Five-dimensional gauge theory via holography

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Introduction

• Standard QFT textbooks tell us:
  – *gauge theory is renormalizable only in four or fewer dimensions*

• Dynamics of branes in M-theory and Type II suggest otherwise:
  – *Six-dimensional superconformal theories with 32 or 16 supercharges*;
  – *Five-dimensional superconformal fixed points with 16 supercharges*
  – *Possess Coulomb branch and Higgs branch deformations by relevant operators to non-conformal theories*
Overview

- **Field theory approach** (review)
  - *Existence of finite Coulomb branch and super-conformal phase*

- **String/branes approach** (review)
  - *D5/NS5 webs in Type IIB*

- **Holographic approach to the super-conformal phase**
  - *Type IIB supergravity on $AdS_6 \times S^2$ warped over Riemann surface $\Sigma$*
  - *Obtain exact local solutions to the BPS equations for 16 supersymmetries*
  - *Construct global solutions*
  - *Open problems*
Bibliography

Key earlier work

- Five-dimensional SUSY Field Theories, Non-trivial Fixed Points, and String Dynamics, N. Seiberg, hepth/9608111;
- Five-Dimensional Supersymmetric Gauge Theories and Degenerations of Calabi-Yau Spaces, K. Intriligator, D.R. Morrison, N. Seiberg, hepth/9702198;

Our papers

- Warped $\text{AdS}_6 \times S^2$ in Type IIB supergravity I: Local Solutions, ED, Michael Gutperle, Andreas Karch, Christoph F. Uhlemann, arXiv:1606.01254;
- Holographic duals for five-dimensional superconformal quantum field theories, ED, Michael Gutperle, Christoph F. Uhlemann, arXiv:1611.09411;
- Warped $\text{AdS}_6 \times S^2$ in Type IIB supergravity II: Global Solutions and five-brane webs, ED, Michael Gutperle, Christoph F. Uhlemann, arXiv:1703.08186.
**Five-dimensional gauge theory**

- Five-dimensional gauge theory (e.g. $SU(N)$ gauge group)

\[ \mathcal{L} \sim g^{-2} \text{tr}(F^2) + \frac{c}{24\pi^2} \text{tr}(A \wedge F \wedge F + \cdots) \]

- $[g^{-2}]$ = mass and hence perturbatively non-renormalizable;
- $c$ quantized in integers by gauge invariance;
- View $\text{tr}(F^2)$ as relevant deformation on marginal CS term?
- Theory naturally contains solitonic states
  * 4-dim instantons $\rightarrow$ 5-dim particles, $U(1)_I$ charge $J = \text{tr}(F \wedge F)$
  * 4-dim magnetic monopoles $\rightarrow$ 5-dim strings.

- cfr Three-dimensional gauge theory

\[ \mathcal{L} \sim g^{-2} \text{tr}(F^2) + \frac{c}{2\pi} \text{tr}(A \wedge F + \cdots) \]

- $g^2$ has dimensions of mass, hence perturbatively renormalizable;
- $c$ quantized in integers by gauge invariance;
- View $\text{tr}(F^2)$ as irrelevant deformation on marginal CS term.
Five-dimensional supersymmetry

- Minimal Poincaré supersymmetry in five dimensions
  - has 2 supersymmetry spinor generators = 8 real supersymmetries
  - and thus $SU(2)_R$ symmetry
  - under $S^1$ compactification reduces to $\mathcal{N} = 2$ in $d = 4$.

- Supermultiplets
  - gauge multiplet $A = (A_\mu, \lambda_\alpha, \phi)$ with $\phi$ a real scalar;
  - hypermultiplet $H = (H_a, \psi)$ with $H_a$ four real scalars;

- Super-conformal symmetry
  - the conformal algebra in $d \geq 3$ dimensions is $SO(2, d)$
  - the superconformal algebra contains $SO(2, d)$, the R-symmetry algebra, and fermionic generators which are spinors under $SO(2, d)$

\[
\begin{align*}
  d = 3 &\quad OSp(2m|4) &\quad SO(2, 3) = Sp(4, \mathbb{R}), &\quad m = 1, 2, 3, 4 \\
  d = 4 &\quad SU(2, 2|m) &\quad SO(2, 4) = SU(2, 2), &\quad m = 1, 2, 3, 4 \\
  d = 5 &\quad F(4) &\quad SO(2, 5) \oplus SU(2) \text{ max bosonic subalgebra} \\
  d = 6 &\quad OSp(8^*|2m) &\quad SO(2, 6) = SO(8^*), &\quad m = 1, 2 \\
\end{align*}
\]

- maximal 32 supersymmetries in $d = 3, 4, 6$ but only 16 in $d = 5$. 
The Coulomb and Higgs branches

- When hypermultiplets are included, both branches exist,
  - Coulomb branch: gauge scalars acquire vevs \(\langle \phi \rangle \neq 0\)
  - Higgs branch: hypermultiplet scalars acquire vevs \(\langle H \rangle \neq 0\)

- Reach super-conformal fixed point via Coulomb branch (Seiberg, 1996)
  - generically \(SU(N) \rightarrow U(1)^{N-1}/\text{Weyl}\)
  - \(U(1)\) gauge supermultiplets \(A^i = (A^i_\mu, \lambda^i_\alpha, \phi^i)\)
    with \(i = 1, \ldots, N, \sum_i A^i = 0\)
The pre-potential

• Dynamics in the Coulomb branch is governed by a pre-potential $\mathcal{F}(A^i)$
  – just as for $d = 4, \mathcal{N} = 2$ (Seiberg-Witten 1994)
  – bosonic part of the Lagrangian dictated by supersymmetry

\[
\mathcal{L} \sim \sum_{i,j} \partial_i \partial_j \mathcal{F}(\phi) \left( F^i F^j + \partial \phi^i \partial \phi^j \right) + \sum_{i,j,k} \partial_i \partial_j \partial_k \mathcal{F}(\phi) \left( A^i \wedge F^j \wedge F^k + \cdots \right)
\]

  – Gauge invariance $A^i \rightarrow A^i + d\theta^i$ requires $\partial^3 \mathcal{F}$ to be constant.
  – Hence the pre-potential is at most cubic in $\phi^i, A^i$.
  – Quantum corrections at one-loop only are due to massive BPS states,
  – no instanton contributions (in contrast with $d = 4, \mathcal{N} = 2$).

• Exact pre-potential for $SU(N)$ with $N_f$ hypermultiplets in the $N$ of $SU(N)$

\[
\mathcal{F}(\phi) = \frac{1}{2g_0^2} \sum_i \phi_i^2 + \frac{c}{6} \sum_i (\phi_i)^3 + \frac{1}{6} \sum_{i<j} |\phi_i - \phi_j|^3 - \frac{1}{12} \sum_{f=1}^{N_f} \sum_i |\phi_i + m_f|^3
\]

  – the bare coupling $g_0^2$ is a UV cutoff, $m_f$ are hypermultiplet masses.
Dynamics on the Coulomb branch

- Regularity requires the gauge kinetic energy to have positive sign,
  \[ \partial_i \partial_j F \text{ must be positive for } \phi \in \mathbb{R}^{N-1}/\text{Weyl} \]

- For \( SU(2) \) gauge group \( \phi = \phi_1 = -\phi_2 \), and \( N_f \) hypermultiplets,
  \[
  \frac{1}{g^2(\phi)} = \frac{1}{g_0^2} + 2|\phi| - \frac{1}{4} \sum_{f=1}^{N_f} |\phi - m_f| \quad \frac{1}{g^2(\phi)} = \partial^2 F(\phi)
  \]

- Requiring \( g^2(\phi) > 0 \) for regularity,
  - we can remove the UV cutoff, \( g_0^2 \to \infty \), provided \( N_f \leq 7 \),
  - leaving a UV finite theory on the Coulomb branch.
  - Superconformal fixed point as \( \langle \phi \rangle, m_f \to 0 \) is strongly coupled.
  - Global \( SO(2N_f) \) symmetry of hypermultiplets, and \( U(1)_I \) of “instantons”
    enhanced to \( E_{N_f+1} \) global symmetry \( (E_8, E_7, E_6, SO(10), SU(5), \ldots) \)

- Extends to \( SU(N) \) with \( N_f \) hypermultiplets provided \( N_f \leq 2N \).
Supersymmetric field theories from branes

- Standard cases have maximal supersymmetry
  - 16 Poincaré supercharges
  - in the near-horizon limit enhanced to 32 conformal supercharges

<table>
<thead>
<tr>
<th>dim</th>
<th>theory</th>
<th>brane</th>
<th>near-horizon geometry</th>
<th>asymptotic symmetry</th>
</tr>
</thead>
<tbody>
<tr>
<td>d=3</td>
<td>M-theory</td>
<td>M2</td>
<td>$AdS_4 \times S^7$</td>
<td>$SO(2, 3) \times SO(8)$</td>
</tr>
<tr>
<td>d=4</td>
<td>Type IIB</td>
<td>D3</td>
<td>$AdS_5 \times S^5$</td>
<td>$SO(2, 4) \times SO(6)$</td>
</tr>
<tr>
<td>d=6</td>
<td>M-theory</td>
<td>M5</td>
<td>$AdS_7 \times S^4$</td>
<td>$SO(2, 6) \times SO(5)$</td>
</tr>
</tbody>
</table>

- For $d = 5$, superconformal $F(4)$ is unique and has 16 supercharges (8 Poincaré)

- Brane approaches to five-dimensional gauge theory
  - D4 probe brane and parallel D8 branes in massive Type IIA
    (Seiberg, 1996) and (Brandhuber, Oz 1999)
  - D5 intersecting NS5 branes in Type IIB
    (Aharony, Hanany 1997)

- M-theory on 6-dim Calabi-Yau approach to five-dimensional fixed points
  (Morrison, Seiberg, 1996)
Five-branes in Type IIB string theory

- D5 and NS5 branes intersecting along a five-dimensional space-time
  - Poincaré $ISO(1, 4)$ invariant along 01234 parallel directions
  - $SO(3)$ invariant along 789-transverse directions
  - has 8 Poincaré supersymmetries

\[
\begin{array}{cccccccccc}
& 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\hline
\text{D5} & \times & \times & \times & \times & \times & \times & \times & & & \\
\text{NS5} & \times & \times & \times & \times & \times & \times & \times & & & \\
\end{array}
\]

- D5 and NS5 transform under $SL(2, \mathbb{Z})$ duality of Type IIB (Schwarz 1995)
  - $(p, q)$ five-branes with $p, q \in \mathbb{Z}$
  - $x_5$ labels positions of NS5 branes,
  - $x_6$ labels positions of D5 branes

\[\begin{array}{cc}
x_6 \\
x_5 \\
\end{array}
\begin{array}{cc}
(1, 0) & \text{D5} \\
& \text{NS5} \\
\end{array}
\begin{array}{cc}
(0, 1) \\
(1, 1) \\
\end{array}\]
(p, q) brane webs

• (p, q)-brane intersections conserve p, q-charges due to $SL(2, \mathbb{Z})$ symmetry

• $N$ parallel D5 branes suspended between two semi-infinite branes
  – non-coincident: $U(1)^{N-1}$ gauge theory plus massive $W$-bosons
  – coincident: $SU(N)$ gauge theory
  – superconformal: web collapses to a single point
Near-horizon limit

• Take the near-horizon limit of a \((p, q)\) web configuration
  – with a large number \(N\) of coincident D5 branes

<table>
<thead>
<tr>
<th>branes</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>D5</td>
<td>×</td>
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<td>×</td>
<td>×</td>
<td>×</td>
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<tr>
<td>NS5</td>
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</table>

– radial coordinate in 789 direction combines with 01234 to \(AdS_6\)
– remaining angular directions of 789 give \(S^2\)
– with combined isometries \(SO(2, 5) \times SO(3)\)

• Total space-time geometry

\[AdS_6 \times S^2 \times \Sigma\]

– where \(AdS_6 \times S^2\) is warped over the two-dimensional surface \(\Sigma\)
– \(\Sigma\) contains the structure of the web in the near-horizon limit

• Our approach: obtain the Type IIB supergravity solutions directly
  – several earlier attempts (no physically interesting solutions)
**Type IIB supergravity**

- The fields of Type IIB supergravity are

<table>
<thead>
<tr>
<th>Field</th>
<th>Description</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_{MN}$</td>
<td>metric</td>
<td>$P, Q \sim dB$ (contains $\chi, \Phi$)</td>
</tr>
<tr>
<td>$B$</td>
<td>axion/dilaton</td>
<td>$G$ (contains NSNS, RR)</td>
</tr>
<tr>
<td>$C_2$</td>
<td>complex 2-form</td>
<td>$F_5 \ast F_5 = F_5$</td>
</tr>
<tr>
<td>$C_4$</td>
<td>real 4-form</td>
<td></td>
</tr>
<tr>
<td>$\psi_M$</td>
<td>gravitino</td>
<td>Weyl spinor</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>dilatino</td>
<td>Weyl spinor</td>
</tr>
</tbody>
</table>

- Type IIB supergravity is invariant under global $SL(2, \mathbb{R}) = SU(1, 1)$
  - Einstein-frame metric and $F_5$ are invariant,
  - dilaton/axion $B$ in coset $SU(1, 1)/U(1)$, complex $C_2$ transforms linearly,

\[
B \rightarrow \frac{uB + v}{\bar{v}B + \bar{u}} \quad C_2 \rightarrow uC_2 + v\bar{C}_2 \quad |u|^2 - |v|^2 = 1
\]

- Bianchi identities and field equations.
Supersymmetric solutions and BPS equations

• Susy variations in Type IIB at vanishing Fermi fields

$$\delta \lambda = iP \cdot \Gamma B^{-1} \varepsilon^* - \frac{i}{4} (G \cdot \Gamma) \varepsilon$$

$$\delta \psi_M = D_M \varepsilon + \frac{i}{4} (F_5 \cdot \Gamma) \Gamma_M \varepsilon - \frac{1}{16} \left( \Gamma_M (G \cdot \Gamma) + 2 (G \cdot \Gamma) \Gamma_M \right) B^{-1} \varepsilon^*$$

- $\Gamma_M$ are Dirac matrices, $(\Gamma_M)^* = B \Gamma_M B^{-1}$ effects charge conjugation.

- A configuration is supersymmetric if $\delta \psi_M = \delta \lambda = 0$ has solutions with $\varepsilon \neq 0$

- A configuration is half-BPS if there are 16 linearly independent solutions $\varepsilon$

• BPS equations remind of Lax equations in integrable systems

field equations $\Leftrightarrow$ integrability of system of linear differential eqs

- with 32 susys, BPS eqs imply all Bianchi and field equations;
- with $\geq 28$ susy, interesting results (Gran, Gutowski, Papadopoulos)
- with 16 susys, BPS eqs plus some Bianchi identities imply all the field eqs;
The supergravity Ansatz

- The $SO(2,5) \times SO(3)$ symmetry dictates the space-time structure,

$$AdS_6 \times S^2 \text{ warped over a Riemann surface } \Sigma$$

- The metric and flux fields are restricted by symmetry,

\[
\begin{align*}
 ds^2 &= f_6^2 \, d\hat{s}_{AdS_6}^2 + f_2^2 \, d\hat{s}_{S^2}^2 + ds_{\Sigma}^2 \\
 F_3 &= g_a \, e^a \wedge e^6 \wedge e^7 \\
 P &= p_a \, e^a \\
 Q &= q_a \, e^a \\
 F_5 &= 0
\end{align*}
\]

- $d\hat{s}_{AdS_6}^2$ and $d\hat{s}_{S^2}^2$ have unit radius “round” metrics;

- $e^A$ is orthonormal frame, $A = 6, 7$ for $S^2$ and $A = a = 8, 9$ for $\Sigma$

- $ds_{\Sigma}^2 = e^a \otimes e^b \delta_{ab}$ with $a, b = 8, 9$. 
Reducing the BPS equations

• Use Killing spinors on $AdS_6 \times S^2$ as basis for the susy parameter $\varepsilon$,

$$\varepsilon = \sum_{\eta_1, \eta_2} \chi^{\eta_1, \eta_2} \otimes \zeta_{\eta_1, \eta_2}$$

- $\chi^{\eta_1, \eta_2}$ fixed basis of Killing spinors, $\eta_1 = \pm$ and $\eta_2 = \pm$ independently;
- $\zeta_{\eta_1, \eta_2}$ are 2-component spinors on $\Sigma$.

• The BPS equations reduce to a system of 4 spinor equations,

\begin{align*}
0 &= 4p_a \gamma^a \gamma^9 \zeta^* - g_a \tau^{(2)}_3 \gamma^a \zeta \\
0 &= -\frac{i}{2f_6} \tau^{(1)}_2 \otimes \tau^{(2)}_1 \zeta + \frac{D_a f_6}{2f_6} \gamma^a \zeta - \frac{1}{16} g_a \tau^{(2)}_3 \gamma^a \gamma^9 \zeta^* \\
0 &= \frac{1}{2f_2} \tau^{(2)}_2 \zeta + \frac{D_a f_2}{2f_2} \gamma^a \zeta + \frac{3}{16} g_a \tau^{(2)}_3 \gamma^a \gamma^9 \zeta^* \\
0 &= \left( D_a + \frac{i}{2} \omega_a \sigma^3 - \frac{i}{2} q_a \right) \zeta + \frac{3}{16} g_a \tau^{(2)}_3 \gamma^9 \zeta^* - \frac{1}{16} g_b \tau^{(2)}_3 \gamma^b \gamma^9 \zeta^*
\end{align*}

- Derivative $D_a$ and connection $\omega_a$ are defined with respect to the frame $e^a$,
- $\tau^{(1,2)}$ are Pauli matrices acting on indices $\eta_{1,2}$.
Decoupling the reduced BPS equations

- Algebraic methods used to restrict range of $\zeta$ (ED, Estes, Gutperle 2007)

\[
\bar{\alpha} = \zeta_{+++} = -\zeta_{---} = -i\nu\zeta_{++-} = +i\nu\zeta_{--+} \quad \nu^2 = 1
\]

\[
\bar{\beta} = \zeta_{---} = +\zeta_{+++} = -i\nu\zeta_{--+} = -i\nu\zeta_{+-+}
\]

- The radii $f_6$ and $f_2$ may be obtained algebraically in terms of $\alpha, \beta$,

\[
f_6 = 3(|\alpha|^2 + |\beta|^2) \quad f_2 = -\nu(|\alpha|^2 - |\beta|^2)
\]

- Choose local complex coordinates $(w, \bar{w})$ with $e^z = e^8 + ie^9 = \rho \, dw$

- Use Bianchi identities to express the fields $p_z, q_z, p_{\bar{z}}, q_{\bar{z}}$ in terms of $B$

- Two of the four differential equations may be integrated exactly,

\[
\rho\bar{\alpha}^2 = f(\kappa_+ + B \, \kappa_-) \quad \kappa_\pm = \partial_w A_\pm
\]

\[
\rho\bar{\beta}^2 = f(\bar{B} \, \kappa_+ + \kappa_-) \quad f^{-2} = 1 - |B|^2
\]

- where $A_\pm$ are arbitrary locally holomorphic functions on $\Sigma$. 
Secret to integrability

- The remaining reduced equations for $B, \bar{B}, \rho$ are as follows

\[ 2 \partial_w \ln \rho - f^2 (\partial_w \bar{B}) \frac{\kappa_+ + B \kappa_-}{\bar{B} \kappa_+ + \kappa_-} - 2f^2 (\partial_w \bar{B}) e^{+i\vartheta} = \frac{\bar{B} \partial_w \kappa_+ + \partial_w \kappa_-}{\bar{B} \kappa_+ + \kappa_-} \]

\[ 2 \partial_w \ln \rho - f^2 (\partial_w B) \frac{\bar{B} \kappa_+ + \kappa_-}{\kappa_+ + B \kappa_-} - 2f^2 (\partial_w B) e^{-i\vartheta} = \frac{\partial_w \kappa_+ + B \partial_w \kappa_-}{\kappa_+ + B \kappa_-} \]

\[ (\partial_w B) \frac{(\bar{\kappa}_+ + \bar{B} \bar{\kappa}_-)^{3/2}}{(B \bar{\kappa}_+ + \bar{\kappa}_-)^{1/2}} - (\partial_w \bar{B}) \frac{(B \bar{\kappa}_+ + \bar{\kappa}_-)^{3/2}}{(\bar{\kappa}_+ + \bar{B} \bar{\kappa}_-)^{1/2}} + \frac{2\rho^2}{3f^3} = 0 \]

- where the phase angle $\vartheta$ is defined by,

\[ e^{2i\vartheta} = \left(\frac{\kappa_+ + B \kappa_-}{\bar{\kappa}_+ + \bar{B} \bar{\kappa}_-}\right) \left(\frac{B \bar{\kappa}_+ + \bar{\kappa}_-}{\bar{B} \kappa_+ + \kappa_-}\right) \]

- This system is actually solvable,
  - Effectively a Lax system on $\Sigma$, and thus integrable in principle,
  - Three fields $(B, \bar{B}, \rho)$ version of the sine-Gordon-Liouville-Toda type
Local solutions to the BPS equations

- Metric components of the solution are given as follows,

\[
\begin{align*}
\rho^4 &= \frac{R(1 + R)(\kappa^2)^3}{|\partial_w G|^2(1 - R)} \\
f_2^2 &= \frac{\kappa^2(1 - R)}{\rho^2(1 + R)} \\f_6^2 &= \frac{9\kappa^2(1 + R)}{\rho^2(1 - R)}
\end{align*}
\]

- in terms of the following combinations,

\[
\begin{align*}
\kappa^2 &= -|\partial_w A_+|^2 + |\partial_w A_-|^2 \\
R + \frac{1}{R} &= 2 + \frac{6\kappa^2 G}{|\partial_w G|^2} \\
\partial_w B &= A_+ \partial_w A_- - A_- \partial_w A_+ \\
G &= |A_+|^2 - |A_-|^2 + B + \bar{B}
\end{align*}
\]

- SU(1, 1) symmetry of Type IIB acts naturally,

\[
B \rightarrow \frac{uB + v}{\bar{v}B + \bar{u}} \quad \left( \begin{array}{c} A_+ \\ A_- \end{array} \right) \rightarrow \left( \begin{array}{cc} u & -v \\ -\bar{v} & \bar{u} \end{array} \right) \left( \begin{array}{c} A_+ \\ A_- \end{array} \right) \quad |u|^2 - |v|^2 = 1
\]

- manifestly leaves \( \kappa^2, G \) and thus the Einstein frame metric invariant

- Positivity of metric functions \( f_6^2, f_2^2, \rho^4 \) when \( \kappa^2, G > 0 \) choosing \( R < 1 \).
Strategy for global solutions

- Summary of the associated mathematical problem
  - Riemann surface $\Sigma$ of unknown type (genus $?$, boundaries $?$)
  - Locally holomorphic functions $A_+, A_-$ on $\Sigma$
    - with linear transformation law under $SU(1, 1)$ symmetry of Type IIB
    - subject to positivity conditions
      \[
      0 < \kappa^2 = -|\partial_w A_+|^2 + |\partial_w A_-|^2 \\
      0 < G = |A_+|^2 - |A_-|^2 + B + \overline{B}
      \]
- No (regular) solutions when $\Sigma$ is compact without boundary,
  \[
  \partial_w \partial_w G = -\kappa^2 \quad \Rightarrow \quad \int_{\Sigma} \kappa^2 = 0
  \]
- The boundary $\partial \Sigma$ of $\Sigma$ has vanishing $S^2$ radius
  \[
  \partial \Sigma : \quad f_2 \to 0 \quad f_6 \neq 0
  \]
  - $\partial \Sigma$ is not a boundary of the solution's space-time manifold,
  - $\partial \Sigma$ corresponds to $S^2$ slice of $S^3$ cycle,
  - requires $\kappa^2 = G = 0$ on $\partial \Sigma$. 
Inspiration from Electro-statics

- Holomorphic $SU(1, 1)$-vector bundles give unproductive hint.

- Map this onto an electro-statics problem.
  - Consider the locally meromorphic ratio $\lambda$ on $\Sigma$ (it can have poles)
    \[ \lambda = \frac{\partial_w A_+}{\partial_w A_-} \]
    \[ \kappa^2 = -|\partial_w A_+|^2 + |\partial_w A_-|^2 \]
    - in the interior of $\Sigma$ the condition $\kappa^2 > 0$ requires $|\lambda|^2 < 1$
    - on the boundary of $\Sigma$ the condition $\kappa^2 = 0$ requires $|\lambda|^2 = 1$
  - Consider the “electro-static potential”
    \[ \Phi = -\ln |\lambda|^2 \]
    - $\Phi$ is real harmonic on $\Sigma$
    - $\Phi > 0$ in the interior of $\Sigma$, and $\Phi = 0$ on the boundary of $\Sigma$

- Place an array of positive electric charges in the interior of $\Sigma$
  and opposite image charges in the mirror image of $\Sigma$
**Σ of genus zero and one boundary component**

- With a single boundary component, and genus zero, 
  - \( \partial \Sigma \) may be mapped onto the real line 
  - \( \Sigma \) may be mapped onto the upper half plane

\[
\Phi(w) = -\ln |\lambda|^2 = -\sum_{n=1}^{N} q_n \left( \ln |w - s_n|^2 - \ln |w - \bar{s}_n|^2 \right) \quad q_n > 0
\]

- for arbitrary \( N, q_n, s_n \).
Solving for the differentials

• Regularity of the meromorphic function $\lambda$ requires $q_n = 1$ for all $n$,

$$\lambda(w) = \prod_{n=1}^{N} \frac{w - s_n}{w - \bar{s}_n}$$

• Assuming $\partial_w A_{\pm}$ meromorphic, $\partial_w A_+ = \lambda \partial_w A_-$ and regularity require,

$$\partial_w A_+ = \frac{1}{R(w)} \prod_{n=1}^{N} (w - s_n)$$

$$\partial_w A_- = \frac{1}{R(w)} \prod_{n=1}^{N} (w - \bar{s}_n)$$

– $R(w)$ is polynomials with only real roots $r_\ell$

$$R(w) = \prod_{\ell=1}^{\deg R} (w - r_\ell)$$

– real zeros are also allowed but may be viewed as the limit of $\text{Im}(s_n) \to 0$
– regularity at $\infty$ requires $\deg R = N + 2$
Satisfying the regularity conditions

- Alternative form of \( \partial_w A_\pm \),

\[
\partial_w A_\pm (w) = \sum_{\ell=1}^{N+2} \frac{Z_\pm^\ell}{w - r_\ell}
\]

\( Z_+ = (Z_-)^* = \frac{1}{P'(r_\ell)} \prod_{n=1}^{N} (r_\ell - \bar{s}_n) \)

- allows us to integrate up to \( A_\pm \),

\[
A_\pm (w) = \sum_{\ell=1}^{N+2} Z_\pm^\ell \ln(w - r_\ell)
\]

- and to obtain \( B \) in terms of “dilogarithm integrals”

\[
B(w) = \sum_{\ell, \ell'=1}^{N+2} \left( Z_+^\ell Z_-^{\ell'} - Z_+^{\ell'} Z_-^\ell \right) \int_{w_0}^{w} dw \frac{\ln(w - r_\ell)}{w - r_{\ell'}}
\]

- judicious choice of branch cuts allows one to show that

\[
G = |A_+|^2 - |A_-|^2 + B + \bar{B}
\]

- obeys \( G = 0 \) on the boundary of \( \Sigma \)
- obeys \( G > 0 \) in the interior of \( \Sigma \)
Asymptotics near pole $= \text{near } (p, q)$ five-brane

- The solution is regular everywhere on $\Sigma$, except at the poles $r_\ell$
  \[ w = r_\ell + u e^{i\theta} \]
- The dilaton diverges and the Einstein-frame metric becomes,
  \[ ds^2 = (\ln u) d\hat{s}^2_{AdS6} + |Z^\ell_+ - Z^\ell_-| \left( \frac{du^2}{u^2} + d\hat{s}^2_{S3} \right) \]
  - $AdS_6$ expands to infinite radius, by rescaling tends to $\mathbb{R}^6$;
  - $(p, q)$-charges at the pole given by $p_\ell = \text{Re}(Z^\ell_+)$ and $q_\ell = -\text{Im}(Z^\ell_+)$
- Stack of $N$ coincident NS5 branes produces string frame metric and dilaton,
  \[ ds^2 = dx^\mu dx_\mu + e^{2\Phi} dy^2 \quad \quad e^{2\Phi(y)} = e^{2\Phi(\infty)} + N/y^2 \]
  - $x^\mu$ along 5-brane, $y$ perpendicular to 5-brane, near-horizon $u^2 = y^2 \to 0$
  \[ ds^2 \sim dx^\mu dx_\mu + \frac{du^2}{u^2} + d\hat{s}^2_{S3} \quad \quad e^{2\Phi(y)} \sim N/u^2 \]
  - agrees with behavior near the poles of our solutions
Poles represent semi-infinite “heavy” branes

- Conformally map the upper half plane to the unit disc;
  - real axis to unit circle
  - points $r_\ell \in \mathbb{R}$ to points $\tilde{r}_\ell$ on unit circle

Riemann surface $\Sigma$

$(p, q)$ five-brane web
Correlators holographically?

- Key motivation for obtaining our Type IIB supergravity solutions
  - access the superconformal phase of five-dimensional gauge theories
  - compute operator dimensions and correlators

- For standard cases, asymptotically region has enhanced symmetry
  - eg asymptotically $SU(2,2|4)$ for asymptotic $AdS_5 \times S^5$
  - In five dimensions superconformal algebra $F(4)$ throughout

- The “heavy” effectively six-dimensional branes are part of the solution (as poles)
  - Effects of warping persist to the holographic boundary
  - For five-dim holography one must prevent access to six-dim regions
  - impose boundary conditions on red “walls”?
Outlook

• We constructed exactly a wealth of $AdS_6 \times S^2 \times \Sigma$ solutions in Type IIB
  – regular except for expected asymptotics of “heavy” $(p, q)$ branes,
  – precise matching of parameters in brane and supergravity constructions,
  – solutions with D7-branes (ED, Gutperle, Uhlemann arXiv:1706.00433)
  – Local solutions to the “double analytic continuation” $AdS_2 \times S^6 \times \Sigma$

• Dynamics of five-dimensional superconformal gauge theories
  – spectrum of operator dimensions around the solutions,
  – do these solutions have exceptional global symmetries $E_8, E_7, E_6, \cdots$?