Lectures on Superstring Amplitudes

Part 1: Bosonic String

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Outline of lectures

- **Lecture 1**

  Bosonic strings and conformal field theory

- **Lecture 2**

  Superstring amplitudes

- **Lecture 3**

  Low energy effective interactions

Strings

- **A string is a 1-dimensional object**
  - open string = topology of an interval;
  - closed string = topology of a circle;
  - physical size Planck length $\ell_P \approx 10^{-33}\mathrm{cm} \approx 10^{-19}\times$ size of the proton.

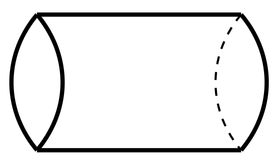
- **Ultimate goal: unified theory of particle physics and gravity**
  - elementary particles correspond to strings and their excited states;
  - consistently with quantum mechanics and general relativity;
  - remarkably unique structure.

- **Immediate goal: relating string amplitudes and field theory amplitudes**
  - at distance scales larger than the Planck length (low energy)
    a string effectively behaves as a point particle
  - string theory exhibits powerful structure of amplitudes

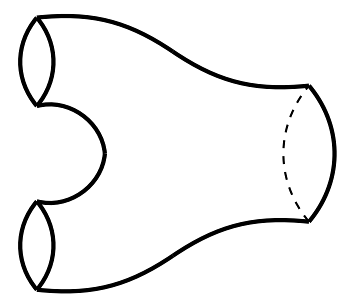
String Topology

- **Consistent interacting string theories**
  - only closed strings (Type IIA,B and heterotic)
  - closed and open strings (Type I)
  - Type II theories have open strings in the presence of D-branes
- **Strings live in a physical space-time M**
  - M may be a manifold or an orbifold (with mild isolated singularities)
  - superstring theory predicts 10-dim
  - but space-time visible to us is 4-dim. $\Rightarrow$ requires "compactification"
- **Under time-evolution strings sweep out a 2-dim. surface**

closed strings



time-evolution
(freely propagating)



basic interaction
(purely topological)

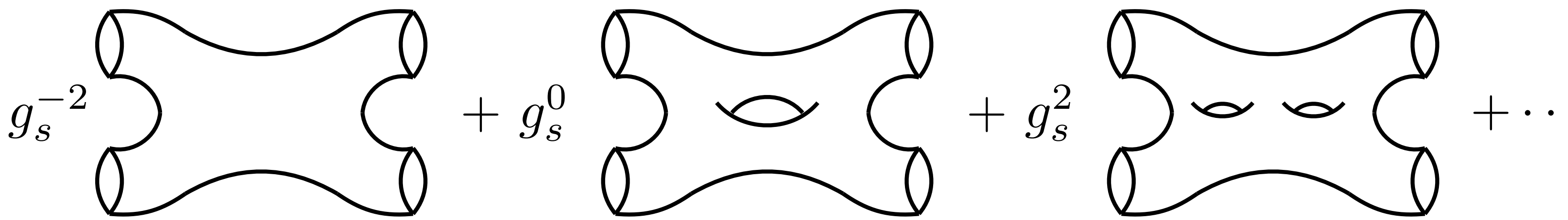
Perturbative String Amplitudes

- **Quantum probability "scattering" amplitudes**

  = Feynman functional integral/sum over all surfaces
  with given boundary components for initial and final strings

- **Closed oriented string perturbation theory**

  – The only remaining topological characterization is the genus $h \geq 0$
    - probability amplitude includes sum over all genera
    - weighed by a factor g_s^{2h-2} where g_s is the "string coupling"



  – genus h = number of "loops"

Structure of string amplitudes

- **Perturbative part of string amplitude decomposes into a sum over topologies**

$$\mathcal{A}_{\rm perturbative} = \sum_{h=0}^{\infty} g_s^{2h-2} \times \mathcal{A}^{(h)}$$

  – $\mathcal{A}^{(h)}$ is the amplitude at genus h

  – The perturbative expansion in g_s is asymptotic but not convergent
  (just as in field theory)

- **Non-perturbative part** (not considered here)

  ⋆ instantons $\approx e^{-1/g_s^2}$
  ⋆ D-branes contribute $\approx e^{-1/g_s}$.

String Data (closed oriented bosonic strings)

- **Assume fixed space-time M, with fixed metric G**
  - Physical space-time has Minkowski signature metric G
  - Starting point for string theory is often a Riemannian metric (if needed to be analytically continued to Minkowski signature)

- **The 2-dimensional worldsheet Σ is mapped into space-time M**
  - The space of all such maps $x : \Sigma \to M$ is denoted $\mathrm{Map}(\Sigma)$.

- **Riemannian metric G induces a Riemannian metric $x^*(G)$ on Σ**
  - Hence Σ is a Riemann surface (i.e. complex manifold with holó transition functions)

- **Polyakov formulation invokes an independent metric**
  - Riemannian metric g on Σ
  - Denote the ∞-dim. Riemannian manifold of such metrics by $\mathrm{Met}(\Sigma)$
  - String amplitude at fixed genus h obtained by weighed sum over g, x

$$\mathcal{A}^{(h)} = \int_{\mathrm{Met}(\Sigma)} Dg \int_{\mathrm{Map}(\Sigma)} Dx \; e^{-I_G[x,g]}$$

The worldsheet action I_G and the measure Dx

- **Basic Criteria**
  - Intrinsic = invariant under "reparametrizations" Diff(Σ) of Σ
  - lead to a well-defined QFT (renormalizable)

- **e.g. Non-linear sigma model action** with Riemannian metric G

$$I_G[x,g] = \frac{1}{\alpha'} \int_\Sigma d^2\xi \sqrt{g}\, g^{mn} \partial_m x^\mu \partial_n x^\nu G_{\mu\nu}(x)$$

$m,n = 1,2$ worldsheet indices

$\mu,\nu = 1,\cdots,D$ space-time Einstein indices

- **The measure is governed by the L^2-norm**

$$\|\delta x\|^2_G = \int_\Sigma d^2\xi \sqrt{g}\, \delta x^\mu\, \delta x^\nu\, G_{\mu\nu}(x)$$

  - manifestly intrinsic
  - renormalizable in a generalized sense (the metric G is renormalized)

Weyl(Σ)-invariance

- **Weyl transformations:** $g_{mn} \to e^{2\sigma} g_{mn}$ **leaving** x^μ **and** G **unchanged**

- **The classical action** I_G **is Weyl-invariant for any metric** G
  - but the measure Dx is not Weyl-invariant
  - which gives rise to a "Weyl-anomaly"
    - $=$ symmetry of classical action not preserved by quantization

- **The action** I_G **defines a conformal quantum field theory**

$$e^{-W_G[g]} = \int_{\mathrm{Map}(\Sigma)} Dx \; e^{-I_G[x,g]}$$

  - provided W_G is Diff(Σ)-invariant
  - obeys the following Ward identity under Weyl transformations

$$\delta W_G[g] = \frac{c}{24} \int_\Sigma d^2\xi \sqrt{g}\, R_g \, \delta\sigma$$

  - where R_g is the scalar curvature of the metric g on the surface Σ

- **The measure** Dg **is not Weyl-invariant, but the combined amplitude**
  - is Weyl invariant for central charge $c = 26 = \dim(M)$
  - later we shall see for the superstring $D = 10 = \dim(M)$

Conformal Field Theory

- **Stress tensor encodes response of field theory to change in metric**

$$T^c_{mn} = \frac{\delta W_G[g]}{\sqrt{g}\, \delta g^{mn}} \qquad\qquad T^c_{mn} = T^c_{nm}$$

  - Diff(Σ)-invariance requires "a conserved stress tensor" $\nabla^m T^c_{mn} = 0$
  - Weyl anomaly requires $g^{mn} T^c_{mn} = -\frac{c}{12} R_g$

- **Traceless stress tensor T_{mn} obtained by adding a local counter-term**
  - In local complex coordinates $(z, \tilde{z})$ we have $T_{z\tilde{z}} = T_{\tilde{z}z} = 0$ and

$$T_{zz} = T^c_{zz} + \frac{c}{6}\left(2\partial_z \Gamma^z_{zz} - (\Gamma^z_{zz})^2\right) \qquad\qquad \Gamma^z_{zz} = \partial_z \ln g_{z\tilde{z}}$$

  - Successive derivatives of W in g_{mn} give correlators of T_{mn}
  - Their singular part is governed by the OPE and the Ward identities

$$T_{zz} T_{ww} = \frac{c/2}{(z-w)^4} + \frac{2T_{ww}}{(z-w)^2} + \frac{\partial_w T_{ww}}{z-w} + \text{regular}$$

  - The mode expansion $T_{zz} = \sum_m z^{-2-m} L_m$ gives the Virasoro algebra

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0}$$

Negative norm states

- **Consider flat Minkowski $M = \mathbb{R}^{26}$ with metric $\eta = \mathrm{diag}(- + \cdots +)$**
  - Maps $x : \Sigma \to M$ satisfy Laplace equation $\partial_{\bar z} \partial_z x^\mu = 0$ for $\mu = 1, \cdots, 26$
  - Concentrate on holomorphic field

$$\partial_z x^\mu = \sum_{m \in \mathbb{Z}} x_m^\mu \, z^{-m-1} \qquad [x_m^\mu, x_n^\nu] = m \, \delta_{m+n,0} \, \eta^{\mu\nu} \qquad (x_n^\mu)^\dagger = x_{-n}^\mu$$

  - Similarly anti-holomorphic field $\partial_{\bar z} x^\mu$ produces modes $\tilde x^\mu$

- **Single string ground state $|0, k\rangle$ labeled by its momentum k satisfies**

$$x_0^\mu |0, k\rangle = k^\mu |0, k\rangle \qquad x_m^\mu |0, k\rangle = 0 \text{ for } m > 0$$

  - Fock space (holo sector) generated by linear combinations of

$$x_{m_1}^{\mu_1} \cdots x_{m_p}^{\mu_p} |0, k\rangle \qquad m_1, \cdots, m_p < 0$$

  - Lowest excited state $\varepsilon_\mu(k) x_{-1}^\mu |0, k\rangle$ has norm

$$\| \varepsilon_\mu(k) x_{-1}^\mu |0, k\rangle \|^2 = \varepsilon_\mu(k) \varepsilon_\nu(k) \eta^{\mu\nu} \, \| |0, k\rangle \|^2$$

  - component $\varepsilon^\mu = \delta^{\mu,0}$ produces *negative norm state* (assuming $\| |0, k\rangle \|^2 > 0$)
    $=$ inconsistent with quantum mechanical probability interpretation

Eliminating negative norm states – conformal symmetry

- **Conformal symmetry guarantees the existence of Virasoro algebra**

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0}$$

  – for the bosonic string $c = 26$ and

$$L_m = \sum_{n \in \mathbb{Z}} \frac{1}{2} x_{m-n} \cdot x_n \qquad\qquad L_0 = \frac{1}{2} x_0^2 + \sum_{n \in \mathbb{N}} x_{-n} \cdot x_n$$

- **A state $|\psi\rangle$ is "physical" if $(L_0 - 1)|\psi\rangle = L_m|\psi\rangle = 0$ for $m \in \mathbb{N}$**
  – Eliminates all negative norm states;
  – Decouples all null states produced by gauge transformations;

  – e.g. on states $|\psi\rangle = \varepsilon(k) \cdot x_{-1}|0, k\rangle$
    ⋆ L_1 constraint imposes $k \cdot \varepsilon(k) = 0$
    ⋆ L_0 constraint imposes $k^2 = 0$
    ⋆ L_m constraints are automatic for $m \geq 2$ for this particular state
  – the state $|0, k\rangle$ itself is a tachyon (to be absent in the superstring)

$\Rightarrow$ **Negative norm and null states eliminated by conformal symmetry**

Conformal symmetry in curved space-times

- **Condition for Weyl-invariance on the metric G**
  - Infinitesimal Weyl variation for arbitrary G to one-loop order in α'

$$\delta W_G[g] = \int_\Sigma d^2\xi \sqrt{g}\, g^{mn} \partial_m x^\mu \partial_n x^\nu R_{\mu\nu}(x) \delta\sigma + \cdots + \mathcal{O}(\alpha')$$

    where $R_{\mu\nu}$ is the Ricci tensor of the metric $G_{\mu\nu}$
  - Thus, to leading order in α' conformal invariance requires $R_{\mu\nu} = 0$

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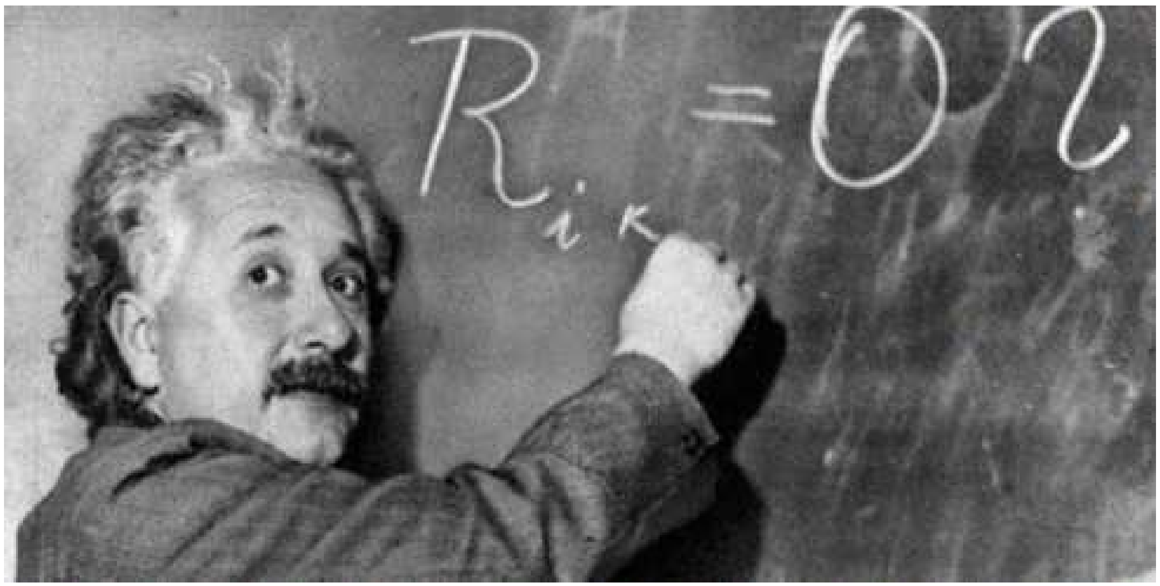


Vertex operators

- **Small fluctuations in the metric are gravitons**
  - A string couples to N gravitons in flat space by slightly perturbing the metric

$$G_{\mu\nu}(x) = \eta_{\mu\nu} + \sum_{i=1}^{N} \varepsilon_{i\mu\nu}(k_i)\, e^{ik_{i\mu}x^\mu} + \mathcal{O}(\varepsilon^2)$$

  - conformal invariance requires G to satisfy the linearized Einstein equations

$$k_i^2 = 0 \qquad\qquad k_i^\mu \varepsilon_{i\mu\nu}(k_i) = 0 \qquad \text{for } i = 1, \cdots, n$$

- **Vertex operator formulation is obtained by expanding in powers of ε_i**

$$\mathcal{A} = \sum_{h=0}^{\infty} g_s^{2h-2} \int_{\mathrm{Met}(\Sigma)} Dg \int_{\mathrm{Map}(\Sigma)} Dx\ \mathcal{V}_1[x,g] \cdots \mathcal{V}_N[x,g]\, e^{-I_\eta[x,g]}$$

  - where the vertex operator for an on-shell physical graviton is given by

$$\mathcal{V}_i[x,g] = \varepsilon_{i\mu\nu}(k_i) \int_\Sigma d^2\xi \sqrt{g}\, g^{mn} \partial_m x^\mu \partial_n x^\nu\, e^{ik_\mu x^\mu}$$

  - On-shell conditions $k_i^2 = k_i \cdot \varepsilon_i = 0$ guarantee conformal invariance

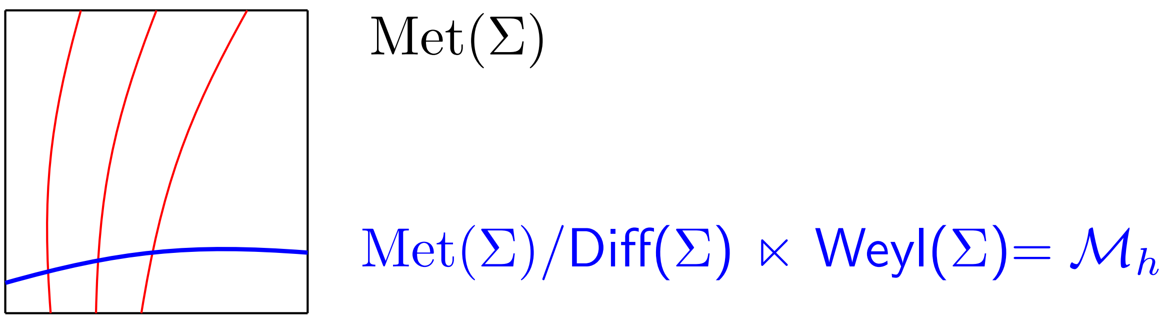
Diff(Σ) $\ltimes$ Weyl(Σ) and Moduli space

- **Fix topology of Σ**
  - Diff(Σ) re-parametrizes ξ^m on Σ by vector field $\delta\xi^m = -\delta v^m$

$$\delta g_{mn} = \nabla_m \delta v_n + \nabla_n \delta v_m$$

  - Weyl (Σ) $\quad \delta g_{mn} = 2\delta\sigma \, g_{mn}$ with $\delta\sigma$ an arbitrary real function of Σ

- **Orbits of Diff(Σ) $\ltimes$ Weyl(Σ) acting on the space Met(Σ)**



- **Moduli space $\mathcal{M}_h$ of compact Riemann surfaces of genus h** (no boundaries)

  = space of conformal structures (= space of complex structures)

$$\dim_{\mathbb{C}} \mathcal{M}_h = \begin{cases} 0 & h = 0 \\ 1 & h = 1 \\ 3h-3 & h \geq 2 \end{cases}$$

Some trivial moduli spaces

- **Given an infinitesimal δg_{mn} can one solve for $\delta\sigma$ and δv_m ?**

$$\delta g_{mn} = 2\delta\sigma g_{mn} + \nabla_m \delta v_n + \nabla_n \delta v_m$$

  – Eliminate the trace part by choosing $\delta\sigma = g^{mn}\delta g_{mn} + \nabla_m \delta v^m$
  – In local complex coordinates $(z, \tilde{z})$, remaining eqs for traceless part

$$\delta g_{zz} = \nabla_z v_z \qquad\qquad \delta g_{\tilde{z}\tilde{z}} = \nabla_{\tilde{z}} v_{\tilde{z}}$$

  – Integrability automatic since ∇_z and $\nabla_{\tilde{z}}$ act on different functions
  $\Rightarrow$ locally, or in any simply connected set, you can always solve

- **The sphere S^2 has no moduli (compact)**
  – Its stereographic projection onto $\mathbb{C}$ admits a globally conformally flat metric

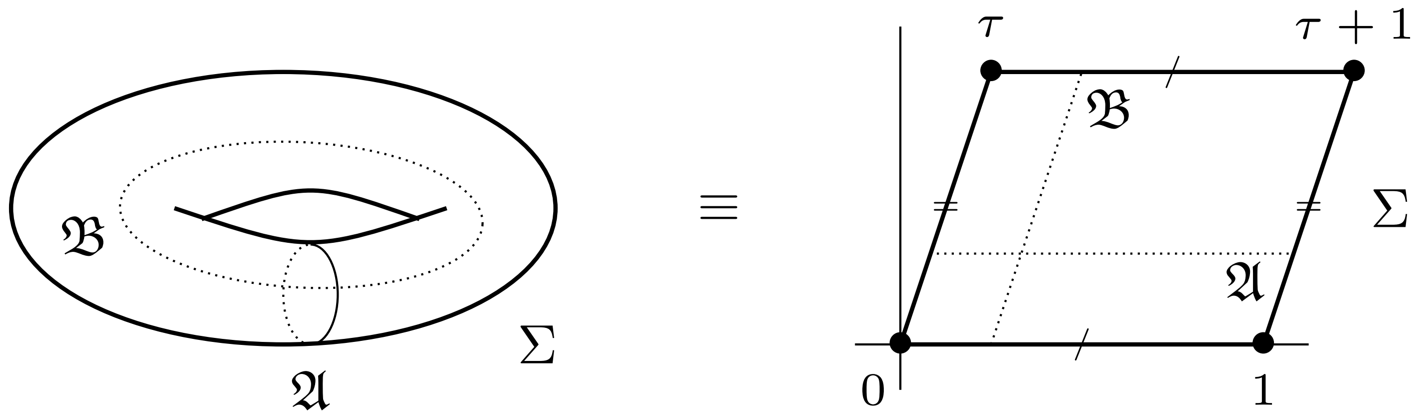
$$ds^2 = \frac{|dz|^2}{(1+|z|^2)^2}$$

- **The Poincaré upper half plane $\mathcal{H}$ has no moduli (non-compact)**

$$ds^2 = \frac{|dz|^2}{(\mathrm{Im}\, z)^2} \qquad\qquad \mathrm{Im}\, z > 0$$

Moduli deformations of the torus

- **The torus may be viewed as the product of two circles $\mathfrak{A}$ and $\mathfrak{B}$**
  - The ratio of their lengths and relative angle provide two real moduli
  - equivalently represented by parallelogram in $\mathbb{C}$ with sides pairwise identified



  - The complex number τ contains the information of relative lengths and angle

- **Constant metric deformations equivalently provide a complex modulus**
  - translation invariance on the circles induces translation invariance on the torus
  - by translation invariance, metric is constant on Σ
  - constant trace-part of δg_{mn} eliminated by constant σ
  - but constant $\delta g_{zz} = \partial_z v_z$ has no *periodic* solutions v_z

  $\Rightarrow$ constant δg_{zz} provides the deformation of the complex modulus of the torus.

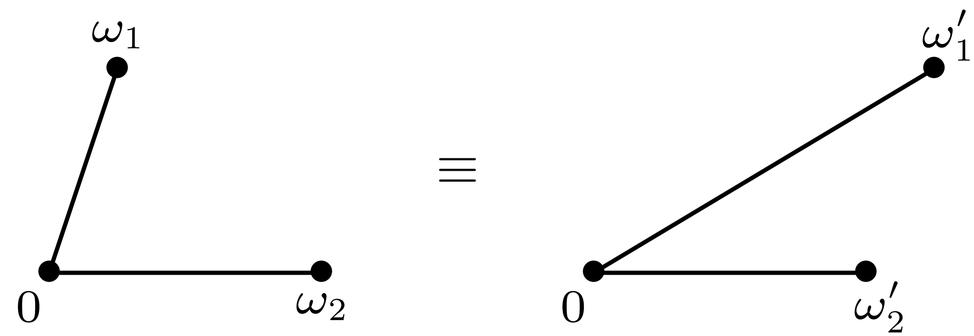
Moduli space of the torus

- **Oriented Riemann surfaces: cycles $\mathfrak{A}$ and $\mathfrak{B}$ ordered**
  - equivalently choose orientation $\tau \in \mathcal{H}_1 = \{\tau \in \mathbb{C}, \mathrm{Im}(\tau) > 0\}$

- **Space of inequivalent tori = space of inequivalent lattices** $\Lambda_\tau = \mathbb{Z} \oplus \tau\mathbb{Z}$
  - but different values of τ may give the same lattice

$$\omega_1' = a\,\omega_1 + b\,\omega_2$$
$$\omega_2' = c\,\omega_1 + d\,\omega_2$$

$$\tau = \omega_1/\omega_2$$
$$\tau' = (a\tau + b)/(c\tau + d)$$



  - identical lattices requires $\Lambda_{\tau'} \subset \Lambda_\tau$ and $\Lambda_\tau \subset \Lambda_{\tau'}$
  - so that $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{Z})$
  - generated by $\tau \to \tau + 1$ and $\tau \to -\tau^{-1}$

- **Moduli space of tori = space of inequivalent lattices =** $\mathcal{H}_1/SL(2,\mathbb{Z})$
  - standard fundamental domain

$$\mathcal{H}_1/SL(2,\mathbb{Z}) \equiv \left\{\tau \in \mathcal{H}_1,\ |\tau| \geq 1,\ |\mathrm{Re}(\tau)| \leq \frac{1}{2}\right\}$$

Decomposing the measure Dg

- **At any point $g \in \text{Met}(\Sigma)$ the measure Dg factors**

$$Dg \quad = \quad Z_g \quad \times \quad D\sigma \quad \times \quad Dv \quad \times \quad d\mu_{\mathcal{M}_h}$$

$$\qquad\qquad \text{Jacobian} \qquad \text{Weyl} \qquad \text{Diff}_0 \qquad \mathcal{M}_h$$

  - infinitesimal Weyl $\qquad \delta g_{mn} = \delta\sigma\, g_{mn}$
  - infinitesimal Diff$_0$ $\qquad \delta g_{mn} = \nabla_m \delta v_n + \nabla_n \delta v_m$
  - infinitesimal moduli deformations δg_{mn}

- **Goal**
  - compute Z_g
  - formulate Z_g in terms of ghosts
  - omit volume factors $D\sigma\, Dv$ of the group Diff$^+(\Sigma) \ltimes$ Weyl(Σ)

- **To decompose Dg we study tensor spaces (alias line bundles) on Σ**

Tensor Spaces - Line Bundles on Σ

- **A one-form $\phi = \phi_z dz + \phi_{\tilde z} d\tilde z$ on Σ decomposes into $K \oplus \bar K$**

  $K = \{\phi_z dz\}$ is the (space of sections of the) canonical bundle on Σ

  for $m \in \mathbb{Z}$ define $K^m = \{\phi_{z\dots z} dz^m\}$ and $\bar K^m = \{\phi_{\tilde z \dots \tilde z} d\tilde z^m\} \approx K^{-m}$

- **L^2 inner product for $\phi_1, \phi_2 \in K^m$**

$$(\phi_1, \phi_2) = \int_\Sigma d\tilde z dz \sqrt{g}\, (g_{z\tilde z})^{-m} \phi_1^* \phi_2$$

  The spaces K^m and K^n with $m \neq n$ are mutually orthogonal

- **Covariant derivative on $\phi \in K^m$ decomposes $\nabla \phi = \nabla_z^{(m)} \phi + \nabla_{\tilde z}^{(m)} \phi$**

  $\nabla_z^{(m)} : K^m \to K^{m+1}$ mutual adjoint operators $(\nabla_z^{(m)})^\dagger = -\nabla^z_{(m+1)}$

  $\nabla^z_{(m)} : K^m \to K^{m-1}$ with $\nabla_{\tilde z}^{(m)} = g_{z\tilde z} \nabla^z_{(m)}$

- **Riemann-Roch and Vanishing Theorems**

$$\dim_{\mathbb{C}} \mathrm{Ker}\, \nabla_{\tilde z}^{(m)} - \dim_{\mathbb{C}} \mathrm{Ker}\, \nabla_{\tilde z}^{(1-m)} = (2m-1)(h-1)$$

  $\mathrm{Ker} \nabla_{\tilde z}^{(m)} = 0$ for $h \geq 2$ and $m \leq -1$ (no holó vector fields for $h \geq 2$)

  $\mathrm{Ker} \nabla_{\tilde z}^{(m)} = 0$ for $h = 0$ and $m \geq 1$ (no holó forms on the sphere)

Decomposing the tangent space to Met(Σ)

- **Orthogonal decomposition of** $T_g(\mathrm{Met}(\Sigma))$

$$T_g(\mathrm{Met}(\Sigma)) = \{\delta\sigma\, g_{z\tilde z}\} \oplus \left\{\delta g_{zz} = g_{z\tilde z}\, \delta\eta_z{}^{\tilde z}\right\} \oplus \{\delta g_{\tilde z\tilde z} = g_{z\tilde z}\, \delta\eta_{\tilde z}{}^{z}\}$$

$$\delta\sigma \in K^0 \qquad \delta\eta_z{}^{\tilde z} \in K \otimes \bar K^{-1} \qquad \delta\eta_{\tilde z}{}^{z} \in \bar K \otimes K^{-1}$$

- **Diff$_0$ acts by** $\delta\eta_z{}^{\tilde z} = \nabla_z^{(1)} \delta v^{\tilde z}$
  - For $h \geq 1$, the range of the operator $\nabla_z^{(1)}$ is NOT all of $K \otimes \bar K^{-1}$
  - The orthogonal complement of the range of $\nabla_z^{(1)}$ is given by

$$\mathrm{Range}\, \nabla_z^{(1)} \oplus \mathrm{Ker}\big(\nabla_z^{(1)}\big)^\dagger = K \otimes \bar K^{-1} \approx K^2$$

- **Holomorphic quadratic differentials** $\phi^j \in \mathrm{Ker} \nabla_{\tilde z}^{(2)} \approx \mathrm{Ker}\big(\nabla_z^{(1)}\big)^\dagger$
  - Hence we may identify $\mathrm{Ker} \nabla_{\tilde z}^{(2)} = T^*_{(1,0)}(\mathcal{M}_h)$
  - One-forms $\delta m^j \in T^*_{(1,0)}(\mathcal{M}_h)$ given by linear forms on $\bar K \otimes K^{-1}$

$$\delta m^j = (\delta\eta, \phi^j) = \int_\Sigma d\tilde z dz\, \delta\eta_{\tilde z}{}^{z}\, \phi^j_{zz}$$

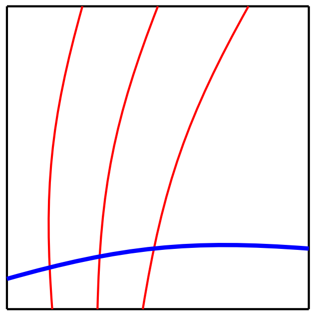
  - Weyl-invariant pairing and vanishes on $\delta\eta \in \mathrm{Range}\, \nabla_z^{(1)}$
  - Riemann-Roch and Vanishing give $\dim_{\mathbb{C}} \mathcal{M}_h = 3h - 3$ for $h \geq 2$

Decomposing the measure Dg (cont'd)

- **Parametrize $\mathcal{M}_h$ by a slice in $\mathrm{Met}(\Sigma)$ transverse to Weyl $\ltimes$ Diff$_0$**

orbits of Diff(Σ) $\ltimes$ Weyl(Σ)

$m^j, \tilde m^j$ local coordinates on $\mathcal{M}_h$



- **Carry out a change of integration variables**

$$T_g(\mathrm{Met}(\Sigma)) = \{\delta\sigma\, g_{z\bar z}\} \oplus \{\delta\eta_z{}^{\bar z}\} \oplus \{\delta\eta_{\bar z}{}^{z}\}$$

  - Orthogonality implies that the measure factorizes $Dg = D\sigma\, D\eta\, D\bar\eta$
  - The change of variables is given by (repeated indices j are summed)

$$\delta\eta_{\bar z}{}^{z} = \nabla^{(-1)}_{\bar z}\delta v^z + (\mu_j)_z{}^{\bar z}\delta m^j \qquad (\mu_j)_{\bar z}{}^{z} = g^{z\bar z}\frac{\partial g_{\bar z\bar z}}{\partial m^j}$$

$$\delta\eta_z{}^{\bar z} = \nabla^{(1)}_{z}\delta v^{\bar z} + (\tilde\mu_j)_z{}^{\bar z}\delta \tilde m^j \qquad (\tilde\mu_j)_{z}{}^{\bar z} = g^{z\bar z}\frac{\partial g_{zz}}{\partial \tilde m^j}$$

Ghosts

- **Use standard rules to introduce ghosts for the determinant**
  - gauge transformations $(\delta v^z, \delta v^{\tilde z}) \to (c^z, \tilde c^{\tilde z})$ Grassmann-odd ghosts
  - conjugate $(\delta \eta_z{}^{\tilde z}, \delta \eta_{\tilde z}{}^z) \to (b_{zz}, \tilde b_{\tilde z \tilde z})$ Grassmann-odd anti-ghosts
  - extended ghost action

$$\int_\Sigma d^2 z \Big[b_{zz} \left(\partial_{\tilde z} c^z + \mu_j \, \delta m^j \right) + \tilde b_{\tilde z \tilde z} \left(\partial_z \tilde c^{\tilde z} + \tilde \mu_j \, \delta \tilde m^j \right) \Big]$$

  - Here $\delta m^j, \delta \tilde m^j$ are differential one-forms which are Grassmann odd

- **Integrating out $\delta m^j, \delta \tilde m^j$ gives the standard ghost representation**

$$\int D(x^\mu, b, \tilde b, c, \tilde c) \mathcal{V}_1 \cdots \mathcal{V}_N e^{-I_G - I_{gh}} \prod_j \delta(\langle b, \mu_j \rangle) \delta(\langle \tilde b, \tilde \mu_j \rangle) dm^j d\tilde m^j$$

  - where I_{gh} is the standard ghost action

$$I_{gh} = \int_\Sigma d^2 z \Big[b_{zz} \partial_{\tilde z} c^z + \tilde b_{\tilde z \tilde z} \partial_z \tilde c^{\tilde z} \Big]$$

  - gauge fixed formulation has BRST invariance
  - for the sphere and the torus, quotient out by conformal automorphisms

Bosonic string has tachyon and no fermions: unphysical

- **Warm-up : tree-level tachyon scattering amplitude**
  - tachyon vertex operator $\mathcal{V}(k_i) = \int_\Sigma d^2 z_i \sqrt{g(z_i)} : e^{ik_i \cdot x(z_i)} :$
  - scalar Green function on the sphere with metric $|dz|^2/(1+|z|^2)^2$

$$\langle x^\mu(z) x^\nu(w) \rangle = \eta^{\mu\nu} G(z,w) \qquad\qquad G(z,w) = -\ln \frac{|z-w|^2}{(1+|z|^2)(1+|w|^2)}$$

- **Sphere has no moduli, ghost and scalar partition functions are constant**

$$\langle \prod_{i=1}^N d^2 z_i \sqrt{g(z_i)} : e^{ik_i \cdot x(z_i)} : \rangle = \prod_{i=1}^N d^2 z_i \prod_{i<j}^N |z_i - z_j|^{\alpha' k_i \cdot k_j}$$

  - Integrand invariant under $z_i \to (\alpha z_i + \beta)/(\gamma z_i + \delta)$ with $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{C})$
  - Factor out volume of $SL(2,\mathbb{C})$ by fixing $z_N = \infty, z_{N-1} = 1, z_{N-2} = 0$

- **The 4-tachyon amplitude with** $s_{ij} = -\alpha'(k_i + k_j)^2/4$

$$\frac{1}{g_s^2} \int_\Sigma d^2 z \, |z|^{\alpha' k_1 \cdot k_2} |z-1|^{\alpha' k_1 \cdot k_3} = \frac{\Gamma(-1-s)\Gamma(-1-t)\Gamma(-1-u)}{g_s^2 \, \Gamma(2+s)\Gamma(2+t)\Gamma(2+u)}$$

  - Tachyon poles at $s,t,u = -1$

Kawai-Lewellen-Tye (KLT) relations

- **Tree-level closed string amplitudes are bilinears in open string amplitudes**
  - Closed string amplitudes on the sphere, vertex operators in interior
  - Open string amplitude on upper half plane, vertex operators on boundary
  - Consider open and closed string 4-tachyon amplitudes

$$\mathcal{A}^{(0)}_{\rm open}(s,t) = \int_0^1 d\xi |\xi|^{k_1 \cdot k_2} |1-\xi|^{k_2 \cdot k_3} \qquad \mathcal{A}^{(0)}_{\rm closed}(s,t,u) = \int_{S^2} d^2 z |z|^{2k_1 \cdot k_2} |1-z|^{2k_2 \cdot k_3}$$

  - Parametrize $z = \alpha + i\beta$ then z-integrand is analytic function of β with branch points at $\beta = \pm i\alpha$ and $\beta = \pm i(1-\alpha)$
  - Deform β-contour from real to imaginary axis, but pick up phases

$$\int_{S^2} d^2 z |z|^{2k_1 \cdot k_2} |1-z|^{2k_2 \cdot k_3} = \sin(\pi k_2 \cdot k_3) \int_0^1 d\xi |\xi|^{k_1 \cdot k_2} |1-\xi|^{k_2 \cdot k_3} \int_1^\infty d\eta |\eta|^{k_1 \cdot k_2} |1-\eta|^{k_2 \cdot k_3}$$

  - Converting the second integral back to $\mathcal{A}_{\rm open}$, we obtain the KLT relation

$$\mathcal{A}^{(0)}_{\rm closed}(s,t,u) = \sin(\pi k_2 \cdot k_3) \mathcal{A}^{(0)}_{\rm open}(s,t) \mathcal{A}^{(0)}_{\rm open}(t,u)$$

  - Does the worldsheet secretly have a Minkowski signature structure ?
  - No generalization known to loop level

Lectures on Superstring Amplitudes

Part 2: Superstrings

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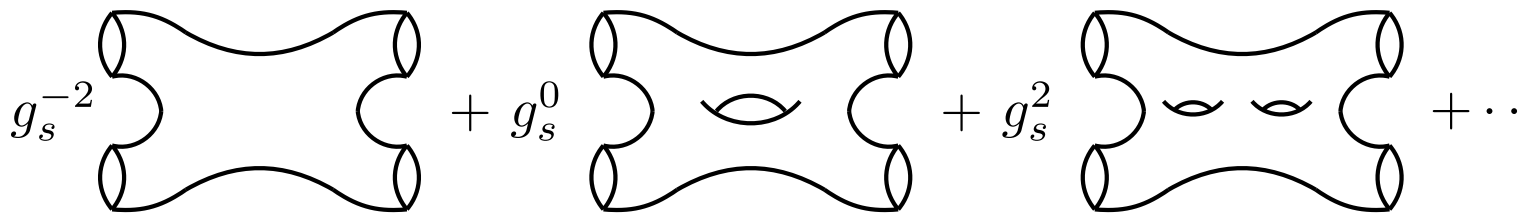
Amplitudes 2018 Summer School

Superstring Perturbation Theory

- **Theory of fluctuating random surfaces** (closed strings shown)
  - governed by topological expansion in the genus h weighed by g_s^{2h-2}



- **Bosonic string**
  - unstable with closed string tachyon
  - Nature has fermions !
- **Superstrings generalize bosonic string**
  - they have fermions
  - no tachyon
  - supersymmetry

Approaches to Superstring Perturbation Theory

- **Goal is to obtain superstring amplitudes at all genera**
  - Ramond-Neveu-Schwarz formulation of fermionic strings;
    w/ Gliozzi-Scherk-Olive projection to supersymmetric spectrum;
  - Green-Schwarz space-time supersymmetric formulation;
  - Mandelstam light-cone formulation;
  - String field theory;
  - Topological string theory;
  - Berkovits pure spinor formulation.

- **Different perturbative superstring theories** (in 10 dimensions)
  - Type I — open & closed, orientable & non-orientable, D-branes
  - Type IIA,B — closed orientable, D-branes
  - Heterotic — closed orientable $E_8 \times E_8$, $Spin(32/\mathbb{Z}_2)$

- Here: **RNS formulation, closed orientable superstrings, dimension 10**

Genus-zero four-graviton superstring amplitude

- **Kinematics of the four-graviton amplitude**
  - momenta of gravitons k_i^μ are conserved $\sum_i k_i^\mu = 0$
  - choose basis of factorized polarization tensors $\varepsilon_i^{\mu\nu} = \varepsilon_i^\mu \tilde\varepsilon_i^\nu$
  - masslessness $k_i^2 = 0$ and transversality $k_i^\mu \varepsilon_i^\mu = k_i^\mu \tilde\varepsilon_i^\mu = 0$ for $i = 1, 2, 3, 4$
  - kinematic invariants $s = s_{12} = s_{34}$, $t = s_{14} = s_{23}$, $u = s_{13} = s_{24}$

$$s_{ij} = -\alpha'(k_i + k_j)^2/4$$

- **Tree-level four-graviton amplitude is given by**

$$\mathcal{A}^{(0)}(\varepsilon_i, \tilde\varepsilon_i, k_i) = \frac{1}{g_s^2} \times \mathcal{K}\tilde{\mathcal{K}} \times \frac{1}{stu} \frac{\Gamma(1-s)\Gamma(1-t)\Gamma(1-u)}{\Gamma(1+s)\Gamma(1+t)\Gamma(1+u)}$$

  - Kinematical factor $\mathcal{K}$ given in terms of $f_i^{\mu\nu} = k_i^\mu \varepsilon_i^\nu - k_i^\nu \varepsilon_i^\mu$ by

$$\begin{aligned} \mathcal{K} &= (f_1 f_2)(f_3 f_4) + (f_1 f_3)(f_2 f_4) + (f_1 f_4)(f_2 f_3) \\ &\quad -4(f_1 f_2 f_3 f_4) - 4(f_1 f_2 f_4 f_3) - 4(f_1 f_3 f_2 f_4) \end{aligned}$$

  - for $\tilde{\mathcal{K}}$ replace ε_i by $\tilde\varepsilon_i$
  - Equivalently, $\mathcal{K} \times \tilde{\mathcal{K}} = \mathcal{R}^4$ with $\mathcal{R}$ the linearized Weyl tensor
  - String duality: symmetric in s, t, u
  - Poles in each channel, at $s, t, u = 0, 1, 2, \cdots$

Genus-one four-graviton superstring amplitude

- **Type II four-graviton amplitude to one-loop order** (Green, Schwarz 1982)

$$\mathcal{A}^{(1)}(\varepsilon_i, \tilde\varepsilon_i, k_i) = \mathcal{R}^4 \int_{\mathcal{M}_1} \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2} \mathcal{B}^{(1)}(s_{ij}|\tau)$$

  – Partial amplitude $\mathcal{B}^{(1)}$ is a modular function in $\tau \in \mathcal{M}_1 = \mathcal{H}_1/SL(2,\mathbb{Z})$

$$\mathcal{B}^{(1)}(s_{ij}|\tau) = \int_{\Sigma^4} \prod_{i=1}^4 \frac{d^2 z_i}{\mathrm{Im}\,\tau} \exp\Big(\sum_{i<j} s_{ij}\, G(z_i - z_j|\tau) \Big)$$

  – $G(z|\tau)$ is the scalar Green function on the torus Σ of modulus τ.
  – Analogous formulas for Heterotic strings and more external states.

- **Singularity structure**
  – For fixed τ integrations over Σ produce poles in $\mathcal{B}^{(1)}$ at positive integers s_{ij}.
  – The integral over τ converges absolutely only for $\mathrm{Re}(s_{ij}) = 0$.
  – Analytic continuation to $s_{ij} \in \mathbb{C}$ via decomposition of $\mathcal{M}_1$.
  – Branch cuts in s_{ij} starting at integers ≥ 0 are produced by $\tau \to i\infty$ region.

Loop momenta

- **Loop momenta may be exposed**
  - Choose a canonical basis of homology cycles $\mathfrak{A}, \mathfrak{B}$.
  - Choose loop momentum p flowing through the cycle $\mathfrak{A}$,

$$\int_{\mathcal{M}_1} \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2} \mathcal{B}^{(1)}(s_{ij}|\tau) = \int_{\mathbb{R}^{10}} d^{10}p \int_{\mathcal{M}_1} \int_{\Sigma^4} \Big| \mathcal{F}(z_i, k_i, p|\tau) \Big|^2$$

- **Chiral amplitude $\mathcal{F}$ is locally holomorphic in τ and z_i**

$$\mathcal{F}(z_i, k_i, p|\tau) = e^{i\pi\tau p^2 + 2\pi i p \sum_i k_i z_i} \prod_{i<j} \vartheta_1(z_i - z_j|\tau)^{-s_{ij}} \, d\tau \prod_{i=1}^{4} dz_i$$

  - at the cost of non-trivial monodromy

$$\begin{aligned} \mathcal{F}(z_i + \delta_{i,\ell}\mathfrak{A}, k_i, p|\tau) &= e^{2\pi i k_\ell \cdot p} \mathcal{F}(z_i, k_i, p|\tau) \\ \mathcal{F}(z_i + \delta_{i,\ell}\mathfrak{B}, k_i, p|\tau) &= \mathcal{F}(z_i, k_i, p + k_\ell|\tau) \end{aligned}$$

  - Modular invariance of $\mathcal{A}^{(1)}$ guarantees independence of choices.
  - Hermitian pairing of $\mathcal{F}$ and $\bar{\mathcal{F}}$ is familiar from 2-d CFT where loop momentum p labels conformal blocks of 10 copies of $c = 1$.

RNS formulation of superstrings

- $M = \mathbb{R}^{10}$ **flat Minkowski space-time with Lorentz group** $SO(1,9)$
  - x^μ scalars on worldsheet Σ, map Σ into M
  - ψ^μ spinors on Σ but Lorentz vector under $SO(1,9)$
    - ⋆ Worldsheet supersymmetry $\Longrightarrow$ Σ is a super Riemann surface
    - ⋆ Two sectors : NS bosons $SO(1,9)$-tensors
      R fermions $SO(1,9)$-spinors

- **With Minkowski signature** Σ
  - ψ^μ and $\tilde\psi^\mu$ are *independent* Majorana-Weyl spinors of opposite chirality

- **With Euclidean signature** Σ
  - ψ^μ and $\tilde\psi^\mu$ must be *independent* complex Weyl spinors
  - Globally, on a compact Riemann surface of genus h,
    - ⋆ All ψ^μ are sections of a the same spin bundle S (and $\tilde\psi^\mu$ of $\tilde S$)
    - ⋆ 2^{2h} distinct spin structures for S (and 2^{2h} independently for $\tilde S$)

- **GSO projection requires independent summation over spin structures**

Quantization of worldsheet spinor fields

- **Illustrate**
  - Ramond and Neveu-Schwarz sectors
  - independence of chiralities

- **Dirac action and equation for flat $M = \mathbb{R}^{10}$ with metric η**
  - All components of ψ_+^μ are sections of the same spin bundle S
  - Complex structure J with local complex coordinates $(z, \tilde{z})$
  - Dirac action,

$$I_\psi[\psi, J] = \frac{1}{2\pi} \int_\Sigma d\tilde{z} dz \, \psi_+^\mu \partial_{\tilde{z}} \psi_+^\nu \eta_{\mu\nu}$$

  - Dirac equation $\partial_{\tilde{z}} \psi_+^\mu = 0$ has locally holomorphic solutions,
  - but products of operators produce singularities

$$\psi_+^\mu(z) \psi_+^\nu(w) = \frac{\eta^{\mu\nu}}{z - w} + \text{regular}$$

  - each component ψ^μ generates a CFT with central charge $c = \frac{1}{2}$.

Quantization of worldsheet spinor fields (cont'd)

- **Quantization on flat cylinder or conformal equivalent flat annulus**
  - cylinder $w = \tau + i\sigma$ with identification $\sigma \approx \sigma + 2\pi$
  - annulus centered at $z = 0$, conformally mapped by $z = e^w$
  - one-forms related by $dz = e^w \, dw$, spinors by $(dz)^{\frac{1}{2}} = e^{w/2} \, (dw)^{\frac{1}{2}}$
  - fields related by conformal transformation $\psi_{\rm cyl}(z) = e^{w/2} \, \psi_{\rm ann}(w)$

- **Two possible spin structures**

NS $\quad \psi^\mu_{\rm cyl}(\tau, \sigma + 2\pi) = - \psi^\mu_{\rm cyl}(\tau, \sigma) \quad$ or $\quad \psi^\mu_{\rm ann}(e^{2\pi i} \, z) = + \psi^\mu_{\rm ann}(z)$

R $\quad \psi^\mu_{\rm cyl}(\tau, \sigma + 2\pi) = + \psi^\mu_{\rm cyl}(\tau, \sigma) \quad$ or $\quad \psi^\mu_{\rm ann}(e^{2\pi i} \, z) = - \psi^\mu_{\rm ann}(z)$

- **Free field quantization in annulus representation**

NS $\quad \psi^\mu(z) = \sum_{r \in \frac{1}{2} + \mathbb{Z}} b^\mu_r \, z^{-\frac{1}{2} - r} \qquad \{b^\mu_r, b^\nu_s\} = \eta^{\mu\nu} \delta_{r+s,0}$

R $\quad \psi^\mu(z) = \sum_{n \in \mathbb{Z}} d^\mu_n \, z^{-\frac{1}{2} - n} \qquad \{d^\mu_m, d^\nu_n\} = \eta^{\mu\nu} \delta_{m+n,0}$

Quantization of worldsheet spinor fields (cont'd)

- **Lorentz generators of $SO(1,9)$:** $[J^{\mu\nu}, \psi^\kappa(z)] = \eta^{\nu\kappa}\psi^\mu(z) - \eta^{\mu\kappa}\psi^\nu(z)$

$$J^{\mu\nu}_{\rm NS} = \sum_{r \in \mathbb{N}-\frac{1}{2}} \left(b^\mu_{-r} b^\nu_r - b^\nu_{-r} b^\mu_r\right)$$

$$J^{\mu\nu}_{\rm R} = \frac{1}{2}[d^\mu_0, d^\nu_0] + \sum_{n \in \mathbb{N}} \left(d^\mu_{-n} d^\nu_n - d^\nu_{-n} d^\mu_n\right)$$

- **Fock space construction produces two sectors**

⋆ NS ground state defined by $b^\mu_r|0;{\rm NS}\rangle = 0$ for all $r > 0$
- $|0;{\rm NS}\rangle$ is unique and in trivial representation of $SO(1,9)$
- Fock space $=$ linear combinations of $b^{\mu_1}_{-r_1} \cdots b^{\mu_p}_{-r_p}|0;{\rm NS}\rangle$, $r_i > 0$
- All states in tensor reps of $SO(1,9)$ are space-time bosons.

⋆ R ground state defined by $d^\mu_n|0,\alpha;{\rm R}\rangle = 0$ for all $n > 0$
- $|0,\alpha;{\rm R}\rangle$ is degenerate and in spinor rep. of $SO(1,9)$, states labelled by α
- Fock space $=$ linear combinations of $d^{\mu_1}_{-n_1} \cdots d^{\mu_p}_{-n_p}|0,\alpha;{\rm R}\rangle$, $n_i > 0$
- All states in spinor reps of $SO(1,9)$ are space-time fermions.

Summation over spin structures

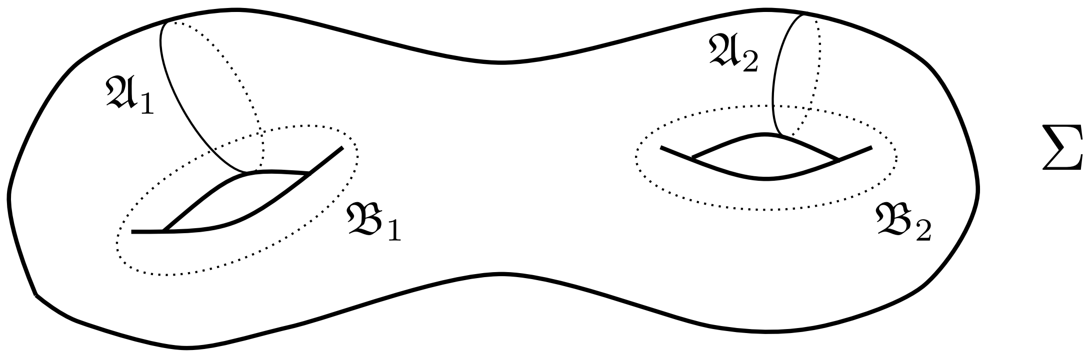
- **Theory with bosons and fermions requires both NS and R sectors**
  - to include both, one must sum over two spin structures of the annulus
- **Type II spin structures of $\psi^\mu_\pm$ are independent of one another**
  - space-time fermions are in the R $\otimes$ NS and NS $\otimes$ R sectors which could never arise if spin structures for opposite chiralities coincided
- **On the torus, viewed as cylinder + identification**
  - spin structures along cycle of cylinder produce R and NS sectors
  - sum over spin structures along conjugate cycle produces GSO-projection
    - ⋆ reduces to half the states in both R and NS sectors
    - ⋆ R-sector: space-time spinor of definite chirality
    - ⋆ NS-sector: eliminates the tachyon

  $\Rightarrow$ sum over *all* spin structures

Summation over spin structures (cont'd)

- **Fix a canonical homology basis of cycles $\mathfrak{A}_I, \mathfrak{B}_I$ of $H_1(\Sigma, \mathbb{Z})$** $I = 1, \cdots, h$
  - with canonical intersection pairing

    $\#(\mathfrak{A}_I, \mathfrak{A}_J) = \#(\mathfrak{B}_I, \mathfrak{B}_J) = 0$ and $\#(\mathfrak{A}_I, \mathfrak{A}_J) = \delta_{IJ}$



- **Transformations which maps one canonical basis into another**
  - linear with integer coefficients
  - preserve the intersection matrix: $Sp(2h, \mathbb{Z})$

- **On Riemann surface of higher genus h sum over all spin structures**
  - along $\mathfrak{A}$-cycles produces R and NS sectors
  - along $\mathfrak{B}$-cycles produces GSO-projection
  - mapped into one another by $Sp(2h, \mathbb{Z}_2)$

Super Riemann surfaces

- **Ordinary Riemann surface** (locally $\mathbb{C}$ with coordinate z)
  - complex manifold: holomorphic transition functions $z \to z'(z)$;
  - complex structure = conformal structure J
  - Moduli space $\mathcal{M}_h = \{J\}/\mathrm{Diff}(\Sigma)$ of genus h compact Riemann surfaces

- **Complex super manifold** (locally $\mathbb{C}^{1|1}$ with coordinates $z|\theta$)
  - holó transition functions $z|\theta \to z'(z,\theta)|\theta'(z,\theta)$ generate $\mathcal{N}=2$ super conformal

- **Super Riemann surface** (locally $\mathbb{C}^{1|1}$ with coordinates $z|\theta$)
  - holó transition functions $z|\theta \to z'|\theta'$ rescale $D_\theta = \partial_\theta + \theta\partial_z$
  - Transition functions define $\mathcal{N}=1$ superconformal structure $\mathcal{J}$
  - Globally: $T\Sigma$ has a completely non-integrable subbundle of rank $0|1$

- **Moduli space of compact super Riemann surfaces**: $\mathfrak{M}_h = \{\mathcal{J}\}/\mathrm{Diff}(\Sigma)$
  $=$ equivalence classes of superconformal structures $\mathcal{J}$

$$\dim_{\mathbb{C}} \mathfrak{M}_h = \begin{cases} 0|0 & h=0 \\ 1|0 \text{ or } 1|1 & h=1 \quad \text{even or odd spin structure} \\ 3h-3|2h-2 & h\geq 2 \end{cases}$$

  - odd modulus at $h=1$ odd spin structure is a book keeping device;
  - odd moduli really first appear at genus 2, as curved super spaces.

Superstring worldsheets and moduli spaces

- **Heterotic**
  - Left : RS Σ_L, moduli space $\mathcal{M}_L$ coord resp. $\tilde z$ and $\tilde m^i$
  - Right : SRS Σ_R, moduli space $\mathfrak{M}_R$ coord resp. (z,θ) and (m^i,ζ^α)
  - Worldsheet is a cycle $\Sigma \subset \Sigma_L \times \Sigma_R$ of dim $1|1$

    subject to $\Sigma_{\rm red} = {\rm diag}(\Sigma_{L\,{\rm red}} \times \Sigma_{R\,{\rm red}})$: $\tilde z^* = z + {\rm nilpotent}$
  - Moduli space is a cycle $\Gamma \subset \mathcal{M}_L \times \mathfrak{M}_R$ of dim $3h-3|2h-2$ for $h \geq 2$

    subject to $\Gamma_{\rm red} = {\rm diag}(\mathcal{M}_{L\,{\rm red}} \times \mathfrak{M}_{R\,{\rm red}})$: $(\tilde m^i)^* = m^i + {\rm nilpotent}$

    (reduced space obtained by setting all nilpotent variables to zero)

- **Type II**
  - Left : SRS Σ_L, moduli space $\mathfrak{M}_L$ coord resp. $(\tilde z,\tilde\theta)$ and $(\tilde m^i,\tilde\zeta^\alpha)$
  - Right : SRS Σ_R, moduli space $\mathfrak{M}_R$ coord resp. (z,θ) and (m^i,ζ^α)
  - Worldsheet is a cycle $\Sigma \subset \Sigma_L \times \Sigma_R$ of dim $1|2$
  - Moduli space is cycle $\Gamma \subset \mathfrak{M}_L \times \mathfrak{M}_R$ of dim $3h-3|4h-4$ for $h \geq 2$

    subject to $\tilde z^* = z + {\rm nilpotent}$ and $(\tilde m^i)^* = m^i + {\rm nilpotent}$

- **Super-Stokes theorem ensures independence of the choice of cycles**
  - in amplitudes with BRST invariant vertex operators
  - consistent definition of superstring amplitudes to all genera (Witten 2012)

Worldsheet action for Type II superstrings

- **Worldsheet is** $\Sigma \subset \Sigma_L \times \Sigma_R$
  - Σ_L has superconformal structure $\tilde{\mathcal{J}}$ with local coordinates $\tilde z|\tilde\theta$
  - Σ_R has superconformal structure $\mathcal{J}$ with local coordinates $z|\theta$

- **Superconformal invariant matter action**
  - worldsheet matter field

$$X^\mu(\tilde z, z|\tilde\theta, \theta) = x^\mu(\tilde z, z) + \theta\psi^\mu(\tilde z, z) + \tilde\theta\tilde\psi^\mu(\tilde z, z) + \tilde\theta\theta F^\mu(\tilde z, z)$$

  - Worldsheet action in local coordinates $(D_\theta = \partial_\theta + \theta\partial_z)$

$$I_m[X^\mu, \tilde{\mathcal{J}}, \mathcal{J}] = \int_\Sigma [d\tilde z dz|d\tilde\theta d\theta] \tilde D_{\tilde\theta} X^\mu D_\theta X_\mu$$

  - Superconformal algebra on fields generated by

$$\mathcal{S}_{z\theta} = S_{z\theta} + \theta T_{zz} \qquad S_{z\theta} = \frac{1}{2}\psi^\mu \partial_z x_\mu \qquad T_{zz} = -\frac{1}{2}\partial_z x^\mu \partial_z x_\mu + \frac{1}{2}\psi^\mu \partial_z \psi_\mu$$

$$\tilde{\mathcal{S}}_{\tilde z\tilde\theta} = \tilde S_{\tilde z\tilde\theta} + \tilde\theta \tilde T_{\tilde z\tilde z} \qquad \tilde S_{\tilde z\tilde\theta} = \frac{1}{2}\tilde\psi^\mu \partial_{\tilde z} x_\mu \qquad \tilde T_{\tilde z\tilde z} = -\frac{1}{2}\partial_{\tilde z} x^\mu \partial_{\tilde z} x_\mu + \frac{1}{2}\tilde\psi^\mu \partial_{\tilde z} \tilde\psi_\mu$$

Deformations of superconformal structures

- **Under deformation of $\tilde{\mathcal{J}}$ for Σ_L and $\mathcal{J}$ for Σ_R**

$$\delta I = \int_\Sigma [d\tilde{z}dz|d\tilde{\theta}d\theta] \left(H_{\tilde{\theta}}{}^z \mathcal{S}_{z\theta} + \tilde{H}_\theta{}^{\tilde{z}} \tilde{\mathcal{S}}_{\tilde{z}\tilde{\theta}} \right)$$

  – in components by integrating out $\tilde{\theta}, \theta$,

$$\delta I = \int_{\Sigma_{\rm red}} d\tilde{z}dz \left(\mu_{\tilde{z}}{}^z \, T_{zz} + \chi_{\tilde{z}}{}^\theta \, S_{z\theta} + \tilde{\mu}_z{}^{\tilde{z}} \, T_{\tilde{z}\tilde{z}} + \tilde{\chi}_z{}^{\tilde{\theta}} \, \tilde{S}_{\tilde{z}\tilde{\theta}} \right)$$

  – recover Beltrami differentials $\mu, \tilde{\mu}$ and worldsheet gravitino fields $\chi, \tilde{\chi}$

$$H_{\tilde{\theta}}{}^z = \tilde{\theta}(\mu_{\tilde{z}}{}^z + \theta \chi_{\tilde{z}}{}^\theta) \qquad\qquad \tilde{H}_\theta{}^{\tilde{z}} = \theta(\tilde{\mu}_z{}^{\tilde{z}} + \tilde{\theta}\tilde{\chi}_z{}^{\tilde{\theta}})$$

  – Finite deformations of the metric with $\tilde{\mu} = \bar{\mu}$ and $\tilde{\chi} = \bar{\chi}$ integrate to the standard 2-dim $\mathcal{N} = 1$ supergravity action (Brink, Di Vecchia, Howe; Deser, Zumino 1976)

- **Type II superstring perturbation theory requires $\tilde{\mu} \neq \bar{\mu}$ and $\tilde{\chi} \neq \bar{\chi}$**

Type II string amplitude

- **Parametrize deformations $\tilde{H}_\theta{}^{\tilde{z}}, H_{\tilde{\theta}}{}^z$ by slice $\{\tilde{\mathcal{J}}(\tilde{\mathfrak{m}}), \mathcal{J}(\mathfrak{m})\}$ in $\mathfrak{M}_L \times \mathfrak{M}_R$**

$$H_{\tilde{\theta}}{}^z = \tilde{D}_{\tilde{\theta}} V^z + H_A \delta \mathfrak{m}^A \qquad H_A = \partial \mathcal{J}_{\tilde{\theta}}{}^z / \partial \mathfrak{m}^A \qquad \mathfrak{m}^A = (m^i, \zeta^\alpha)$$

$$\tilde{H}_\theta{}^{\tilde{z}} = D_\theta \tilde{V}^{\tilde{z}} + \tilde{H}_{\tilde{A}} \delta \tilde{\mathfrak{m}}^{\tilde{A}} \qquad \tilde{H}_{\tilde{A}} = \partial \mathcal{J}_\theta{}^{\tilde{z}} / \partial \tilde{\mathfrak{m}}^{\tilde{A}} \qquad \tilde{\mathfrak{m}}^A = (\tilde{m}^i, \tilde{\zeta}^\alpha)$$

ghost fields $\quad V^z \to C^z = c^z + \theta \gamma^\theta \qquad H_{\tilde{\theta}}{}^z \to B_{z\theta} = \beta_{z\theta} + \theta b_{zz}$

$$V^{\tilde{z}} \to \tilde{C}^{\tilde{z}} = \tilde{c}^{\tilde{z}} + \tilde{\theta} \tilde{\gamma}^{\tilde{\theta}} \qquad H_\theta{}^{\tilde{z}} \to \tilde{B}_{\tilde{z}\tilde{\theta}} = \tilde{\beta}_{\tilde{z}\tilde{\theta}} + \tilde{\theta} \tilde{b}_{\tilde{z}\tilde{z}}$$

  – Super conformal invariant ghost action

$$I_{\rm gh} = \int_\Sigma [d\tilde{z} dz | d\tilde{\theta} d\theta] \Big(B_{z\theta} \tilde{D}_{\tilde{\theta}} C^z + \tilde{B}_{\tilde{z}\tilde{\theta}} D_\theta C^{\tilde{z}} + B_{z\theta} H_A \delta \mathfrak{m}^A + \tilde{B}_{\tilde{z}\tilde{\theta}} \tilde{H}_{\tilde{A}} \delta \tilde{\mathfrak{m}}^{\tilde{A}} \Big)$$

- **The integrand for the full amplitude is given by**

$$\int D(XB\tilde{B}C\tilde{C})\, \mathcal{V}_1 \cdots \mathcal{V}_n \prod_{\tilde{A},A} [d\tilde{\mathfrak{m}}^{\tilde{A}} d\mathfrak{m}^A]\, \delta(\langle \tilde{B}, \tilde{H}_{\tilde{A}} \rangle) \delta(\langle B, H_A \rangle)\, e^{-I_m - I_{\rm gh}}$$

  – $\mathcal{V}_1 \cdots \mathcal{V}_n$ are BRST-invariant vertex operators.
  – Picture Changing Operator formalism (Friedan, Martinec, Shenker 1986)
    ⋆ may be obtained as singular limit for χ supported at points
    ⋆ globally regular reformulation via "vertical integration" (Sen, Witten 2016)

Loop momenta and Chiral amplitudes

- h **independent loop momenta** p_I^μ **defined to flow across** $\mathfrak{A}_I$ **cycles**

$$p_I^\mu = \oint_{\mathfrak{A}_I} dz \partial_z x^\mu$$

- **Chiral Amplitudes** (ED, Phong 1988)
  - Massless NS bosons with factorized polarization tensor $\varepsilon_i^{\mu\tilde\mu} = \varepsilon_i^\mu \tilde\varepsilon_i^{\tilde\mu}$
  - Chiral amplitude at fixed loop momenta is given by

$$\mathcal{F}_R(\mathcal{J}, \varepsilon_i, k_i, p_I) = \left\langle \mathcal{V}_1 \cdots \mathcal{V}_N \, e^{p_I^\mu \oint_{\mathfrak{B}_I} dz \partial_z x^\mu} \, e^{\int_\Sigma H_{\tilde\theta}{}^z S_{z\theta}} \prod_A \delta(\langle B, H_A \rangle) d\mathfrak{m}^A \right\rangle$$

  - Correlation functions $\langle \cdots \rangle$ computed with chiral Green functions

- **Full Superstring Amplitudes**
  - obtained by pairing left and right and integrating over $\Gamma \in \mathfrak{M}_L \times \mathfrak{M}_R$

$$\mathcal{A}^{(h)}(\varepsilon_i, \tilde\varepsilon_i, k_i) = \int_{\mathbb{R}^{10}} dp_I^\mu \int_\Gamma \mathcal{F}_L(\tilde{\mathcal{J}}, \tilde\varepsilon_i, k_i, p_I^\mu) \, \mathcal{F}_R(\mathcal{J}, \varepsilon_i, k_i, p_I^\mu)$$

  - integration over vertex operator insertion points included in integration over Γ
  - cfr "double copy construction" in supergravity calculations

Parametrization of super moduli

- **Superconformal structure $\mathcal{J} \in \mathfrak{M}_h$ specified by transition functions**
  - Concrete calculations use parametrization by gravitino field $\chi_{\tilde z}{}^\theta$
- **Local parametrization of moduli** (in conformal-invariant theory)
  - Conformal structure J with metric $g = |dz|^2$ in local coordinates $(z, \tilde z)$
  - deform conformal structure by Beltrami differential to $g' = |dz + \mu d\tilde z|^2$
  - realized in CFT by inserting $\int_\Sigma d\tilde z dz \, \mu_{\tilde z}{}^z T_{zz}$ to all orders in μ
- **Local parametrization of supermoduli** (in superconformal-invariant theory)
  - Start with $\Sigma_{\rm red}$ with complex structure given by $J \in \mathfrak{M}_{\rm red}$
  - Deform super conformal structure by inserting T and S

$$\int_{\Sigma_{\rm red}} d\tilde z dz \left(\mu_{\tilde z}{}^z T_{zz} + \chi_{\tilde z}{}^\theta S_{z\theta} \right)$$

  - χ and μ parametrized by local odd coordinates on $\mathfrak{M}_h$
- **For $h = 2$, even spin structures, holó projection $\mathfrak{M}_2 \to \mathcal{M}_2$ exists**
  - via the super period matrix (ED, Phong 2001)
- **For $h \geq 5$ no holó projection $\mathfrak{M}_h \to \mathcal{M}_h$ exists** (Donagi, Witten 2013)

The super period matrix (even spin structures)

- **Start from conformal structure J for $\Sigma_{\rm red}$ with holó 1-forms ω_I**

$$\oint_{\mathfrak{A}_I} \omega_J = \delta_{IJ} \qquad \oint_{\mathfrak{B}_I} \omega_J = \Omega_{IJ} \qquad I, J = 1, 2$$

- **Deform to superconformal structure $\mathcal{J}$ on Σ with superholó forms $\hat\omega_I$**

$$\oint_{\mathfrak{A}_I} \hat\omega_J = \delta_{IJ} \qquad \oint_{\mathfrak{B}_I} \hat\omega_J = \hat\Omega_{IJ} \qquad I, J = 1, 2$$

  – Explicit formula for the super period matrix $\hat\Omega$ for even spin structure δ

$$\hat\Omega_{IJ} = \Omega_{IJ} - \frac{i}{8\pi} \int_{\Sigma^2_{\rm red}} \omega_I(z)\chi(z)S_\delta(z,w|\Omega)\chi(w)\omega_J(w) + \int_{\Sigma_{\rm red}} \mu\,\omega_I\omega_J$$

  – $\hat\Omega_{IJ}$ is locally supersymmetric; $\hat\Omega_{IJ} = \hat\Omega_{JI}$; and ${\rm Im}\,\hat\Omega > 0$
  – Every $\hat\Omega$ corresponds to an ordinary Riemann surface
  – Szegö kernel $S_\delta(z,w|\Omega)$ is non-singular in the interior of $\mathcal{M}_2$

$\Rightarrow$ **Projection using $\hat\Omega$ is holomorphic and natural for genus 2**

Projecting and pairing Chiral Amplitudes

- **Chiral Amplitudes on $\mathfrak{M}_2$**
  - Natural parametrization of $\mathfrak{M}_2$ by $(\hat\Omega_{IJ}, \zeta^\alpha)$ (even spin structure δ)
  - involves measure $d\kappa[\delta](\hat\Omega, \zeta)$ and correlation functions $\mathcal{C}[\delta](\varepsilon_i, k_i, p_I|\hat\Omega, \zeta)$

- **Projection to chiral amplitudes on $\mathcal{M}_2$**
  - by integrating over ζ and summing over δ at fixed $\hat\Omega$

$$\begin{aligned}
\mathcal{R}(\varepsilon_i, k_i, p_I|\hat\Omega) &= \sum_\delta \int_\zeta d\kappa[\delta](\hat\Omega, \zeta)\, \mathcal{C}[\delta](\varepsilon_i, k_i, p_I|\hat\Omega, \zeta) \\
\mathcal{L}(\tilde\varepsilon_i, k_i, p_I|\hat\Omega) &= \sum_{\tilde\delta} \int_{\tilde\zeta} d\kappa[\tilde\delta](\hat\Omega, \tilde\zeta)\, \mathcal{C}[\tilde\delta](\tilde\varepsilon_i, k_i, p_I|\hat\Omega, \tilde\zeta)
\end{aligned}$$

  - for heterotic, $\mathcal{L}$ is chiral half of bosonic string, has no integral in $\tilde\zeta$
  - phase factors determined by $Sp(4,\mathbb{Z})$ modular invariance

- **Pairing left and right chiral amplitudes, integrating over p_I and $\hat\Omega$**

$$\mathcal{A}^{(2)}(\varepsilon_i, \tilde\varepsilon_i, k_i) = \int_{\mathcal{M}_2} d\hat\Omega \int dp_I^\mu\, \mathcal{R}(\varepsilon_i, k_i, p_I|\hat\Omega)\, \overline{\mathcal{L}(\tilde\varepsilon_i, k_i, p_I|\hat\Omega)}$$

  - Integral over p_I is Gaussian and can be carried out explicitly.

Genus two

- **Siegel Upper half space $\mathcal{S}_2$**

$$\mathcal{S}_2 = \{\Omega_{IJ} = \Omega_{JI} \in \mathbb{C} \text{ with } I,J = 1,2 \text{ and } Y = \mathrm{Im}\Omega > 0\}$$

  – $Sp(4,\mathbb{R})$ acts by $\Omega \to (A\Omega + B)(C\Omega + D)^{-1}$

$$M^t J M = J \qquad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

  – $\mathcal{S}_2$ has $Sp(4,\mathbb{R})$-invariant metric ds_2^2 and volume form $d\mu_2$

$$ds_2^2 = \sum_{I,J,K,L=1,2} Y^{-1}_{IJ}\, d\bar\Omega_{JK}\, Y^{-1}_{KL}\, d\Omega_{LI}$$

- **Compact Riemann surfaces Σ**
  – Choose canonical homology basis of $\mathfrak{A}_I, \mathfrak{B}_I$ cycles for $H_1(\Sigma,\mathbb{Z})$.
  – ω_I dual holomorphic (1,0) forms,

$$\oint_{\mathfrak{A}_I} \omega_J = \delta_{IJ} \qquad \oint_{\mathfrak{B}_I} \omega_J = \Omega_{IJ}$$

  – Riemann relations imply $\Omega \in \mathcal{S}_2$;
  – Modular group $Sp(4,\mathbb{Z})$; moduli space $\mathcal{M}_2 = \mathcal{S}_2 / Sp(4,\mathbb{Z})$.

Genus-two Type II four-graviton amplitude

- **Type II four-graviton amplitude** (ED, Phong 2001 – 2005)

$$\mathcal{A}^{(2)}(\varepsilon_i, \tilde\varepsilon_i, k_i) = g_s^2 \, \mathcal{K}\tilde{\mathcal{K}} \int_{\mathcal{M}_2} d\mu_2 \, \mathcal{B}^{(2)}(s_{ij}|\Omega)$$

$$\mathcal{B}^{(2)}(s_{ij}|\Omega) = \int_{\Sigma^4} \frac{\mathcal{Y} \wedge \bar{\mathcal{Y}}}{(\det \mathrm{Im}\, \Omega)^2} \exp \Big(\sum_{i<j} s_{ij} \, G(z_i, z_j|\Omega) \Big)$$

  – $G(z_i, z_j)$ is the genus-two scalar Green function;
  – $\Delta(z_i, z_j)$ is a bi-holomorphic form independent of s, t, u.

$$\Delta(z,w) = \omega_1(z) \wedge \omega_2(w) - \omega_2(z) \wedge \omega_1(w)$$

$$\begin{aligned} \mathcal{Y} = {} & (t-u)\Delta(z_1,z_2) \wedge \Delta(z_3,z_4) + (s-t)\Delta(z_1,z_3) \wedge \Delta(z_4,z_2) \\ & +(u-s)\Delta(z_1,z_4) \wedge \Delta(z_2,z_3) \end{aligned}$$

  – reproduced (with fermions) in pure spinor formulation (Berkovits, Mafra 2005)

- **Singularity structure**
  – For fixed Ω integrations over Σ produce poles in $\mathcal{B}$ at positive integers s_{ij}.
  – The integral over Ω requires analytic continuation beyond $\mathrm{Re}(s_{ij}) = 0$.
  – Branch cuts in s_{ij} starting at integers produced from $\Omega_{11}, \Omega_{22} \to i\infty$

Genus-two Heterotic four-graviton amplitude

- **Heterotic four NS boson amplitude at genus 2** (ED, Phong 2005)

$$\mathcal{A}^{(2)}_{\mathcal{O}}(\varepsilon_i, \tilde\varepsilon_i, k_i) = g_s^2 \, \mathcal{K} \int_{\mathcal{M}_2} d\mu_2 \, \mathcal{B}^{(2)}_{\mathcal{O}}(\tilde\varepsilon_i, k_i | \Omega)$$

$$\mathcal{B}^{(2)}_{\mathcal{O}}(\tilde\varepsilon_i, k_i | \Omega) = \int_{\Sigma^4} \frac{\mathcal{Y} \wedge \overline{\mathcal{W}_{\mathcal{O}}(\tilde\varepsilon_i, k_i)}}{(\det \mathrm{Im}\Omega)^2 \, \overline{\Psi_{10}(\Omega)}} \exp\Big(\sum_{i<j} s_{ij} \, G(z_i, z_j) \Big)$$

  – $\Psi_{10}(\Omega)$ is the Igusa cusp form.

- **Dependence of the operator $\mathcal{O}$ on the channel:**
  - ⋆ 4 gravitons $\mathcal{R}^4$
  - ⋆ 2 gravitons + 2 gauge bosons $\mathcal{R}^2 \mathrm{tr}(\mathcal{F}^2)$;
  - ⋆ 4 gauge bosons $(\mathrm{tr}\mathcal{F}^2)^2$
  - ⋆ 4 gauge bosons $\mathrm{tr}(\mathcal{F}^4)$

  – For example,

$$\mathcal{W}_{\mathcal{R}^4}(\tilde\varepsilon_i, k_i) = \frac{\langle \prod_{i=1}^4 \tilde\varepsilon_i \cdot \bar\partial \tilde x(z_i) \, e^{ik_i \cdot \tilde x(z_i)} \rangle}{\langle \prod_{i=1}^4 e^{ik_i \cdot \tilde x(z_i)} \rangle}$$

  – Gauge parts are obtained by the correlators of the current $(0,1)$-forms.

UV-finiteness and one-loop amplitudes

- **Thanks to modular invariance, all string amplitudes are UV-finite**
  - shown for the closed bosonic string at genus one (Shapiro 1972)
  - holds for all modular invariant superstrings to all loops (i.e. all genera)

- **All chiral amplitudes have a universal loop momentum factor**

$$\mathcal{F}_R(z_i, \varepsilon_i, k_i, p_I|\Omega) = e^{i\pi p_I^\mu \Omega_{IJ} p_J^\mu} \times \cdots$$

  - Modular invariance allows one to choose
    a fundamental domain where $\mathrm{Im}(\Omega)$ bounded from below
  - For genus one, choose the standard fundamental domain

$$\mathcal{H}_1/SL(2,\mathbb{Z}) = \{\tau \in \mathbb{C}, \mathrm{Im}(\tau) > 0, \, |\tau| \geq 1, \, |\mathrm{Re}(\tau)| \leq \frac{1}{2}\}$$

  - Analogous, more complicated, choices to higher genus

$\Rightarrow$ **Uniform Gaussian suppression at large loop momenta**

$\Rightarrow$ **UV finiteness to all genera**

Singularities in the projection $\overline{\mathfrak{M}}_2 \to \overline{\mathcal{M}}_2$

- **Projection $\mathfrak{M}_2 \to \mathcal{M}_2$ is holó, but integration extends to boundary**
  - are there singularities in the projection $\overline{\mathfrak{M}}_2 \to \overline{\mathcal{M}}_2$?

$$\Omega = \begin{pmatrix} \tau & u \\ u & \sigma \end{pmatrix} \qquad \begin{matrix} u \to 0 & \text{separating node} \\ \sigma \to i\infty & \text{non-separating node} \end{matrix}$$

  - Key ingredient in $\hat{\Omega}$ is the Szegö kernel

$$S_\delta(z, w|\Omega) = \frac{\vartheta[\delta](z-w|\Omega)}{\vartheta[\delta](0|\Omega)E(z,w)}$$

  - As $u \to 0$ we have $\vartheta[\delta](0|\Omega) \to \vartheta[\delta_1](0|\tau)\,\vartheta[\delta_2](0|\tau)$
  - Even $\delta = [\delta_1, \delta_2]$ with δ_1, δ_2 odd produces a singularity in S_δ and $\hat{\Omega}$

- **Physical effects**
  - singularity killed by ψ-zero modes in $\mathbb{R}^{10}$ (space-time susy)
  - contribution when susy is broken by radiative corrections (Witten 2013)
  - Two-loop vacuum energy in Heterotic strings on CY orbifold $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$
    - ⋆ is zero for $E_8 \times E_8 \to E_6 \times E_8$ with unbroken susy
    - ⋆ non-zero for $\mathrm{Spin}(32)/\mathbb{Z}_2 \to SO(26) \times U(1)$ with broken susy

(Atick, Sen 1988; $\cdots$; ED, Phong 2013; Berkovits, Witten 2014)

Singularities in the projection $\mathfrak{M}_3 \to \mathcal{M}_3$

- **Some basic structure theorems**
  - A hyper-elliptic surface is a branched double cover of the sphere S^2;
  - All genus 1 and all genus 2 surfaces are hyper-elliptic;
  - Hyper-elliptic surfaces form a co-dim 1 sub-variety in the interior of $\mathcal{M}_3$
    (referred to as the hyper-elliptic divisor)

- **The genus-three period matrix (for even spin structure)**

$$\hat{\Omega}_{IJ} = \Omega_{IJ} - \frac{i}{8\pi} \int\!\!\int \omega_I(z)\chi(z) S_\delta(z,w|\Omega)\chi(w)\omega_J(w) + \mathcal{O}(\chi^4)$$

  - For Ω on the hyper-elliptic divisor of $\mathfrak{M}_3$
    there always exists an even spin structure δ such that $\vartheta[\delta](0|\Omega) = 0$
  - the presence of the extra Dirac zero modes kills effects of this singularity

$\Rightarrow$ Beautiful proposal for the genus 3 superstring measure
(Cacciatori, Dalla Piazza, van Geemen 2008)

- Another even δ does produce a *subtle singularity* in $\hat{\Omega}$ (Witten 2015)

Lectures on Superstring Amplitudes

Part 3: Low energy effective interactions

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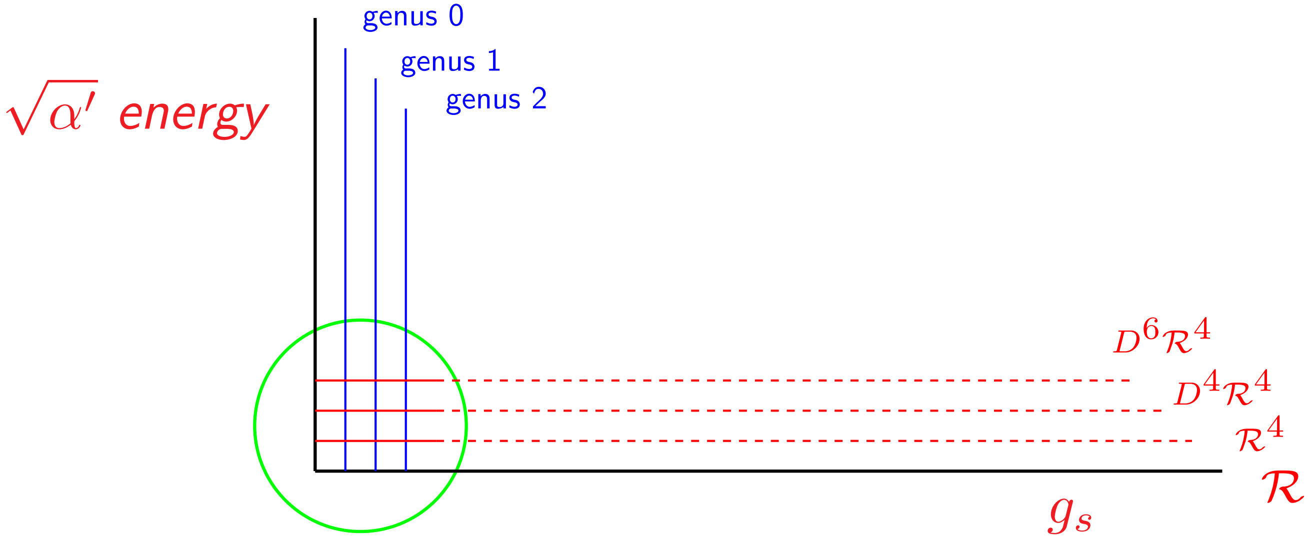
Center for Quantum Mathematics and Physics - 2018
Amplitudes 2018 Summer School

Superstring Perturbation Theory and Supergravity



- **Superstring perturbation theory in powers of the string coupling** g_s
  - holds for weak coupling g_s
  - and for all energies
- **Classical supergravity** "$\mathcal{R}$"
  - leading low energy expansion of string theory
  - holds for all couplings g_s
- **String induced effective interactions** $\mathcal{R}^4, D^4\mathcal{R}^4, D^6\mathcal{R}^4$
  - Evaluated in perturbation theory for $g_s \ll 1$

Low energy expansion of tree-level amplitudes

- **Closed superstring tree-level four-graviton amplitude**

$$\mathcal{A}^{(0)}(\varepsilon_i, \tilde\varepsilon_i, k_i) = \frac{1}{g_s^2} \frac{\mathcal{R}^4}{stu} \frac{\Gamma(1-s)\Gamma(1-t)\Gamma(1-u)}{\Gamma(1+s)\Gamma(1+t)\Gamma(1+u)} \qquad s_{ij} = -\frac{\alpha'}{4}(k_i+k_j)^2$$

  – $\mathcal{R}$ symbolically stands for the Weyl tensor
  – $\mathcal{R}^4$ symbolically stands for a scalar contraction dictated by supersymmetry

- **At low energy** $|s_{ij}| \ll 1$
  – massless string exchanges produce non-local contributions;
  – massive string exchanges produce local effective interactions
  – string-induced corrections to supergravity; eg. in Type II

$$\frac{1}{stu} + 2\zeta(3) + \zeta(5)(s^2+t^2+u^2) + 2\zeta(3)^2 stu + \frac{1}{2}\zeta(7)(s^2+t^2+u^2)^2 + \cdots$$

massless $\quad \mathcal{R}^4 \quad D^4\mathcal{R}^4 \quad D^6\mathcal{R}^4 \quad D^8\mathcal{R}^4$

  – $D^{2k}\mathcal{R}^4$ contraction of covariant derivatives D and $\mathcal{R}^4$

Effective interactions from Type IIB superstrings

- $SL(2,\mathbb{Z})$**-duality symmetry of Type IIB superstrings**
  - requires effective interactions to be $SL(2,\mathbb{Z})$-invariant;
  - Einstein frame metric G_E and $\mathcal{R}_E^4$ invariant under $SL(2,\mathbb{Z})$
  - combine axion χ dilaton Φ in $\rho = \chi + ie^{-\Phi}$
  - transforms by Möbius transformations under $SL(2,\mathbb{Z})$

$$\rho \to \frac{a\rho + b}{c\rho + d} \qquad\qquad a,b,c,d, \in \mathbb{Z}, \quad ad - bc = 1$$

  - Flux fields F_3^R, F_3^{NS} transform linearly; F_5 is invariant

- **Effective interactions from four-graviton amplitude in Type IIB**

$$\int \sqrt{G_E}\left(\mathcal{E}_0(\rho)\mathcal{R}_E^4 + \mathcal{E}_4(\rho)D_E^4\mathcal{R}_E^4 + \mathcal{E}_6(\rho)D_E^6\mathcal{R}_E^4 + \mathcal{E}_8(\rho)D_E^8\mathcal{R}_E^4 + \cdots\right)$$

  - For each p the real-valued function $\mathcal{E}_p(\rho)$ is $SL(2,\mathbb{Z})$-invariant

$$\mathcal{E}_p\left(\frac{a\rho + b}{c\rho + d}\right) = \mathcal{E}_p(\rho)$$

  - namely it is a *real-analytic modular function*
    (not to be confused with meromorphic modular functions)

Real-analytic Eisenstein series

- **A famous family of real-analytic modular functions**
  - For $\mathrm{Re}(s) > 1$ one defines E_s by Kronecker-Eisenstein sums

$$E_s(\rho) = \sum_{m,n\in\mathbb{Z}}{}' \frac{\rho_2^s}{\pi^s |m+\rho n|^{2s}} \qquad \rho = \rho_1 + i\rho_2,\ \rho_1, \rho_2 \in \mathbb{R}$$

  - They are $SL(2,\mathbb{Z})$-invariant and eigenfunctions of the Laplacian

$$\Delta E_s(\rho) = s(1-s)E_s \qquad \Delta = 4\rho_2^2 \partial_\rho \partial_{\bar\rho}$$

  - Their asymptotic expansion for $\rho_2 \to \infty$ = weak string coupling

$$E_s(\rho) = 2\zeta(2s)\frac{\rho_2^s}{\pi^s} + \frac{2\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)\pi^{s-\frac{1}{2}}\rho_2^{s-1}} + \mathcal{O}(e^{-2\pi\rho_2})$$

Effective interactions and Eisenstein series

- **String perturbation theory calculations in string frame**
  - Convert Einstein metric $G_{E\mu\nu}$ to string metric $G_{\mu\nu} = e^{\Phi/2} G_{E\mu\nu}$

$$\sqrt{G_E}\, \mathcal{E}_{2k}(\rho) D_E^{2k} \mathcal{R}_E^4 = \mathbf{e}^{(\mathbf{k}-\mathbf{1})\mathbf{\Phi}/\mathbf{2}} \sqrt{G} \mathcal{E}_{2k}(\rho) D^{2k} \mathcal{R}^4$$

  - Consider combinations involving Eisenstein series

$$\begin{aligned}
\sqrt{G_E}\, E_{\frac{3}{2}}(\rho) \mathcal{R}_E^4 & \approx e^{-2\Phi} \zeta(3) \mathcal{R}^4 + \frac{\pi^2}{3} \mathcal{R}^4 \\
\sqrt{G_E}\, E_{\frac{5}{2}}(\rho) D_E^4 \mathcal{R}_E^4 & \approx e^{-2\Phi} \zeta(5) D^4 \mathcal{R}^4 + \frac{2\pi^4}{135} e^{2\Phi} D^4 \mathcal{R}^4 \\
\sqrt{G_E}\, E_{\frac{3}{2}}(\rho)^2 D_E^6 \mathcal{R}_E^4 & \approx e^{-2\Phi} \zeta(3)^2 D^6 \mathcal{R}^4 + \frac{2\pi^2}{3} \zeta(3) D^6 \mathcal{R}^4 + \frac{\pi^4}{9} e^{-2\Phi} D^6 \mathcal{R}^4 \\
\sqrt{G_E}\, E_{\frac{7}{2}}(\rho) D_E^8 \mathcal{R}_E^4 & \approx e^{-2\Phi} \zeta(7) D^8 \mathcal{R}^4 + \frac{16\pi^6}{14175} e^{-4\Phi} D^8 \mathcal{R}^4
\end{aligned}$$

  - Compare with low energy expansion of tree-level

$$\frac{1}{stu} + 2\zeta(3) + \zeta(5)(s^2+t^2+u^2) + 2\zeta(3)^2 stu - \frac{1}{2}\zeta(7)(s^2+t^2+u^2)^2 + \cdots$$

$$\mathcal{R}^4 \qquad\qquad D^4\mathcal{R}^4 \qquad\qquad D^6\mathcal{R}^4 \qquad\qquad D^8\mathcal{R}^4$$

D-instantons, S-duality and supersymmetry

- **Space-time supersymmetry and S-duality**
  - D-instantons (Green, Gutperle, Vanhove 1996), space-time susy (Green, Sethi 1997)

$$\mathcal{E}_0(\rho) = E_{\frac{3}{2}}(\rho)$$

  - matches tree-level and genus-one results from string perturbation theory
  - Vanishing contribution from genus-two (ED, Gutperle, Phong 2005)

- **M-theory perturbation theory on torus** (Green, Kwon, Vanhove 1999; GV 2005)

$$\begin{aligned} \mathcal{E}_4(\rho) &= E_{\frac{5}{2}}(\rho) \\ (\Delta - 12)\mathcal{E}_6(\rho) &= E_{\frac{3}{2}}(\rho)^2 \end{aligned}$$

  - $\mathcal{E}_4$ matches genus two (ED, Gutperle, Phong 2005)
  - $\mathcal{E}_6$ matches genus-two (ED, Green, Pioline, R. Russo 2014)
    genus three (Gomez, Mafra 2015)

- **Non-renormalization theorems**: no perturbative corrections
  - for $\mathcal{E}_0$ for $h \geq 2$
  - for $\mathcal{E}_4$ for $h \geq 3$
  - for $\mathcal{E}_6$ for $h \geq 4$

Low energy expansion at genus one

- **Recall genus-one Type II four-graviton amplitude** $(\mathcal{M}_1 = \mathcal{H}_1/SL(2,\mathbb{Z}))$

$$\mathcal{A}^{(1)}(\varepsilon_i, \tilde{\varepsilon}_i, k_i) = \mathcal{R}^4 \int_{\mathcal{M}_1} \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2} \mathcal{B}^{(1)}(s_{ij}|\tau)$$

- **Expand the partial amplitude** $\mathcal{B}^{(1)}$ **for** $|s_{ij}| \ll 1$ **for fixed** τ

$$\mathcal{B}^{(1)}(s_{ij}|\tau) = \int_{\Sigma^4} \prod_{i=1}^{4} \frac{d^2 z_i}{\mathrm{Im}\,\tau} \exp\Big(\sum_{i<j} s_{ij}\, G(z_i - z_j|\tau)\Big)$$

  – Scalar Green function $G(z|\tau)$ given by "Kronecker-Eisenstein" Fourier sum

$$G(z|\tau) = \sum_{m,n\in\mathbb{Z}}{}' \frac{\tau_2}{\pi} \frac{e^{2\pi i(m\beta - n\alpha)}}{|m+\tau n|^2} \qquad\qquad z = \alpha + \tau\beta,\ \alpha,\beta \in \mathbb{R}$$

  – For fixed τ the Taylor expansion of $\mathcal{B}^{(1)}$ in s_{ij} converges for $|s_{ij}| < 1$

- **Graphical expansion of** $\mathcal{B}^{(1)}(s_{ij}|\tau) \Longrightarrow$ **Modular Graph Functions of** τ

Modular graph functions

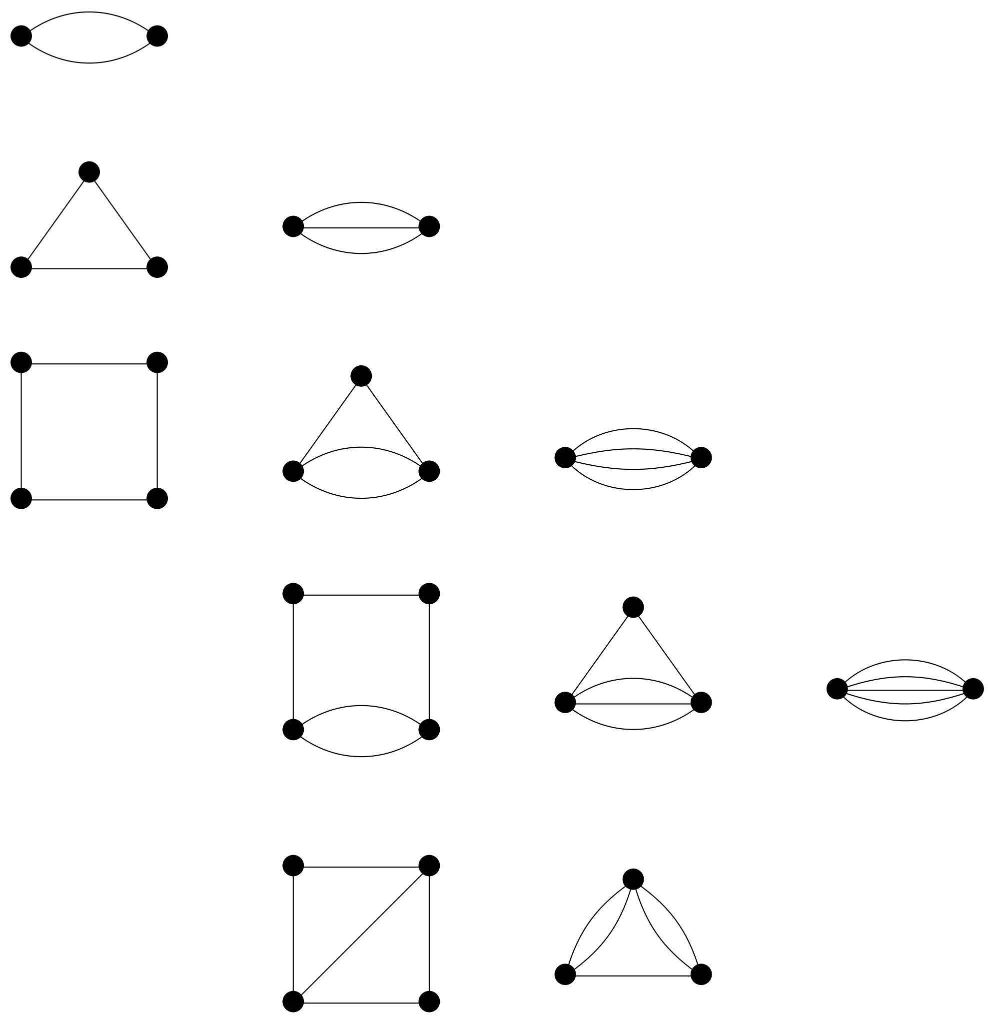
- **Graph in the expansion of $D^{2w}\mathcal{R}^4 \Longrightarrow$ Modular Function**

$D^4\mathcal{R}^4$

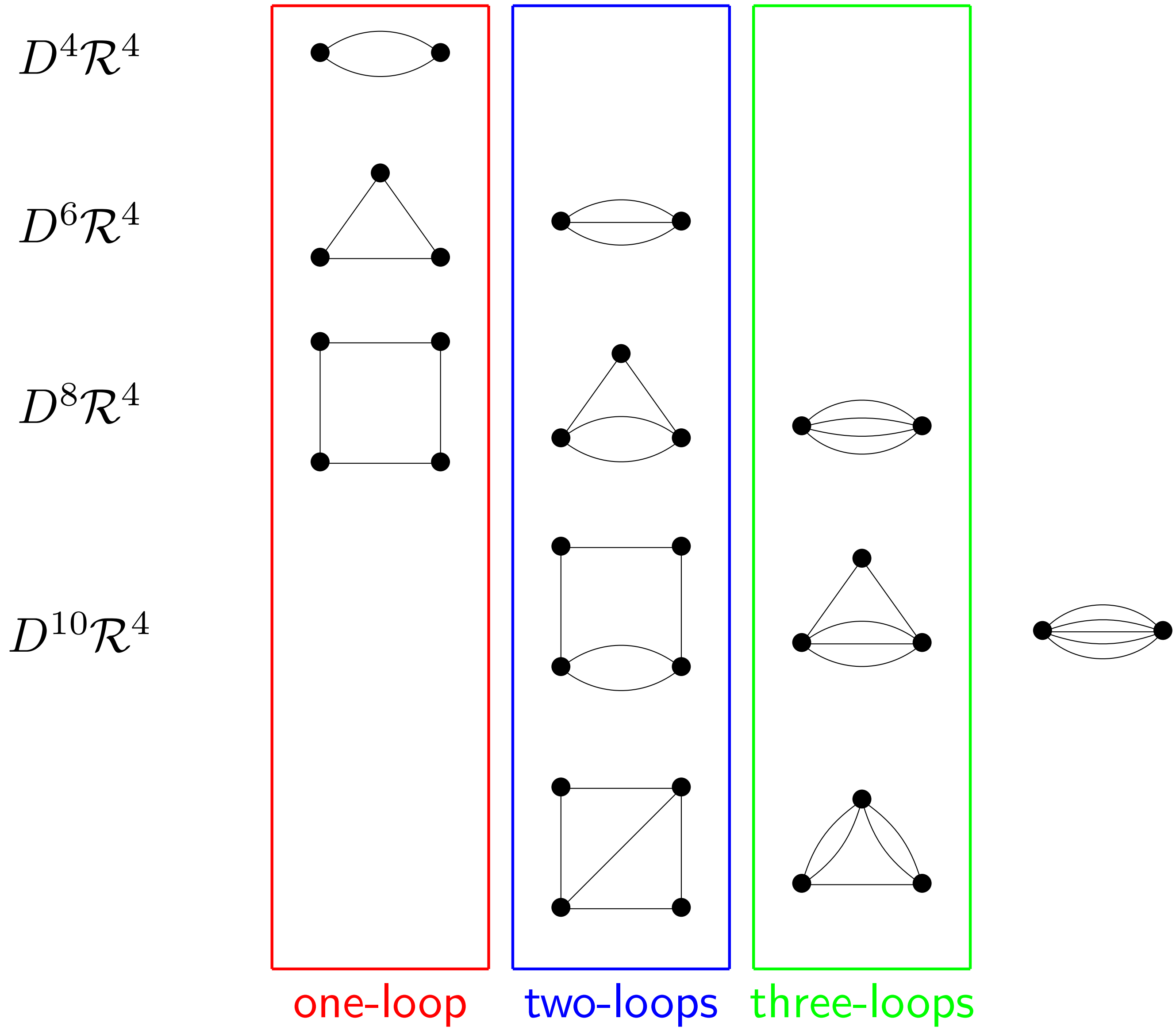
$D^6\mathcal{R}^4$

$D^8\mathcal{R}^4$

$D^{10}\mathcal{R}^4$



Modular graph functions



One-loop : Eisenstein series

- **One-loop worldsheet Feynman diagram with k bivalent vertices**

$$\prod_{i=1}^{k} \int_\Sigma \frac{d^2 z_i}{\tau_2} G(z_i - z_{i+1}|\tau) = \sum_{m,n \in \mathbb{Z}}{}' \frac{\tau_2^k}{\pi^k |m+n\tau|^{2s}} = E_k(\tau)$$

  – Our old friend: non-holomorphic Eisenstein series for integer index k

- **Recall properties of $E_s(\tau)$**
  – absolutely convergent for $\mathrm{Re}(s) > 1$; analytically continue to $s \in \mathbb{C}$
  – reflection relation $\Gamma(s)E_s(\tau) = \Gamma(1-s)E_{1-s}(\tau)$
  – satisfies a Laplace-eigenvalue equation on $\mathcal{H}_1$

$$\Big(\Delta - s(s-1)\Big)E_s(\tau) = 0 \qquad\qquad \Delta = 4\tau_2^2 \partial_\tau \partial_{\bar\tau}$$

  – modular invariant $E_s(\frac{a\tau+b}{c\tau+d}) = E_s(\tau)$ under $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$

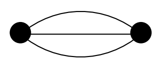
Two-loops : modular graph functions

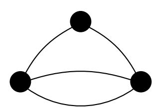
- **Feynman diagrams evaluate to the modular functions**

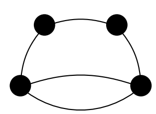
$$C_{a_1,a_2,a_3}(\tau) = \sum_{\substack{m_r,n_r\in\mathbb{Z},\\ r=1,2,3}}{}^{\prime} \, \delta\left(\sum_{r=1}^{3} m_r\right)\left(\sum_{r=1}^{3} n_r\right)\prod_{r=1}^{3}\left(\frac{\tau_2}{\pi|m_r+n_r\tau|^2}\right)^{a_r}$$

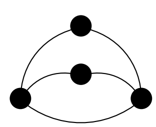
– contribute to $D^{2w}\mathcal{R}^4$ with the *weight* given by $w = a_1 + a_2 + a_3$

– satisfy (inhomogeneous) Laplace-eigenvalue equations

$w = 3$ $\quad C_{1,1,1} =$  $\quad (\Delta - 0)C_{1,1,1} = 6E_3$

$w = 4$ $\quad C_{2,1,1} =$  $\quad (\Delta - 2)C_{2,1,1} = 9E_4 - E_2^2$

$w = 5$ $\quad C_{3,1,1} =$  $\quad (\Delta - 6)C_{3,1,1} = 3C_{2,2,1} + 16E_5 - 4E_2E_3$

$w = 5$ $\quad C_{2,2,1} =$  $\quad (\Delta - 0)C_{2,2,1} = 8E_5$

– Note that eigenvalues are of the form $s(s-1)$ for $s = 1, 2, 3$;

Structure Theorem

- $C_{a,b,c}(\tau)$ **are linear combinations of** $\mathfrak{C}_{w;s;\mathfrak{p}}(\tau)$ **satisfying** (ED, Green, Vanhove 2015)

$$\big(\Delta - s(s-1)\big)\mathfrak{C}_{w;s;\mathfrak{p}} = \mathfrak{F}_{w;s;\mathfrak{p}}\big(E_{s'}\big)$$

  - $\mathfrak{C}_{w;s;\mathfrak{p}}$ and $\mathfrak{F}_{w;s;\mathfrak{p}}$ of weight $w = a+b+c$ (with $E_{s'}$ assigned weight s');
  - $\mathfrak{F}_{w;s;\mathfrak{p}}$ is a polynomial of total degree 2 in $E_{s'}$ with $2 \leq s' \leq w$;

$$s = w - 2\mathfrak{m} \qquad \mathfrak{m} = 1, \cdots, \left[\frac{w-1}{2}\right] \qquad \mathfrak{p} = 0, \cdots, \left[\frac{s-1}{3}\right]$$

- **Examples at low weight**

$w=3$	$s=1$	$0^{(1)}$
$w=4$	$s=2$	$2^{(1)}$
$w=5$	$s=1,3$	$0^{(1)} \oplus 6^{(1)}$
$w=6$	$s=2,4$	$2^{(1)} \oplus 12^{(2)}$
$w=7$	$s=1,3,5$	$0^{(1)} \oplus 6^{(1)} \oplus 20^{(2)}$

- **System of differential relations to all loop orders** (ED, Green, Kaidi, Vanhove 2016)
- **Relation with polylogarithms & multiple zeta values**
  (ED, Green, Vanhove 2015; Francis Brown 2017)

Type IIB effective interactions at genus-two

- **Recall Type II four-graviton amplitude at genus 2,**

$$\mathcal{A}^{(2)}(\varepsilon_i, k_i) = \mathcal{R}^4 \int_{\mathcal{M}_2} d\mu_2\, \mathcal{B}^{(2)}(s_{ij}|\Omega)$$

$$\mathcal{B}^{(2)}(s_{ij}|\Omega) = \int_{\Sigma^4} \mathcal{Y} \wedge \bar{\mathcal{Y}} \exp \sum_{i<j} s_{ij}\, G(z_i, z_j)$$

– $\mathcal{Y} = (s-t)\Delta(z_1, z_3) \wedge \Delta(z_4, z_2) + 2$ permutations;
– $\Delta(z_i, z_j)$ is a holomorphic form independent of s, t, u.

- **Contributions to local effective interactions,**

– $\mathcal{R}^4$: zero, since $\mathcal{Y}$ vanishes for $s = t = u = 0$;
– $D^4\mathcal{R}^4$: non-zero, $\mathcal{B}^{(2)}$ constant on $\mathcal{M}_2$;
– $D^6\mathcal{R}^4$: non-zero, one power of G brought down in integral over Σ^4;

$$\mathcal{B}^{(2)}(s_{ij}|\Omega) = 32(s^2 + t^2 + u^2) + 192\, stu\, \varphi(\Omega) + \mathcal{O}(s^4, \cdots)$$

– $\varphi(\Omega)$ coincides with the Kawazumi-Zhang invariant.

The Zhang-Kawazumi invariant for genus-two

- **The ZK-invariant is given as follows**

$$8\varphi(\Omega) = \sum_{I,J,K,L} \left(Y^{-1}_{IJ} Y^{-1}_{KL} - 2 Y^{-1}_{IL} Y^{-1}_{JK} \right) \int_{\Sigma^2} G(x,y) \omega_I(x) \overline{\omega_J(x)} \omega_K(y) \overline{\omega_L(y)}$$

  – equivalent to definition via Arakelov geometry (Zhang 2007, Kawazumi 2008)

- **Coefficient of genus-two $D^6 \mathcal{R}^4$ interaction involves $\int_{\mathcal{M}_2} d\mu_2 \, \varphi(\Omega)$**

  – Direct evaluation appeared completely out of reach ... until ...

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- **Theorem** (ED, Green, Pioline, R. Russo 2014)

$$(\Delta - 5)\varphi = -2\pi \delta_{SN}$$

  - Δ is the Laplace-Beltrami operator on $\mathcal{M}_2$ with Siegel metric ds^2_2;
  - δ_{SN} has support on separating node (into two genus-one surfaces)
  - The integral over $\mathcal{M}_2$ reduces to an integral over $\partial \mathcal{M}_2$

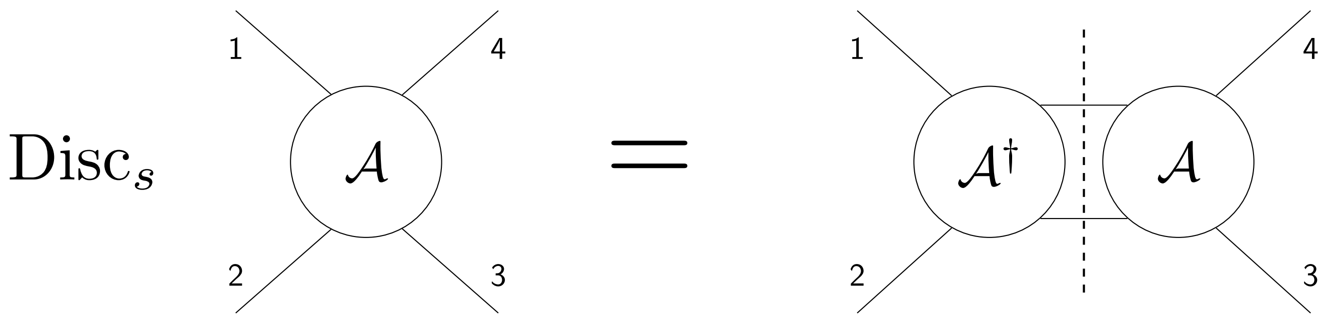
$$\int_{\mathcal{M}_2} d\mu_2 \, \varphi = \frac{1}{5} \int_{\mathcal{M}_2} d\mu_2 \Big(\Delta \varphi + 2\pi \delta_{SN} \Big) = \frac{2\pi^3}{45}$$

  - Exact agreement with predictions from S-duality and supersymmetry

Non-analytic contributions at low energy

- **Non-analytic parts of the amplitudes at low energy**
  - arise from boundary of moduli space contribution to the integral over $\mathcal{B}$
  - dominant contribution at low energy is from supergravity
  - plus string corrections

- **Look at two-particle unitarity cut in the s-channel**



$$i \, \mathrm{Disc}_s \, \mathcal{A}_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(p_1,p_2,p_3,p_4) \; =$$

$$\int \frac{d^{10}k}{(2\pi)^{10}} \delta(k^2)\delta((q-k)^2)\mathcal{A}_{\varepsilon_1,\varepsilon_2,\varepsilon_r,\varepsilon_s}(p_1,p_2,-k,k-q)\mathcal{A}_{\varepsilon_r,\varepsilon_s,\varepsilon_3,\varepsilon_4}(k,q-k,p_3,p_4)$$

Non-analytic part of the genus-one amplitude

- **Obtain the genus-one discontinuity from tree-level**
  - Use the fact that the kinematic factor is the same at all genera h

$$\mathcal{A}^{(h)}_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(p_1,p_2,p_3,p_4) = \mathcal{R}^4_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(p_1,p_2,p_3,p_4)\mathcal{A}^{(h)}_{\rm red}(s,t,u)$$

  - and satisfies the recursive formula (Bern, Dixon, Dunbar, Perelstein, Rozowsky 1998)

$$\sum_{\varepsilon_r,\varepsilon_s} \mathcal{R}^4_{\varepsilon_1,\varepsilon_2,\varepsilon_r,\varepsilon_s}(p_1,p_2,-k,k-q)\mathcal{R}^4_{\varepsilon_r,\varepsilon_s,\varepsilon_3,\varepsilon_4}(k,q-k,p_3,p_4) = s^4\,\mathcal{R}^4_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(p_1,p_2,p_3,p_4)$$

  - to obtain an effective discontinuity formula

$$i\,{\rm Disc}_s\,\mathcal{A}^{(1)}_{\rm red}(s,t,u) = \int \frac{d^{10}k}{(2\pi)^{10}}\delta(k^2)\delta((q-k)^2)\mathcal{A}^{(0)}_{\rm red}(s,t',u')\mathcal{A}^{(0)}_{\rm red}(s,t'',u'')$$

  - where $t' = -(p_1-k)^2,\ u' = -(p_2-k)^4,\ t'' = -(p_4-k)^2,\ u'' = -(p_3-k)^2$

$$\mathcal{A}^{(0)}_{\rm red}(s,t,u) = \frac{1}{stu} + 2\zeta(3) + \zeta(5)(s^2+t^2+u^2) + \cdots$$

- **Substitution into s-channel unitarity relation gives** (by power-counting)

$$\begin{aligned}
{\rm Disc}_s\,\mathcal{A}^{(1)}_{\rm red}(s,t,u) &= \#s + \#\zeta(3)s^4 + \#\zeta(5)s^6 + \cdots \\
\mathcal{A}^{(1)}_{\rm red}(s,t,u) &= \#s\ln(-s) + \#\zeta(3)s^4\ln(-s) + \#\zeta(5)s^6\ln(-s) + \cdots
\end{aligned}$$

Absence of non-analytic contributions

- **Discontinuity relation gives non-analytic contributions**

  – At genus-one use previously obtained result

$$\mathcal{A}^{(1)}_{\rm red}(s,t,u) = \# s \ln(-s) + \#\zeta(3) s^4 \ln(-s) + \#\zeta(5) s^6 \ln(-s) + \cdots$$

  – Effective interaction $D^2\mathcal{R}^4$ vanishes by $s+t+u=0$

  – Genus-one $\mathcal{R}^4$, $D^4\mathcal{R}^4$, $D^6\mathcal{R}^4$, $D^{10}\mathcal{R}^4$ effective interactions are completely determined by the analytic part of the amplitude

  – Local effective interaction $D^8\mathcal{R}^4$ can be fixed only after non-analytic part has been properly normalized

Non-analytic plus analytic parts from genus-one amplitude

- **Derivation of full genus-one $D^8\mathcal{R}^4$ from string theory amplitude**

  – non-analytic part arises from $\tau \to i\infty$: partition moduli space

$$\mathcal{M}_1 = \mathcal{M}_{1L} \cup \mathcal{M}_{1R} \qquad \begin{array}{l} \mathcal{M}_{1L} = \{\tau \in \mathcal{M}_1, \, \mathrm{Im}(\tau) < L\} \\ \mathcal{M}_{1R} = \{\tau \in \mathcal{M}_1, \, \mathrm{Im}(\tau) > L\} \end{array}$$

  – Full amplitude is a sum $\mathcal{A}^{(1)} = \mathcal{A}^{(1)}_L + \mathcal{A}^{(1)}_R$

$$\mathcal{A}^{(1)}_{L,R}(\varepsilon_i, \tilde\varepsilon_i, k_i) = \mathcal{R}^4 \int_{\mathcal{M}_{1L,R}} \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2} \mathcal{B}^{(1)}(s_{ij}|\tau)$$

  – Both $\mathcal{A}^{(1)}_L$, $\mathcal{A}^{(1)}_R$ depend on L, but sum is independent of L

  – $\mathcal{A}^{(1)}_L$ is analytic in s_{ij} but $\mathcal{A}^{(1)}_R$ exhibits non-analyticity at $s_{ij} = 0$

Explicit calculation for $D^8\mathcal{R}^4$

- **Since $\mathcal{A}_L^{(1)}$ is analytic in s_{ij}, evaluate using modular graph functions**

$$\mathcal{A}_L^{(1)} = \frac{2\pi\zeta(3)}{45}\mathcal{R}^4 \left(\ln L - \frac{1}{4} + \ln 2 + \frac{\zeta'(4)}{\zeta(4)} - \frac{\zeta'(3)}{\zeta(3)} \right) + \text{power-behaved in } L$$

- **For $L \gg 1$, approximate integrand of $\mathcal{A}_R^{(1)}$ by supergravity + corrections**

$$\begin{aligned}\mathcal{A}^{(1)}\Big|_{D^8\mathcal{R}^4} \quad = \quad & \frac{4\pi\zeta(3)}{45}\left(\frac{17}{5} - \frac{1}{4} + \ln 2 + \frac{\zeta'(4)}{\zeta(4)} - \frac{\zeta'(3)}{\zeta(3)}\right)(s^4+t^4+u^4)\,\mathcal{R}^4 \\ & -\frac{4\zeta(3)}{45}\Big(s^4\ln(-2\pi s) + t^4\ln(-2\pi t) + u^4\ln(-2\pi u)\Big)\mathcal{R}^4\end{aligned}$$

  - Note: no ambiguities, no infinities, no renormalization required !
  - Transcendentality ... (ED, Green, in progress)

- **Genus-two story ...**

  (ED, Green, Pioline 2017, 2018, and in progress)

Outlook

- **Some additional developments**
  - Clarification of super Riemann surfaces with R-punctures (Witten 2012)
  - There exists a super-period matrix for R-punctures (Witten; ED, Phong 2015)
  - New relations between open and closed string amplitudes (Schlotterer et al.)

- **Some outstanding issues**
  - Systematic structure of low energy effective interactions w/ Green, Pioline
    - ⋆ in terms of properties of modular graph functions
    - ⋆ calculation without requiring subtleties of supermoduli space
    - ⋆ UV divergences in supergravity and effective interactions
  - Ambi-twistor strings
  - string perturbation theory on curved spaces with RR flux, e.g. $AdS_5 \times S^5$