb{Z})$ modular invariance

**Pairing left and right chiral amplitudes, integrating over $p_I$ and $\hat{\Omega}$**
\[
A^{(2)}(\varepsilon_i, \bar{\varepsilon}_i, k_i) = \int_{\mathcal{M}_2} d\hat{\Omega} \int dp_I^\mu \mathcal{R}(\varepsilon_i, k_i, p_I|\hat{\Omega}) \mathcal{L}(\bar{\varepsilon}_i, k_i, p_I|\hat{\Omega})
\]
- Integral over $p_I$ is Gaussian and can be carried out explicitly.
Genus two

- **Siegel Upper half space** $S_2$

  $$S_2 = \{ \Omega_{IJ} = \Omega_{JI} \in \mathbb{C} \text{ with } I, J = 1, 2 \text{ and } Y = \text{Im} \Omega > 0 \}$$

  - $Sp(4, \mathbb{R})$ acts by $\Omega \to (A\Omega + B)(C\Omega + D)^{-1}$
    $$M^t JM = J \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

  - $S_2$ has $Sp(4, \mathbb{R})$-invariant metric $ds_2^2$ and volume form $d\mu_2$
    $$ds_2^2 = \sum_{I, J, K, L = 1, 2} Y_{IJ}^{-1} d\bar{\Omega}_{JK} Y_{KL}^{-1} d\Omega_{LI}$$

- **Compact Riemann surfaces** $\Sigma$

  - Choose canonical homology basis of $\mathcal{A}_I, \mathcal{B}_I$ cycles for $H_1(\Sigma, \mathbb{Z})$.

  - $\omega_I$ dual holomorphic (1,0) forms,
    $$\oint_{\mathcal{A}_I} \omega_J = \delta_{IJ} \quad \oint_{\mathcal{B}_I} \omega_J = \Omega_{IJ}$$

  - Riemann relations imply $\Omega \in S_2$;

  - Modular group $Sp(4, \mathbb{Z})$; moduli space $M_2 = S_2/Sp(4, \mathbb{Z})$. 

Genus-two Type II four-graviton amplitude

**Type II four-graviton amplitude** (ED, Phong 2001 – 2005)

\[
A^{(2)}(\varepsilon_i, \tilde{\varepsilon}_i, k_i) = g_s^2 \mathcal{K}\bar{\mathcal{K}} \int_{M_2} d\mu_2 \mathcal{B}^{(2)}(s_{ij}|\Omega)
\]

\[
\mathcal{B}^{(2)}(s_{ij}|\Omega) = \int_{\Sigma^4} \frac{\mathcal{Y} \wedge \bar{\mathcal{Y}}}{(\det \text{Im} \Omega)^2} \exp \left( \sum_{i<j} s_{ij} G(z_i, z_j|\Omega) \right)
\]

- \(G(z_i, z_j)\) is the genus-two scalar Green function;
- \(\Delta(z_i, z_j)\) is a bi-holomorphic form independent of \(s, t, u\).

\[
\Delta(z, w) = \omega_1(z) \wedge \omega_2(w) - \omega_2(z) \wedge \omega_1(w)
\]

\[
\mathcal{Y} = (t - u)\Delta(z_1, z_2) \wedge \Delta(z_3, z_4) + (s - t)\Delta(z_1, z_3) \wedge \Delta(z_4, z_2) + (u - s)\Delta(z_1, z_4) \wedge \Delta(z_2, z_3)
\]

- reproduced (with fermions) in pure spinor formulation (Berkovits, Mafra 2005)

**Singularity structure**

- For fixed \(\Omega\) integrations over \(\Sigma\) produce poles in \(\mathcal{B}\) at positive integers \(s_{ij}\).
- The integral over \(\Omega\) requires analytic continuation beyond \(\text{Re}(s_{ij}) = 0\).
- Branch cuts in \(s_{ij}\) starting at integers produced from \(\Omega_{11}, \Omega_{22} \rightarrow i\infty\).
Genus-two Heterotic four-graviton amplitude

• Heterotic four NS boson amplitude at genus 2 (ED, Phong 2005)

\[
A^{(2)}_{\mathcal{O}}(\varepsilon_i, \tilde{\varepsilon}_i, k_i) = g_s^2 \mathcal{K} \int_{\mathcal{M}_2} d\mu_2 \mathcal{B}^{(2)}_{\mathcal{O}}(\tilde{\varepsilon}_i, k_i|\Omega)
\]

\[
B^{(2)}_{\mathcal{O}}(\tilde{\varepsilon}_i, k_i|\Omega) = \int_{\Sigma^4} \frac{\mathcal{V} \wedge \mathcal{W}_{\mathcal{O}}(\tilde{\varepsilon}_i, k_i)}{(\det \text{Im}\Omega)^2 \Psi_{10}(\Omega)} \exp \left( \sum_{i<j} s_{ij} G(z_i, z_j) \right)
\]

- \(\Psi_{10}(\Omega)\) is the Igusa cusp form.

• Dependence of the operator \(\mathcal{O}\) on the channel:

  - 4 gravitons \(\mathcal{R}^4\)
  - 2 gravitons + 2 gauge bosons \(\mathcal{R}^2\text{tr}(\mathcal{F}^2)\);
  - 4 gauge bosons \((\text{tr}\mathcal{F}^2)^2\)
  - 4 gauge bosons \(\text{tr}(\mathcal{F}^4)\)

- For example,

\[
\mathcal{W}_{\mathcal{R}^4}(\tilde{\varepsilon}_i, k_i) = \frac{\langle \prod_{i=1}^4 \tilde{\varepsilon}_i \cdot \bar{\partial} \tilde{x}(z_i) e^{ik_i \cdot \tilde{x}(z_i)} \rangle}{\langle \prod_{i=1}^4 e^{ik_i \cdot \tilde{x}(z_i)} \rangle}
\]

- Gauge parts are obtained by the correlators of the current \((0, 1)\)-forms.
UV-finiteness and one-loop amplitudes

• Thanks to modular invariance, all string amplitudes are UV-finite
  – shown for the closed bosonic string at genus one (Shapiro 1972)
  – holds for all modular invariant superstrings to all loops (i.e. all genera)

• All chiral amplitudes have a universal loop momentum factor

\[ \mathcal{F}_R(z_i, \varepsilon_i, k_i, p_I | \Omega) = e^{i\pi p_I^\mu \Omega_{IJ} p_J^\mu} \times \ldots \]

  – Modular invariance allows one to choose a fundamental domain where \( \text{Im}(\Omega) \) bounded from below
  – For genus one, choose the standard fundamental domain

\[ \mathcal{H}_1/\text{SL}(2, \mathbb{Z}) = \{ \tau \in \mathbb{C}, \text{Im}(\tau) > 0, |\tau| \geq 1, |\text{Re}(\tau)| \leq \frac{1}{2} \} \]

  – Analogous, more complicated, choices to higher genus

⇒ Uniform Gaussian suppression at large loop momenta

⇒ UV finiteness to all genera
Singularities in the projection $\overline{M}_2 \to \overline{M}_2$

• Projection $\overline{M}_2 \to \overline{M}_2$ is holó, but integration extends to boundary
  – are there singularities in the projection $\overline{M}_2 \to \overline{M}_2$?
    \[
    \Omega = \begin{pmatrix} \tau & u \\ u & \sigma \end{pmatrix}, \quad u \to 0 \quad \text{separating node} \\
    \sigma \to i\infty \quad \text{non-separating node}
    \]
  – Key ingredient in $\hat{\Omega}$ is the Szegö kernel
    \[
    S_\delta(z, w|\Omega) = \frac{\vartheta(\delta)(z - w|\Omega)}{\vartheta(\delta)(0|\Omega) E(z, w)}
    \]
  – As $u \to 0$ we have $\vartheta(\delta)(0|\Omega) \to \vartheta(\delta_1)(0|\tau) \vartheta(\delta_2)(0|\tau)$
  – Even $\delta = [\delta_1, \delta_2]$ with $\delta_1, \delta_2$ odd produces a singularity in $S_\delta$ and $\hat{\Omega}$

• Physical effects
  – singularity killed by $\psi$-zero modes in $\mathbb{R}^{10}$ (space-time susy)
  – contribution when susy is broken by radiative corrections (Witten 2013)
  – Two-loop vacuum energy in Heterotic strings on CY orbifold $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$
    $\star$ is zero for $E_8 \times E_8 \to E_6 \times E_8$ with unbroken susy
    $\star$ non-zero for $\text{Spin}(32)/\mathbb{Z}_2 \to SO(26) \times U(1)$ with broken susy
      (Atick, Sen 1988; · · ·; ED, Phong 2013; Berkovits, Witten 2014)
Singularities in the projection $\mathcal{M}_3 \to \mathcal{M}_3$

- **Some basic structure theorems**
  - A hyper-elliptic surface is a branched double cover of the sphere $S^2$;
  - All genus 1 and all genus 2 surfaces are hyper-elliptic;
  - Hyper-elliptic surfaces form a co-dim 1 sub-variety in the interior of $\mathcal{M}_3$
    (referred to as the hyper-elliptic divisor)

- **The genus-three period matrix (for even spin structure)**

  $$\hat{\Omega}_{IJ} = \Omega_{IJ} - \frac{i}{8\pi} \int \int \omega_I(z) \chi(z) S_\delta(z, w|\Omega) \chi(w) \omega_J(w) + \mathcal{O}(\chi^4)$$

  - For $\Omega$ on the hyper-elliptic divisor of $\mathcal{M}_3$
    there always exists an even spin structure $\delta$ such that $\vartheta[\delta](0|\Omega) = 0$
  - the presence of the extra Dirac zero modes kills effects of this singularity

$\Rightarrow$ Beautiful proposal for the genus 3 superstring measure
  (Cacciatori, Dalla Piazza, van Geemen 2008)

- Another even $\delta$ does produce a *subtle singularity* in $\hat{\Omega}$ (Witten 2015)
Lectures on Superstring Amplitudes

Part 3: Low energy effective interactions

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Amplitudes 2018 Summer School
Superstring Perturbation Theory and Supergravity

- Superstring perturbation theory in powers of the string coupling \( g_s \)
  - holds for weak coupling \( g_s \)
  - and for all energies
- Classical supergravity “\( \mathcal{R} \)”
  - leading low energy expansion of string theory
  - holds for all couplings \( g_s \)
- String induced effective interactions \( \mathcal{R}^4, D^4R^4, D^6R^4 \)
  - Evaluated in perturbation theory for \( g_s \ll 1 \)
Low energy expansion of tree-level amplitudes

- **Closed superstring tree-level four-graviton amplitude**

\[
\mathcal{A}^{(0)}(\varepsilon_i, \tilde{\varepsilon}_i, k_i) = \frac{1}{g_s^2} \frac{\mathcal{R}^4}{stu} \frac{\Gamma(1 - s)\Gamma(1 - t)\Gamma(1 - u)}{\Gamma(1 + s)\Gamma(1 + t)\Gamma(1 + u)}
\]

\[
s_{ij} = -\frac{\alpha'}{4}(k_i + k_j)^2
\]

- \(\mathcal{R}\) symbolically stands for the Weyl tensor
- \(\mathcal{R}^4\) symbolically stands for a scalar contraction dictated by supersymmetry

- **At low energy** \(|s_{ij}| \ll 1\)

  - massless string exchanges produce non-local contributions;
  - massive string exchanges produce local effective interactions
  - string-induced corrections to supergravity; eg. in Type II

\[
\frac{1}{stu} + 2\zeta(3) + \zeta(5)(s^2 + t^2 + u^2) + 2\zeta(3)^2stu + \frac{1}{2}\zeta(7)(s^2 + t^2 + u^2)^2 + \cdots
\]

massless \(\mathcal{R}^4\) \(D^4\mathcal{R}^4\) \(D^6\mathcal{R}^4\) \(D^8\mathcal{R}^4\)

- \(D^{2k}\mathcal{R}^4\) contraction of covariant derivatives \(D\) and \(\mathcal{R}^4\)
Effective interactions from Type IIB superstrings

- *SL(2, Z*)-duality symmetry of Type IIB superstrings
  - requires effective interactions to be *SL(2, Z*)-invariant;
  - Einstein frame metric *G_E* and *R^4_E* invariant under *SL(2, Z*)
  - combine axion *χ* dilaton *Φ* in \( \rho = \chi + ie^{-\Phi} \)
  - transforms by Möbius transformations under *SL(2, Z*)
    \[
    \rho \rightarrow \frac{a\rho + b}{c\rho + d}, \quad a, b, c, d, \in \mathbb{Z}, \quad ad - bc = 1
    \]
  - Flux fields *F_3^R*, *F_3^{NS*} transform linearly; *F_5* is invariant

- Effective interactions from four-graviton amplitude in Type IIB
  \[
  \int \sqrt{G_E} \left( \mathcal{E}_0(\rho)R^4_E + \mathcal{E}_4(\rho)D^4_E R^4_E + \mathcal{E}_6(\rho)D^6_E R^4_E + \mathcal{E}_8(\rho)D^8_E R^4_E + \cdots \right)
  \]
  - For each *p* the real-valued function *\mathcal{E}_p(\rho)* is *SL(2, Z*)-invariant
    \[
    \mathcal{E}_p \left( \frac{a\rho + b}{c\rho + d} \right) = \mathcal{E}_p(\rho)
    \]
  - namely it is a real-analytic modular function
    (not to be confused with meromorphic modular functions)
Real-analytic Eisenstein series

• A famous family of real-analytic modular functions

  – For $\text{Re}(s) > 1$ one defines $E_s$ by Kronecker-Eisenstein sums

    \[ E_s(\rho) = \sum_{m,n \in \mathbb{Z}} \frac{\rho_2^s}{\pi^s|m + \rho n|^{2s}} \quad \rho = \rho_1 + i \rho_2, \rho_1, \rho_2 \in \mathbb{R} \]

  – They are $SL(2,\mathbb{Z})$-invariant and eigenfunctions of the Laplacian

    \[ \Delta E_s(\rho) = s(1 - s)E_s \quad \Delta = 4 \rho_2^2 \partial_\rho \partial_{\bar{\rho}} \]

  – Their asymptotic expansion for $\rho_2 \to \infty = \text{weak string coupling}$

    \[ E_s(\rho) = 2\zeta(2s)\frac{\rho_2^s}{\pi^s} + \frac{2\Gamma(s - \frac{1}{2})\zeta(2s - 1)}{\Gamma(s)\pi^{s-\frac{1}{2}}\rho_2^{s-1}} + \mathcal{O}(e^{-2\pi \rho_2}) \]
Effective interactions and Eisenstein series

- String perturbation theory calculations in string frame
  - Convert Einstein metric $G_{E \mu \nu}$ to string metric $G_{\mu \nu} = e^{\Phi/2} G_{E \mu \nu}$
  \[
  \sqrt{G_E} \mathcal{E}_{2k}(\rho) D_E^{2k} R_E^4 = e^{(k-1)\Phi/2} \sqrt{G_E} \mathcal{E}_{2k}(\rho) D_E^{2k} R^4
  \]

  - Consider combinations involving Eisenstein series
    \[
    \sqrt{G_E} E_{3/2}(\rho) R_E^4 \approx e^{-2\Phi} \zeta(3) R^4 + \frac{\pi^2}{3} R^4
    \]
    \[
    \sqrt{G_E} E_5(\rho) D_E^4 R_E^4 \approx e^{-2\Phi} \zeta(5) D^4 R^4 + \frac{2\pi^4}{135} e^{2\Phi} D^4 R^4
    \]
    \[
    \sqrt{G_E} E_{3/2}(\rho)^2 D_E^6 R_E^4 \approx e^{-2\Phi} \zeta(3)^2 D^6 R^4 + \frac{2\pi^2}{3} \zeta(3) D^6 R^4 + \frac{\pi^4}{9} e^{-2\Phi} D^6 R^4
    \]
    \[
    \sqrt{G_E} E_{7/2}(\rho) D_E^8 R_E^4 \approx e^{-2\Phi} \zeta(7) D^8 R^4 + \frac{16\pi^6}{14175} e^{-4\Phi} D^8 R^4
    \]

  - Compare with low energy expansion of tree-level
    \[
    \frac{1}{stu} + 2\zeta(3) + \zeta(5)(s^2 + t^2 + u^2) + 2\zeta(3)^2 stu - \frac{1}{2} \zeta(7)(s^2 + t^2 + u^2)^2 + \cdots
    \]
    \[
    R^4 \quad D^4 R^4 \quad D^6 R^4 \quad D^8 R^4
    \]
D-instantons, S-duality and supersymmetry

• **Space-time supersymmetry and S-duality**
  - D-instantons (Green, Gutperle, Vanhove 1996), space-time susy (Green, Sethi 1997)
    \[ \mathcal{E}_0(\rho) = E_{3/2}^3(\rho) \]
    - matches tree-level and genus-one results from string perturbation theory
    - Vanishing contribution from genus-two (ED, Gutperle, Phong 2005)

• **M-theory perturbation theory on torus** (Green, Kwon, Vanhove 1999; GV 2005)
    \[ \mathcal{E}_4(\rho) = E_5^5(\rho) \]
    \[ (\Delta - 12)\mathcal{E}_6(\rho) = E_{3/2}^3(\rho)^2 \]
    - \( \mathcal{E}_4 \) matches genus two (ED, Gutperle, Phong 2005)
    - \( \mathcal{E}_6 \) matches genus-two (ED, Green, Pioline, R. Russo 2014)
      genus three (Gomez, Mafra 2015)

• **Non-renormalization theorems**: no perturbative corrections
  - for \( \mathcal{E}_0 \) for \( h \geq 2 \)
  - for \( \mathcal{E}_4 \) for \( h \geq 3 \)
  - for \( \mathcal{E}_6 \) for \( h \geq 4 \)
Low energy expansion at genus one

• Recall genus-one Type II four-graviton amplitude \((\mathcal{M}_1 = \mathcal{H}_1/SL(2, \mathbb{Z}))\)

\[
\mathcal{A}^{(1)}(\varepsilon_i, \tilde{\varepsilon}_i, k_i) = \mathcal{R}^4 \int_{\mathcal{M}_1} \frac{d^2 \tau}{(\text{Im} \tau)^2} \mathcal{B}^{(1)}(s_{ij}|\tau)
\]

• Expand the partial amplitude \(\mathcal{B}^{(1)}\) for \(|s_{ij}| \ll 1\) for fixed \(\tau\)

\[
\mathcal{B}^{(1)}(s_{ij}|\tau) = \int_{\Sigma^4} \prod_{i=1}^{4} \frac{d^2 z_i}{\text{Im} \tau} \exp \left( \sum_{i<j} s_{ij} G(z_i - z_j|\tau) \right)
\]

– Scalar Green function \(G(z|\tau)\) given by “Kronecker-Eisenstein” Fourier sum

\[
G(z|\tau) = \sum_{m,n \in \mathbb{Z}} \frac{\tau_2}{\pi} \frac{e^{2\pi i (m \beta - n \alpha)}}{|m + \tau n|^2}
\]

\(z = \alpha + \tau \beta, \alpha, \beta \in \mathbb{R}\)

– For fixed \(\tau\) the Taylor expansion of \(\mathcal{B}^{(1)}\) in \(s_{ij}\) converges for \(|s_{ij}| < 1\)

• Graphical expansion of \(\mathcal{B}^{(1)}(s_{ij}|\tau) \implies\) Modular Graph Functions of \(\tau\)
Modular graph functions

- Graph in the expansion of $D^{2w} R^4 \implies$ Modular Function

\[
\begin{align*}
D^4 R^4 & \quad \bullet \quad \bullet \\
D^6 R^4 & \quad \bullet \quad \bullet \\
D^8 R^4 & \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\
D^{10} R^4 & \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\
\end{align*}
\]
Modular graph functions

$D^4 \mathcal{R}^4$

$D^6 \mathcal{R}^4$

$D^8 \mathcal{R}^4$

$D^{10} \mathcal{R}^4$

one-loop  two-loops  three-loops
One-loop : Eisenstein series

- One-loop worldsheet Feynman diagram with $k$ bivalent vertices

$$
\prod_{i=1}^{k} \int \frac{d^2 z_i}{\tau_2} G(z_i - z_{i+1} | \tau) = \sum' \frac{\tau_2^k}{\pi^k |m + n\tau|^2 s} = E_k(\tau)
$$

- Our old friend: non-holomorphic Eisenstein series for integer index $k$

- Recall properties of $E_s(\tau)$

  - absolutely convergent for $\text{Re}(s) > 1$; analytically continue to $s \in \mathbb{C}$
  - reflection relation $\Gamma(s) E_s(\tau) = \Gamma(1 - s) E_{1-s}(\tau)$
  - satisfies a Laplace-eigenvalue equation on $\mathcal{H}_1$

$$
\left( \Delta - s(s - 1) \right) E_s(\tau) = 0 \quad \Delta = 4\tau_2^2 \partial_\tau \partial_{\bar{\tau}}
$$

- modular invariant $E_s\left( \frac{a\tau + b}{c\tau + d} \right) = E_s(\tau)$ under $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$
Two-loops : modular graph functions

- Feynman diagrams evaluate to the modular functions

\[ C_{a_1,a_2,a_3}(\tau) = \sum_{m_r,n_r \in \mathbb{Z}, \atop r=1,2,3} \delta \left( \sum_{r=1}^{3} m_r \right) \left( \sum_{r=1}^{3} n_r \right) \prod_{r=1}^{3} \left( \frac{\tau_2}{\pi |m_r + n_r \tau|^2} \right)^{a_r} \]

- contribute to \(D^2w R^4\) with the weight given by \(w = a_1 + a_2 + a_3\)
- satisfy (inhomogeneous) Laplace-eigenvalue equations

\[ w = 3 \quad C_{1,1,1} = \quad (\Delta - 0)C_{1,1,1} = 6E_3 \]

\[ w = 4 \quad C_{2,1,1} = \quad (\Delta - 2)C_{2,1,1} = 9E_4 - E_2^2 \]

\[ w = 5 \quad C_{3,1,1} = \quad (\Delta - 6)C_{3,1,1} = 3C_{2,2,1} + 16E_5 - 4E_2E_3 \]

\[ w = 5 \quad C_{2,2,1} = \quad (\Delta - 0)C_{2,2,1} = 8E_5 \]

- Note that eigenvalues are of the form \(s(s - 1)\) for \(s = 1, 2, 3;\)
Structure Theorem

- $C_{a,b,c}(\tau)$ are linear combinations of $C_{w;s;p}(\tau)$ satisfying (ED, Green, Vanhove 2015)

$$(\Delta - s(s - 1))C_{w;s;p} = \mathcal{F}_{w;s;p}(E_{s'})$$

- $C_{w;s;p}$ and $\mathcal{F}_{w;s;p}$ of weight $w = a + b + c$ (with $E_{s'}$ assigned weight $s'$);
- $\mathcal{F}_{w;s;p}$ is a polynomial of total degree 2 in $E_{s'}$ with $2 \leq s' \leq w$;

$s = w - 2m$ \quad $m = 1, \ldots, \left[\frac{w - 1}{2}\right]$ \quad $p = 0, \ldots, \left[\frac{s - 1}{3}\right]$

• Examples at low weight

$w = 3 \quad s = 1 \quad 0^{(1)}$
$w = 4 \quad s = 2 \quad 2^{(1)}$
$w = 5 \quad s = 1, 3 \quad 0^{(1)} \oplus 6^{(1)}$
$w = 6 \quad s = 2, 4 \quad 2^{(1)} \oplus 12^{(2)}$
$w = 7 \quad s = 1, 3, 5 \quad 0^{(1)} \oplus 6^{(1)} \oplus 20^{(2)}$

• System of differential relations to all loop orders (ED, Green, Kaidi, Vanhove 2016)

• Relation with polylogarithms & multiple zeta values

(ED, Green, Vanhove 2015; Francis Brown 2017)
Type IIB effective interactions at genus-two

• Recall Type II four-graviton amplitude at genus 2,

\[ A^{(2)}(\varepsilon_i, k_i) = \mathcal{R}^4 \int_{\mathcal{M}_2} \mathrm{d}\mu_2 \mathcal{B}^{(2)}(s_{ij}|\Omega) \]

\[ \mathcal{B}^{(2)}(s_{ij}|\Omega) = \int_{\Sigma^4} \mathcal{Y} \wedge \bar{\mathcal{Y}} \exp \sum_{i<j} s_{ij} G(z_i, z_j) \]

– \( \mathcal{Y} = (s - t) \Delta(z_1, z_3) \wedge \Delta(z_4, z_2) + 2 \) permutations;
– \( \Delta(z_i, z_j) \) is a holomorphic form independent of \( s, t, u \).

• Contributions to local effective interactions,
– \( \mathcal{R}^4 \) : zero, since \( \mathcal{Y} \) vanishes for \( s = t = u = 0 \);
– \( D^4 \mathcal{R}^4 \) : non-zero, \( \mathcal{B}^{(2)} \) constant on \( \mathcal{M}_2 \);
– \( D^6 \mathcal{R}^4 \) : non-zero, one power of \( G \) brought down in integral over \( \Sigma^4 \);

\[ \mathcal{B}^{(2)}(s_{ij}|\Omega) = 32(s^2 + t^2 + u^2) + 192 stu \varphi(\Omega) + \mathcal{O}(s^4, \cdots) \]

– \( \varphi(\Omega) \) coincides with the Kawazumi-Zhang invariant.
The Zhang-Kawazumi invariant for genus-two

• The ZK-invariant is given as follows

\[ 8\varphi(\Omega) = \sum_{I,J,K,L} \left( Y_{IJ}^{-1} Y_{KL}^{-1} - 2Y_{IL}^{-1} Y_{JK}^{-1} \right) \int_{\Sigma^2} G(x, y) \omega_I(x) \omega_J(x) \omega_K(y) \omega_L(y) \]

– equivalent to definition via Arakelov geometry (Zhang 2007, Kawazumi 2008)

• Coefficient of genus-two $D^6R^4$ interaction involves \( \int_{\mathcal{M}_2} d\mu_2 \varphi(\Omega) \)

– Direct evaluation appeared completely out of reach ... until ...
The Zhang-Kawazumi invariant for genus-two

- The ZK-invariant is given as follows

$$8\varphi(\Omega) = \sum_{I,J,K,L} \left( Y_{IJ}^{-1} Y_{KL}^{-1} - 2Y_{IL}^{-1} Y_{JK}^{-1} \right) \int_{\Sigma^2} G(x, y) \omega_I(x) \omega_J(x) \omega_K(y) \omega_L(y)$$

  - equivalent to definition via Arakelov geometry (Zhang 2007, Kawazumi 2008)

- Coefficient of genus-two $D^6R^4$ interaction involves $\int_{\mathcal{M}_2} d\mu_2 \varphi(\Omega)$

  - Direct evaluation appeared completely out of reach ... until ...

- Theorem (ED, Green, Pioline, R. Russo 2014)

  $$(\Delta - 5)\varphi = -2\pi \delta_{SN}$$

  - $\Delta$ is the Laplace-Beltrami operator on $\mathcal{M}_2$ with Siegel metric $ds_2^2$;
  - $\delta_{SN}$ has support on separating node (into two genus-one surfaces)
  - The integral over $\mathcal{M}_2$ reduces to an integral over $\partial \mathcal{M}_2$

$$\int_{\mathcal{M}_2} d\mu_2 \varphi = \frac{1}{5} \int_{\mathcal{M}_2} d\mu_2 \left( \Delta \varphi + 2\pi \delta_{SN} \right) = \frac{2\pi^3}{45}$$

  - Exact agreement with predictions from S-duality and supersymmetry
Non-analytic contributions at low energy

- Non-analytic parts of the amplitudes at low energy
  - arise from boundary of moduli space contribution to the integral over $B$
  - dominant contribution at low energy is from supergravity
  - plus string corrections

- Look at two-particle unitarity cut in the $s$-channel

\[
i \text{Disc}_s A_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(p_1, p_2, p_3, p_4) = \int \frac{d^{10}k}{(2\pi)^{10}} \delta(k^2) \delta((q - k)^2) A_{\varepsilon_1,\varepsilon_2,\varepsilon_r,\varepsilon_s}(p_1, p_2, -k, k - q) A_{\varepsilon_r,\varepsilon_s,\varepsilon_3,\varepsilon_4}(k, q - k, p_3, p_4)
\]
Non-analytic part of the genus-one amplitude

• Obtain the genus-one discontinuity from tree-level
  – Use the fact that the kinematic factor is the same at all genera \( h \)
    \[
    A_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}^{(h)}(p_1, p_2, p_3, p_4) = R_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}^4(p_1, p_2, p_3, p_4)A_{\text{red}}^{(h)}(s, t, u)
    \]
  – and satisfies the recursive formula (Bern, Dixon, Dunbar, Perelstein, Rozowsky 1998)
    \[
    \sum_{\epsilon_r,\epsilon_s} R_{\epsilon_1,\epsilon_2,\epsilon_r,\epsilon_s}^4(p_1, p_2, -k, k-q) R_{\epsilon_r,\epsilon_s,\epsilon_3,\epsilon_4}^4(k, q-k, p_3, p_4) = s^4 R_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}^4(p_1, p_2, p_3, p_4)
    \]

  – to obtain an effective discontinuity formula
    \[
    i \text{Disc}_s A_{\text{red}}^{(1)}(s, t, u) = \int \frac{d^{10}k}{(2\pi)^{10}} \delta(k^2) \delta((q-k)^2) A_{\text{red}}^{(0)}(s, t', u') A_{\text{red}}^{(0)}(s, t'', u'')
    \]
  – where \( t' = -(p_1 - k)^2, \ u' = -(p_2 - k)^4, \ t'' = -(p_4 - k)^2, \ u'' = -(p_3 - k)^2 \)
    \[
    A_{\text{red}}^{(0)}(s, t, u) = \frac{1}{stu} + 2\zeta(3) + \zeta(5)(s^2 + t^2 + u^2) + \cdots
    \]

• Substitution into \emph{s}-channel unitarity relation gives (by power-counting)
  \[
  \text{Disc}_s A_{\text{red}}^{(1)}(s, t, u) = \#s + \#\zeta(3)s^4 + \#\zeta(5)s^6 + \cdots
  \]
  \[
  A_{\text{red}}^{(1)}(s, t, u) = \#s \ln(-s) + \#\zeta(3)s^4 \ln(-s) + \#\zeta(5)s^6 \ln(-s) + \cdots
  \]
Absence of non-analytic contributions

- Discontinuity relation gives non-analytic contributions
  
  - At genus-one use previously obtained result
    \[ A_{\text{red}}^{(1)}(s, t, u) = #s \ln(-s) + #\zeta(3)s^4 \ln(-s) + #\zeta(5)s^6 \ln(-s) + \cdots \]
  
  - Effective interaction \( D^2 R^4 \) vanishes by \( s + t + u = 0 \)
  
  - Genus-one \( R^4, D^4 R^4, D^6 R^4, D^{10} R^4 \) effective interactions are completely determined by the analytic part of the amplitude
  
  - Local effective interaction \( D^8 R^4 \) can be fixed only after non-analytic part has been properly normalized
Non-analytic plus analytic parts from genus-one amplitude

- Derivation of full genus-one $D^8 R^4$ from string theory amplitude

  - non-analytic part arises from $\tau \to i\infty$: partition moduli space
    
    $$\mathcal{M}_1 = \mathcal{M}_{1L} \cup \mathcal{M}_{1R}$$
    
    $\mathcal{M}_{1L} = \{\tau \in \mathcal{M}_1, \text{Im}(\tau) < L\}$
    
    $\mathcal{M}_{1R} = \{\tau \in \mathcal{M}_1, \text{Im}(\tau) > L\}$

  - Full amplitude is a sum $A^{(1)} = A^{(1)}_L + A^{(1)}_R$
    
    $$A^{(1)}_{L,R}(\varepsilon_i, \tilde{\varepsilon}_i, k_i) = R^4 \int_{\mathcal{M}_{1L,R}} \frac{d^2 \tau}{(\text{Im} \tau)^2} B^{(1)}(s_{ij} | \tau)$$

  - Both $A^{(1)}_L, A^{(1)}_R$ depend on $L$, but sum is independent of $L$
  - $A^{(1)}_L$ is analytic in $s_{ij}$ but $A^{(1)}_R$ exhibits non-analyticity at $s_{ij} = 0$
Explicit calculation for $D^8 \mathcal{R}^4$

• Since $\mathcal{A}_L^{(1)}$ is analytic in $s_{ij}$, evaluate using modular graph functions

\[
\mathcal{A}_L^{(1)} = \frac{2\pi \zeta(3)}{45} \mathcal{R}^4 \left( \ln L - \frac{1}{4} + \ln 2 + \frac{\zeta'(4)}{\zeta(4)} - \frac{\zeta'(3)}{\zeta(3)} \right) + \text{power-behaved in } L
\]

• For $L \gg 1$, approximate integrand of $\mathcal{A}_R^{(1)}$ by supergravity + corrections

\[
\left. \mathcal{A}^{(1)} \right|_{D^8 \mathcal{R}^4} = \frac{4\pi \zeta(3)}{45} \left( \frac{17}{5} - \frac{1}{4} + \ln 2 + \frac{\zeta'(4)}{\zeta(4)} - \frac{\zeta'(3)}{\zeta(3)} \right) (s^4 + t^4 + u^4) \mathcal{R}^4
\]

\[
- \frac{4\zeta(3)}{45} \left( s^4 \ln(-2\pi s) + t^4 \ln(-2\pi t) + u^4 \ln(-2\pi u) \right) \mathcal{R}^4
\]

– Note: no ambiguities, no infinities, no renormalization required!
– Transcendentality ... (ED, Green, in progress)

• Genus-two story ...

(ED, Green, Pioline 2017, 2018, and in progress)
Outlook

• Some additional developments
  – Clarification of super Riemann surfaces with R-punctures (Witten 2012)
  – There exists a super-period matrix for R-punctures (Witten; ED, Phong 2015)
  – New relations between open and closed string amplitudes (Schlotterer et al.)

• Some outstanding issues
  – Systematic structure of low energy effective interactions w/ Green, Pioline
    ★ in terms of properties of modular graph functions
    ★ calculation without requiring subtleties of supermoduli space
    ★ UV divergences in supergravity and effective interactions

  – Ambi-twistor strings

  – string perturbation theory on curved spaces with RR flux, e.g. $AdS_5 \times S^5$