

Superstring Amplitudes beyond Tree-level

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Outline

- **QFT: proliferation of the number of Feynman diagrams**
 - with increasing number of external states
 - with the number of loops
- **Closed strings: a single diagram per loop order**
 - compact (super) Riemann surface Σ
 - genus = number of handles = number of loops

(open strings: cfr talks by Oliver Schlotterer and Stefan Stieberger)
- **Main topics**
 - I prototypes of closed superstring loop amplitudes
 - II construction of amplitudes in the RNS formulation
 - III supermoduli
 - IV low energy effective interactions

I. Prototypes

- **Type IIB four-graviton amplitude to one-loop order** (Green, Schwarz 1982)

$$\mathcal{A}^{(1)}(\varepsilon_i, k_i) = \mathcal{K} \tilde{\mathcal{K}} \int_{\mathcal{M}_1} \frac{d^2\tau}{(\text{Im}\tau)^2} \mathcal{B}^{(1)}(s_{ij}|\tau)$$

- Factorized basis of polarization tensors $\varepsilon_i^{\mu\nu} = \varepsilon_i^\mu \tilde{\varepsilon}_i^\nu$

$$s_{ij} = -\alpha'(k_i + k_j)^2/4 \quad \mathcal{K} = t_8 \varepsilon_1 \cdots \varepsilon_4 k_1 \cdots k_4$$

- Partial amplitude \mathcal{B} is a modular function in $\tau \in \mathcal{M}_1 = \mathbb{H}/SL(2, \mathbb{Z})$

$$\mathcal{B}^{(1)}(s_{ij}|\tau) = \int_{\Sigma^4} \prod_{i=1}^4 \frac{d^2z_i}{\text{Im}\tau} \exp\left(\sum_{i<j} s_{ij} G(z_i - z_j|\tau)\right)$$

- $G(z|\tau)$ is the scalar Green function on the torus Σ of modulus τ .
- Analogous formulas for Heterotic strings and more external states.

- **Singularity structure**

- For fixed τ integrations over Σ produce poles in \mathcal{B} at positive integers s_{ij} .
- The integral over τ converges absolutely only for $\text{Re}(s_{ij}) = 0$.
- Analytic continuation requires decomposition of \mathcal{M}_1 .
- Massless singularities are produced by $\tau \rightarrow i\infty$ (cfr ambitwistor string).

Loop momenta

- **Loop momenta were hidden but may be exposed**

- Choose a canonical basis of cycles A, B of $H_1(\Sigma, \mathbb{Z})$.
- Choose loop momentum p_μ flowing through the cycle A ,

$$\int_{\mathcal{M}_1} \frac{d^2\tau}{(\text{Im}\tau)^2} \mathcal{B}^{(1)}(s_{ij}|\tau) = \int_{\mathbb{R}^{10}} d^{10}p \int_{\mathcal{M}_1} \int_{\Sigma^4} \left| \mathcal{F}(z_i, k_i, p|\tau) \right|^2$$

- The partial amplitude \mathcal{F} is **locally holomorphic** in τ and z_i

$$\mathcal{F}(z_i, k_i, p|\tau) = d\tau \prod_{i=1}^4 dz_i e^{i\pi\tau p^2 + 2\pi i p \sum_i k_i z_i} \prod_{i<j} \vartheta_1(z_i - z_j|\tau)^{-s_{ij}}$$

- at the cost of non-trivial monodromy

$$\mathcal{F}(z_i + \delta_{i,\ell} A, k_i, p|\tau) = e^{2\pi i k_\ell \cdot p} \mathcal{F}(z_i, k_i, p|\tau)$$

$$\mathcal{F}(z_i + \delta_{i,\ell} B, k_i, p|\tau) = \mathcal{F}(z_i, k_i, p + k_\ell|\tau)$$

- Modular invariance of $\mathcal{A}^{(1)}$ guarantees independence of choices.
- Hermitian pairing of \mathcal{F} and $\bar{\mathcal{F}}$ is familiar from 2-d CFT where loop momenta p_μ label conformal blocks of 10 copies of $c = 1$.

Genus two

- Siegel Upper half space \mathcal{S}_2

$$\mathcal{S}_2 = \{\Omega_{IJ} = \Omega_{JI} \in \mathbb{C} \text{ with } I, J = 1, 2 \text{ and } Y = \text{Im}\Omega > 0\}$$

– $Sp(4, \mathbb{R})$ acts by $\Omega \rightarrow (A\Omega + B)(C\Omega + D)^{-1}$

$$M^t J M = J \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

– \mathcal{S}_2 has $Sp(4, \mathbb{R})$ -invariant metric ds_2^2 and volume form $d\mu_2$

$$ds_2^2 = \sum_{I, J, K, L=1, 2} Y_{IJ}^{-1} d\bar{\Omega}_{JK} Y_{KL}^{-1} d\Omega_{LI}$$

- Compact Riemann surfaces Σ

– Choose canonical homology basis of A_I, B_I cycles for $H_1(\Sigma, \mathbb{Z})$.

– ω_I dual holomorphic $(1,0)$ forms,

$$\oint_{A_I} \omega_J = \delta_{IJ} \quad \oint_{B_I} \omega_J = \Omega_{IJ}$$

– Riemann relations imply $\Omega \in \mathcal{S}_2$;

– Modular group $Sp(4, \mathbb{Z})$; moduli space $\mathcal{M}_2 = \mathcal{S}_2 / Sp(4, \mathbb{Z})$.

Two-loop Type II superstring amplitudes

- **Type II four-graviton amplitude at genus 2** (ED, Phong 2001 – 2005)

$$\mathcal{A}^{(2)}(\varepsilon_i, k_i) = \kappa \tilde{\kappa} \int_{\mathcal{M}_2} d\mu_2 \mathcal{B}^{(2)}(s_{ij}|\Omega)$$

$$\mathcal{B}^{(2)}(s_{ij}|\Omega) = \int_{\Sigma^4} \frac{\mathcal{Y} \wedge \bar{\mathcal{Y}}}{(\det \text{Im } \Omega)^2} \exp \left(\sum_{i < j} s_{ij} G(z_i, z_j|\Omega) \right)$$

- $G(z_i, z_j)$ is the genus-two scalar Green function;

$$\begin{aligned} \mathcal{Y} = & (t - u)\Delta(z_1, z_2) \wedge \Delta(z_3, z_4) + (s - t)\Delta(z_1, z_3) \wedge \Delta(z_4, z_2) \\ & + (u - s)\Delta(z_1, z_4) \wedge \Delta(z_2, z_3) \end{aligned}$$

- $\Delta(z_i, z_j)$ is a bi-holomorphic form independent of s, t, u .

$$\Delta(z, w) = \omega_1(z) \wedge \omega_2(w) - \omega_2(z) \wedge \omega_1(w)$$

- reproduced (with fermions) in pure spinor formulation (Berkovits, Mafra 2005)

- **Singularity structure**

- For fixed Ω integrations over Σ produce poles in \mathcal{B} at positive integers s_{ij} .
- The integral over Ω requires analytic continuation beyond $\text{Re}(s_{ij}) = 0$.
- Leading massless singularities from $\Omega_{11}, \Omega_{22} \rightarrow i\infty$ (cfr ambitwistor string).

Two-loop Heterotic string amplitudes

- Heterotic four NS boson amplitude at genus 2 (ED, Phong 2005)

$$A_{\mathcal{O}}^{(2)}(\varepsilon_i, \tilde{\varepsilon}_i, k_i) = \mathcal{K} \int_{\mathcal{M}_2} d\mu_2 \mathcal{B}_{\mathcal{O}}^{(2)}(\tilde{\varepsilon}_i, k_i | \Omega)$$

$$\mathcal{B}_{\mathcal{O}}^{(2)}(s_{ij} | \Omega) = \int_{\Sigma^4} \frac{\mathcal{Y} \wedge \overline{\mathcal{W}_{\mathcal{O}}(\tilde{\varepsilon}_i, k_i)}}{(\det \operatorname{Im} \Omega)^2 \overline{\Psi_{10}(\Omega)}} \exp\left(\sum_{i < j} s_{ij} G(z_i, z_j)\right)$$

- $\Psi_{10}(\Omega)$ is the Igusa cusp form.
- Dependence on \mathcal{O} according to channel:
 - ★ 4 gravitons \mathcal{R}^4
 - ★ 2 gravitons + 2 gauge bosons $\mathcal{R}^2 \operatorname{tr}(\mathcal{F}^2)$;
 - ★ or 4 gauge with 2 channels $(\operatorname{tr} \mathcal{F}^2)(\operatorname{tr} \mathcal{F}^2)$ and $\operatorname{tr}(\mathcal{F}^4)$
- For example,

$$\mathcal{W}_{\mathcal{R}^4}(\tilde{\varepsilon}_i, k_i) = \frac{\langle \prod_{i=1}^4 \tilde{\varepsilon}_i \cdot \partial \tilde{x}(z_i) e^{ik_i \cdot \tilde{x}(z_i)} \rangle}{\langle \prod_{i=1}^4 e^{ik_i \cdot \tilde{x}(z_i)} \rangle}$$

$$\langle \tilde{x}^\mu(z) \tilde{x}^\nu(w) \rangle = -\eta^{\mu\nu} \ln E(z, w)$$

- Gauge parts are obtained by the correlators of the current $(0, 1)$ -forms.

II. Construction of amplitudes in RNS formulation

- Spinors ψ^μ on the worldsheet

- With Minkowski worldsheet signature

- ★ opposite chirality spinors $\psi^\mu, \tilde{\psi}^\mu$ are *independent* Majorana-Weyl
- ★ Dirac eq solved by $\psi^\mu(\tau - \sigma)$ and $\tilde{\psi}^\mu(\tau + \sigma)$

- With Euclidean worldsheet signature

- ★ right (-movers) $\tau - \sigma \rightarrow z$
- ★ left (-movers) $\tau + \sigma \rightarrow \tilde{z}$
- ★ worldsheet spinors $\psi^\mu, \tilde{\psi}^\mu$ become complex conjugate Weyl spinors

- On a compact Riemann surface of genus g

- ★ In local complex coordinates (z, \tilde{z}) solved by $\psi^\mu(z)$ and $\tilde{\psi}^\mu(\tilde{z})$
- ★ Globally, ψ^μ are sections of spin bundle S (resp. $\tilde{\psi}^\mu$ of \tilde{S})
- ★ same bundle S (resp. \tilde{S}) for all μ by Lorentz invariance in \mathbb{R}^{10}
- ★ $S^{\otimes 2} \approx \tilde{S}^{\otimes 2} \approx T_{(1,0)}^*(\Sigma)$
- ★ 2^{2g} distinct spin bundles labelled by their spin structure.
- ★ S and \tilde{S} must be independent of one another.

Worldsheet fields

• Type II

- $x^\mu, \psi^\mu, \tilde{\psi}^\mu, \mu = 0, 1, \dots, 9$
- $g_{mn} \longrightarrow b, c, \tilde{b}, \tilde{c}$ ghosts + g_{mn}^0 (parametrizing even moduli)
- $\chi_m, \tilde{\chi}_m \longrightarrow \beta, \gamma, \tilde{\beta}, \tilde{\gamma}$ ghosts + $\chi_m^0, \tilde{\chi}_m^0$ (parametrizing odd moduli)
- $\psi^\mu, \beta, \gamma, \chi_m$ all have the same spin structure δ
- $\tilde{\psi}^\mu, \tilde{\beta}, \tilde{\gamma}, \tilde{\chi}_m$ all have the same spin structure $\tilde{\delta}$

• Heterotic

- $x^\mu, \psi^\mu, \tilde{\lambda}^\alpha, \alpha = 1 \dots 32$
- $g_{mn} \longrightarrow b, c, \tilde{b}, \tilde{c}$ ghosts + g_{mn}^0 (parametrizing even moduli)
- $\chi_m \longrightarrow \beta, \gamma$ ghosts + χ_m^0 (parametrizing odd moduli)
- $\psi^\mu, \beta, \gamma, \chi_m$ all have the same spin structure δ
- $\tilde{\lambda}^\alpha$ grouped into $16_{\tilde{\delta}_1} \oplus 16_{\tilde{\delta}_2}$ for $E_8 \times E_8$ or into $32_{\tilde{\delta}}$ for $Spin(32)/\mathbb{Z}_2$

Chiral amplitudes

- **Both Type II and Heterotic require chiral amplitudes**
 - Link chirality with holomorphicity on (super) moduli space
 - Chiral amplitude from non-chiral by *chiral splitting*
(ED, Phong 1988-91)
 - Approach via holomorphic super-Riemann surfaces
(Witten, arXiv:1209.2459, 1209.5461, 1304.2832, 1306.3621)
- **Full amplitudes will be constructed by pairing left and right**
 - cfr “double copy construction”

Conformal structures and deformations

- **An orientable 2-dim manifold with Riemannian metric is a Riemann surface**
 - complex manifold (holomorphic transition functions)
 - complex structure $J : T(\Sigma) \rightarrow T(\Sigma)$, $J^2 = -I$ is integrable
 - in terms of the metric and local coordinates, $J^m_n = \sqrt{g} g^{mp} \varepsilon_{pn}$
 - Moduli space $\mathcal{M}_h = \{J\}/\text{Diff}(\Sigma)$ of genus h Riemann surfaces
- **Moduli space itself is a complex “manifold” (actually orbifold)**
 - Its tangent space splits $T(\mathcal{M}_g) = T_{(1,0)}(\mathcal{M}_g) \oplus T_{(0,1)}(\mathcal{M}_g)$
 - associated Beltrami differentials $\delta J = \delta J_{\tilde{z}^z} \oplus \delta J_z^{\tilde{z}}$
$$\dim_{\mathbb{C}} \mathcal{M}_h = \begin{cases} 0 & h = 0 \\ 1 & h = 1 \\ 3h - 3 & h \geq 2 \end{cases}$$
- **Beltrami differentials provide a parametrization of finite deformations**
 - Fix complex structure J and associated local complex coordinates (z, \tilde{z})
 - metric is conformally flat $ds^2 = |dz|^2$
 - finite deformations to $ds^2 = |dz + \mu_{\tilde{z}^z} d\tilde{z}|^2$ with $\delta \mu_{\tilde{z}^z} = \delta J_{\tilde{z}^z}$

Super conformal structures

- **Super Riemann surfaces** (Friedan, Martinec, Shenker 1986)
 - complex dimension $1|1$ locally isomorphic to $\mathbb{C}^{1|1}$
 - transition functions are superconformal in $z|\theta$
 - transforming the superderivative $D_\theta = \partial_\theta + \theta\partial_z$ by scaling

$$f : (z|\theta) \rightarrow (\hat{z}|\hat{\theta}) \quad D_\theta = F(z, \theta)D_{\hat{\theta}}$$

- Locally: generates the holomorphic $\mathcal{N} = 1$ superconformal algebra
- Globally: $T\Sigma$ has a completely non-integrable subbundle \mathcal{D} of rank $0|1$
- Moduli space $\mathfrak{M}_h = \{\mathcal{J}\}/\text{Diff}(\Sigma)$
= equivalence classes of superconformal structures \mathcal{J}

$$\dim_{\mathbb{C}} \mathfrak{M}_h = \begin{cases} 0|0 & h = 0 \\ 1|0 \text{ or } 1|1 & h = 1 \text{ even or odd spin structure} \\ 3h - 3|2h - 2 & h \geq 2 \end{cases}$$

- odd modulus at $h = 1$ odd spin structure is a book keeping device
- odd moduli really first appear at genus 2
- Global extension to SRS with NS and R punctures (Witten 2011)

Superstring worldsheet and moduli spaces (Witten 2011)

• Heterotic

- Left : Riemann surface Σ_L , moduli space \mathcal{M}_L
- Right : super Riemann surface Σ_R , moduli space \mathfrak{M}_R
- Worldsheet is a cycle $\Sigma \subset \Sigma_L \times \Sigma_R$ of dimension $1|1$
 subject to $\Sigma_{\text{red}} = \text{diag}(\Sigma_{L \text{red}} \times \Sigma_{R \text{red}}) : \tilde{z} = \bar{z} + \text{nilpotent}$
 (reduced space obtained by setting all nilpotent variables to zero)
- Moduli space is a cycle $\Gamma \subset \mathcal{M}_L \times \mathfrak{M}_R$ of dim $3h - 3|2h - 2$ for $h \geq 2$

• Type II

- Left : super Riemann surface Σ_L , moduli space \mathfrak{M}_L
- Right : super Riemann surface Σ_R , moduli space \mathfrak{M}_R
- Worldsheet is a cycle $\Sigma \subset \Sigma_L \times \Sigma_R$ of dimension $1|2$
 subject to $\Sigma_{\text{red}} = \text{diag}(\Sigma_{L \text{red}} \times \Sigma_{R \text{red}})$
- Moduli space is a cycle $\Gamma \subset \mathfrak{M}_L \times \mathfrak{M}_R$ of dim $3h - 3|4h - 4$ for $h \geq 2$

• Super-Stokes theorem ensures independence of the choice of cycles

- in amplitudes with BRST invariant vertex operators

Worldsheet action for Heterotic strings

- **Worldsheet is** $\Sigma \subset \Sigma_L \times \Sigma_R$
 - superconformal structure \mathcal{J} for Σ_R with local coordinates $z|\theta$
 - conformal structure $\tilde{\mathcal{J}}$ for Σ_L , with local coordinates \tilde{z}

$$X^\mu(\tilde{z}, z, \theta) = x^\mu(\tilde{z}, z) + \theta\psi^\mu(\tilde{z}, z)$$

$$\Lambda^\alpha(\tilde{z}, z, \theta) = \lambda_-^\alpha(\tilde{z}, z) + \theta\ell^\alpha(\tilde{z}, z)$$

- **Superconformal invariant matter action**

$$I_M[X^\mu, \Lambda^\alpha, \mathcal{J}, \tilde{\mathcal{J}}] = \int_\Sigma [d\tilde{z}dz|\theta] \left[\partial_{\tilde{z}} X^\mu D_\theta X_\mu + \sum_\alpha \Lambda^\alpha D_\theta \Lambda^\alpha \right]$$

- Integrating out θ , we recover familiar action

$$I_M = \int d\tilde{z}dz \left[\partial_{\tilde{z}} x^\mu \partial_z x_\mu + \psi^\mu \partial_{\tilde{z}} \psi_\mu + \sum_\alpha (\lambda^\alpha \partial_z \lambda^\alpha + \ell^\alpha \ell^\alpha) \right]$$

- Superconformal algebra on fields generated by

$$T_{zz} = -\frac{1}{2} \partial_z x^\mu \partial_z x_\mu + \frac{1}{2} \psi^\mu \partial_z \psi_\mu$$

$$S_{z\theta} = \psi^\mu \partial_z x_\mu$$

Deformations of superconformal structures

- Under deformation of the conformal structure \tilde{J} on Σ_L

$$\delta I = \int d\tilde{z}dz \delta\tilde{J}_z^{\tilde{z}} T_{\tilde{z}\tilde{z}}$$

- Under deformation of superconformal structure \mathcal{J} on Σ_R

$$\delta I = \int d\tilde{z}dz \left[\delta J_{\tilde{z}}^z T_{zz} + \delta\chi_{\tilde{z}}^\theta S_{z\theta} \right]$$

- $\chi_{\tilde{z}}^\theta$ is the “worldsheet gravitino” field
- δJ and $\delta\tilde{J}$ are even Beltrami differentials

- Assemble deformations into super-conformal invariant action

$$\mathcal{S}_{z\theta} = S_{z\theta} + \theta T_{zz}$$

$$\delta H_{\tilde{z}}^z = \delta J_{\tilde{z}}^z + \theta \delta\chi_{\tilde{z}}^\theta$$

$$\delta H_\theta^{\tilde{z}} = \theta \delta\tilde{J}_z^{\tilde{z}} + \text{auxiliary}$$

$$\delta I = \int_\Sigma [d\tilde{z}dz | d\theta] (\delta H_{\tilde{z}}^z \mathcal{S}_{z\theta} + \delta H_\theta^{\tilde{z}} T_{\tilde{z}\tilde{z}})$$

Full Heterotic string amplitude

- Parametrization of the deformations $\delta H_{\tilde{z}z}$ and $\delta H_{\theta\tilde{z}}$ by slice in $\{\mathcal{J}, \tilde{\mathcal{J}}\}$

$$\delta H_{\tilde{z}z} = \partial_{\tilde{z}} V^z + \sum_A H_A dm_A \quad H_A = \partial \mathcal{J}_{\tilde{z}z} / \partial m_A$$

$$\delta H_{\theta\tilde{z}} = D_{\theta} V^{\tilde{z}} + \sum_a \tilde{H}_a d\tilde{m}_a \quad \tilde{H}_a = \partial(\theta J_z^{\tilde{z}}) / \partial \tilde{m}_a$$

– Introducing ghost fields

$$V^z \rightarrow C^z = c^z + \theta \gamma^\theta \quad \delta H_{\tilde{z}z} \rightarrow B_{z\theta} = \beta_{z\theta} + \theta b_{zz}$$

$$V^{\tilde{z}} \rightarrow \tilde{C}^{\tilde{z}} = c^{\tilde{z}} + \text{aux} \quad \delta H_{\theta\tilde{z}} \rightarrow \tilde{B}_{\tilde{z}\tilde{z}} = b_{\tilde{z}\tilde{z}} + \text{aux}$$

- Super conformal invariant ghost action

$$I_{\text{gh}} = \int_{\Sigma} [d\tilde{z}dz|d\theta] \left[B_{z\theta} \partial_{\tilde{z}} C^z + \tilde{B}_{\tilde{z}\tilde{z}} D_{\theta} \tilde{C}^{\tilde{z}} + B_{z\theta} \sum_A H_A dm_A + \tilde{B}_{\tilde{z}\tilde{z}} \sum_a \tilde{H}_a d\tilde{m}_a \right]$$

- Assembling all factors, we obtain the integrand on $\mathcal{M}_L \times \mathfrak{M}_R$

$$\int D(XB\tilde{B}C\tilde{C}) \mathcal{V}_1 \cdots \mathcal{V}_n \prod_{a,A} [d\tilde{m}_a dm_A] \delta(\langle \tilde{B}, \tilde{H}_a \rangle) \delta(\langle B, H_A \rangle) e^{-I_M - I_{\text{gh}}}$$

– $\mathcal{V}_1 \cdots \mathcal{V}_n$ are BRST-invariant vertex operators.

Chiral amplitudes via Chiral splitting

- In Type II superstrings, both Σ_L and Σ_R are super Riemann surfaces
 - Deformation must involve independent $\chi_{\tilde{z}}^\theta$ and $\chi_z^{\tilde{\theta}}$
 - ghosts B, C couple only to $\chi_{\tilde{z}}^\theta$ and \tilde{B}, \tilde{C} only to $\chi_z^{\tilde{\theta}}$
 - 2-d supergravity action for X^μ couples left to right chiralities

$$I_m = \int_{\Sigma} d^2z \left[\partial_z x^\mu \partial_{\tilde{z}} x_\mu + \psi^\mu \partial_{\tilde{z}} \psi_\mu - \chi_{\tilde{z}}^\theta \psi^\mu \partial_z x_\mu + \cdots + \chi_{\tilde{z}}^\theta \chi_z^{\tilde{\theta}} \psi^\mu \tilde{\psi}^\nu \right]$$

- Chiral splitting is established at fixed internal loop momenta
 - Fix canonical homology basis $A_I, B_I, I = 1, \dots, h$ on Σ of genus h
 - h independent internal loop momenta p_I^μ are defined across the cycles A_I
(Verlinde, Verlinde; ED, Phong 1988)

$$p_I^\mu = \oint_{A_I} dz \partial_z x^\mu + \oint_{A_I} d\tilde{z} \partial_{\tilde{z}} x^\mu$$

- Amplitude is an integral over p_I^μ of a product of chiral amplitudes

$$\int_{\mathbb{R}^{10}} dp_I^\mu \mathcal{F}_L(\mu_z^{\tilde{z}}, \chi_z^{\tilde{\theta}}, \varepsilon_i, k_i, p_I^\mu) \mathcal{F}_R(\mu_{\tilde{z}}^z, \chi_{\tilde{z}}^\theta, \tilde{\varepsilon}_i, k_i, p_I^\mu)$$

- with k_i external momenta, and polarization tensors $\varepsilon_i^{\mu\nu} = \varepsilon_i^\mu \tilde{\varepsilon}_i^\nu$

Chiral amplitudes

- Chiral amplitude \mathcal{F}_R has supermoduli deformations of only Σ_R

(similarly \mathcal{F}_L has supermoduli deformations of only Σ_L)

- \mathcal{F}_R computed with effective rules for chiral fields x_+, ψ_+^μ
- and chiral vertex operators $\mathcal{V}_1^+ \cdots \mathcal{V}_N^+$

$$\mathcal{F}_R = \left\langle \mathcal{V}_1^+ \cdots \mathcal{V}_N^+ e^{p_I^\mu \oint_{B_I} dz \partial_z x_+^\mu} \exp \int_{\Sigma_{\text{red}}} d\tilde{z} dz (\mu_{\tilde{z}}^z T_{zz} + \chi_{\tilde{z}}^\theta S_{z\theta}) \right\rangle_{x_+, \psi_+}$$

- with stress tensor and supercurrent evaluated on chiral fields x_+, ψ_+^μ

$$T_{zz} = -\frac{1}{2} \partial_z x_+^\mu \partial_z x_+^\nu \eta_{\mu\nu} + \frac{1}{2} \psi_+^\mu \partial_z \psi_+^\nu \eta_{\mu\nu} + T_{zz}^{\text{gh}}(b, c, \beta, \gamma)$$

$$S_{z\theta} = -\frac{1}{2} \psi_+^\mu \partial_z x_+^\nu \eta_{\mu\nu} + S_{z\theta}^{\text{gh}}(b, c, \beta, \gamma)$$

- $\langle \cdots \rangle_{x_+, \psi_+}$ indicates Wick contractions of x_+^μ, ψ_+^μ with
- effective chiral Green functions

$$\langle \psi_+^\mu(z) \psi_+^\nu(w) \rangle = -\eta^{\mu\nu} S(z, w) \quad \text{Szegő kernel}$$

$$\langle x_+^\mu(z) x_+^\nu(w) \rangle = -\eta^{\mu\nu} \ln E(z, w) \quad \text{prime form}$$

III. Parametrization of supermoduli

- **Superconformal structure** $\mathcal{J} \in \mathfrak{M}_h$ specifies transition functions
 - Practical calculations mostly use parametrization by gravitino field $\chi_{\tilde{z}}^\theta$
- **Local parametrization of bosonic moduli (in conformal-invariant theory)**
 - Complex structure J with metric $g = |dz|^2$ in local coordinates (z, \tilde{z})
 - deformation of complex structure by Beltrami differential to $g' = |dz + \mu d\tilde{z}|^2$
 - realized in CFT by insertion of $\int_{\Sigma} d\tilde{z} dz \mu_{\tilde{z}^z} T_{zz}$
- **Local parametrization of supermoduli (in superconformal-invariant theory)**
 - Start with Σ_{red} with complex structure given by $J \in \mathfrak{M}_{\text{red}}$
 - Deformation of super conformal structure realized by insertion of T and S

$$\int_{\Sigma_{\text{red}}} d\tilde{z} dz (\mu_{\tilde{z}^z} T_{zz} + \chi_{\tilde{z}}^\theta S_{z\theta})$$

- χ and μ parametrized by local odd coordinates on \mathfrak{M}_h
- nilpotency guarantees that these deformations are exact
- **Globally, there is no holomorphic projection** $\mathfrak{M}_h \rightarrow \mathcal{M}_h$ for $h \geq 3$
(Donagi, Witten 2013)

The super period matrix

- For $h = 2$ and even spin structure there is a natural projection $\mathfrak{M}_2 \rightarrow \mathcal{M}_2$ via the super period matrix (E.D. & Phong 1988)

- Fix canonical homology basis A_I, B_I for $H^1(\Sigma, \mathbf{Z})$
- There exist $h = 2$ superholomorphic forms $\hat{\omega}_I$ with super periods $\hat{\Omega}$

$$\oint_{A_I} \hat{\omega}_J = \delta_{IJ} \qquad \oint_{B_I} \hat{\omega}_J = \hat{\Omega}_{IJ}$$

- Explicit formula in terms of (g, χ) , and Szego kernel S_δ

$$\hat{\Omega}_{IJ} = \Omega_{IJ} - \frac{i}{8\pi} \iint \omega_I(z) \chi(z) S_\delta(z, w) \chi(w) \omega_J(w)$$

- $\hat{\Omega}_{IJ}$ is locally supersymmetric with $\hat{\Omega}_{IJ} = \hat{\Omega}_{JI}$ and $\text{Im } \hat{\Omega} > 0$
- Every $\hat{\Omega}$ corresponds to an ordinary Riemann surface
- Szegö kernel $S_\delta(z, w | \Omega)$ is non-singular on \mathcal{M}_2

⇒ Projection using $\hat{\Omega}$ is holomorphic and natural for genus 2

Projecting and pairing Chiral Amplitudes

- **Chiral Amplitudes on \mathfrak{M}_2**

- Natural supersymmetric parametrization of \mathfrak{M}_2 by $(\hat{\Omega}_{IJ}, \zeta^\alpha)$
- involve measure $d\mu[\delta](\hat{\Omega}, \zeta)$ and correlation functions $\mathcal{C}[\delta](\varepsilon_i, k_i, p_I | \hat{\Omega}, \zeta)$

- **Projection to chiral amplitudes on \mathcal{M}_2**

- by integrating over ζ and summing over δ at fixed $\hat{\Omega}$

$$\mathcal{R}(\varepsilon_i, k_i, p_I | \hat{\Omega}) = \sum_{\delta} \int_{\zeta} d\mu[\delta](\hat{\Omega}, \zeta) \mathcal{C}[\delta](\varepsilon_i, k_i, p_I | \hat{\Omega}, \zeta)$$

$$\mathcal{L}(\tilde{\varepsilon}_i, k_i, p_I | \hat{\Omega}) = \sum_{\tilde{\delta}} \int_{\tilde{\zeta}} d\mu[\tilde{\delta}](\hat{\Omega}, \tilde{\zeta}) \mathcal{C}[\tilde{\delta}](\tilde{\varepsilon}_i, k_i, p_I | \hat{\Omega}, \tilde{\zeta})$$

- for heterotic, \mathcal{L} is chiral half of bosonic string, has no integral in $\tilde{\zeta}$
- phase factors determined by $Sp(4, \mathbb{Z})$ modular invariance

- **Pairing left and right chiral amplitudes, integrating over p_I and Ω**

$$\mathcal{A}^{(2)}(\varepsilon_i, \tilde{\varepsilon}_i, k_i) = \int_{\mathcal{M}_2} d\Omega \int dp_I^\mu \mathcal{R}(\varepsilon_i, k_i, p_I | \Omega) \overline{\mathcal{L}(\tilde{\varepsilon}_i, k_i, p_I | \Omega)}$$

- Integral over p_I can be carried out explicitly.

Singularities in the projection $\mathfrak{M}_2 \rightarrow \mathcal{M}_2$

- **Projection $\mathfrak{M}_2 \rightarrow \mathcal{M}_2$ is holomorphic, but integration extends to boundary**
 - are there singularities on the boundary of \mathfrak{M}_2 ?

$$\Omega = \begin{pmatrix} \tau & u \\ u & \sigma \end{pmatrix} \quad \begin{array}{ll} u \rightarrow 0 & \text{separating node} \\ \sigma \rightarrow i\infty & \text{non-separating node} \end{array}$$

- Key ingredient in $\hat{\Omega}$ is the Szegő kernel

$$S_\delta(z, w|\Omega) = \frac{\vartheta[\delta](z - w|\Omega)}{\vartheta[\delta](0|\Omega) E(z, w)}$$

- As $u \rightarrow 0$ we have $\vartheta[\delta](0|\Omega) \rightarrow \vartheta[\delta_1](0|\tau) \vartheta[\delta_2](0|\tau)$
 - Even $\delta = [\delta_1, \delta_2]$ with δ_1, δ_2 odd produces a singularity in S_δ and $\hat{\Omega}$
- **Physical effects**
 - singularity killed by ψ -zero modes in \mathbb{R}^{10} (space-time susy)
 - contribution when susy is broken by radiative corrections (Witten 2013)
 - Two-loop vacuum energy in Heterotic strings on CY orbifold $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$
 - ★ zero for $E_8 \times E_8 \rightarrow E_6 \times E_8$ with unbroken susy
 - ★ non-zero for $Spin(32)/\mathbb{Z}_2 \rightarrow SO(26) \times U(3)$ with broken susy
 - (ED, Phong 2013; Berkovits, Witten 2014)
 - One-loop susy breaking (Attick, Dixon, Sen; Dine Seiberg, Witten 1987)

Singularities in the projection $\mathfrak{M}_3 \rightarrow \mathcal{M}_3$

- **Some basic structure theorems**

- A hyper-elliptic surface is a branched double cover of the sphere S^2
- All genus 1 and all genus 2 surfaces are hyper-elliptic
- Hyper-elliptic surfaces form a subvariety of \mathcal{M}_3 of complex codimension 1 (referred to as the hyper-elliptic divisor)

- **The period matrix (for even spin structure) for genus 3 is given by**

$$\hat{\Omega}_{IJ} = \Omega_{IJ} - \frac{i}{8\pi} \iint \omega_I(z) \chi(z) S_\delta(z, w) \chi(w) \omega_J(w) + \mathcal{O}(\chi^4)$$

- For genus 3, $\vartheta[\delta](0|\Omega) = 0$ for some δ on the hyper-elliptic divisor of \mathfrak{M}_3 (which crosses the interior of \mathfrak{M}_3)
- the presence of additional Dirac zero modes kills the effect of this singularity
- But it is another δ that produces a singularity in $\hat{\Omega}$ (subtle) (Witten 2015)
- Rules out earlier proposals for the genus 3 superstring measure

IV. Low Energy Effective Interactions

- **Four-graviton amplitude in Type II at genus 0**

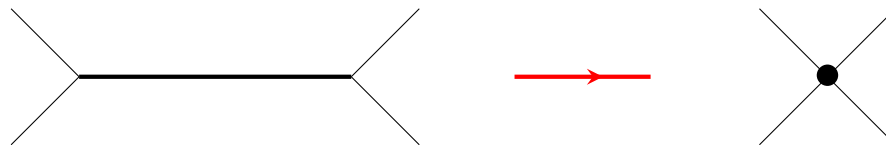
$$\mathcal{A}^{(0)} = \kappa \tilde{\kappa} \frac{1}{stu} \frac{\Gamma(1-s)\Gamma(1-t)\Gamma(1-u)}{\Gamma(1+s)\Gamma(1+t)\Gamma(1+u)}$$

– Low energy expansion corresponds to $|s|, |t|, |u| \ll 1$

$$\frac{1}{stu} + 2\zeta(3) + \zeta(5)(s^2 + t^2 + u^2) + 2\zeta(3)^2 stu + \dots$$

massless
 \mathcal{R}^4
 $D^4\mathcal{R}^4$
 $D^6\mathcal{R}^4$

– Exchange of massive string states produces local effective interactions.



- **Four-graviton amplitudes in Type II at higher genus**

– provide effective interactions of the type $D^{2w}\mathcal{R}^4$ for $w \geq 0$

– interplay with predictions from supersymmetry and S-duality in Type IIB

(cfr talk by Michael Green)

Genus-one effective interactions in Type II

- Recall the Type II genus-one four-graviton amplitude,

$$\mathcal{A}^{(1)}(\varepsilon_i, k_i) = \mathcal{K}\tilde{\mathcal{K}} \int_{\mathcal{M}_1} \frac{d^2\tau}{\tau_2^2} \mathcal{B}^{(1)}(s_{ij}|\tau)$$

$$\mathcal{B}^{(1)}(s_{ij}|\tau) = \prod_{i=1}^4 \int_{\Sigma} \frac{d^2z_i}{\tau_2} \exp \left\{ \sum_{i<j} s_{ij} G(z_i - z_j|\tau) \right\}$$

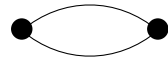
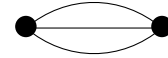
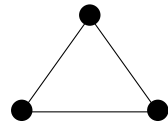
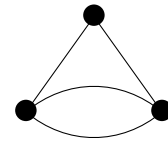
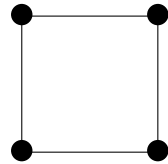
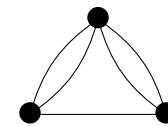
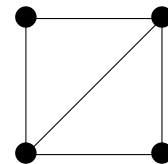
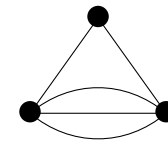
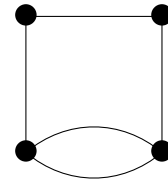
- For fixed τ , expansion of $\mathcal{B}^{(1)}$ in powers of s_{ij}
 - finite radius of convergence $|s_{ij}| < 1$
 - integrals $\int_{\Sigma^4} G^w$ are convergent modular functions, contribute to $D^{2w}\mathcal{R}^4$
 - G is given by a Fourier sum over torus momenta $(m, n) \in \mathbb{Z}^2$

$$G(\alpha + \beta\tau|\tau) = \sum_{(m,n) \neq (0,0)} \frac{\tau_2}{\pi|m + \tau n|^2} e^{2\pi i(m\beta - n\alpha)}$$

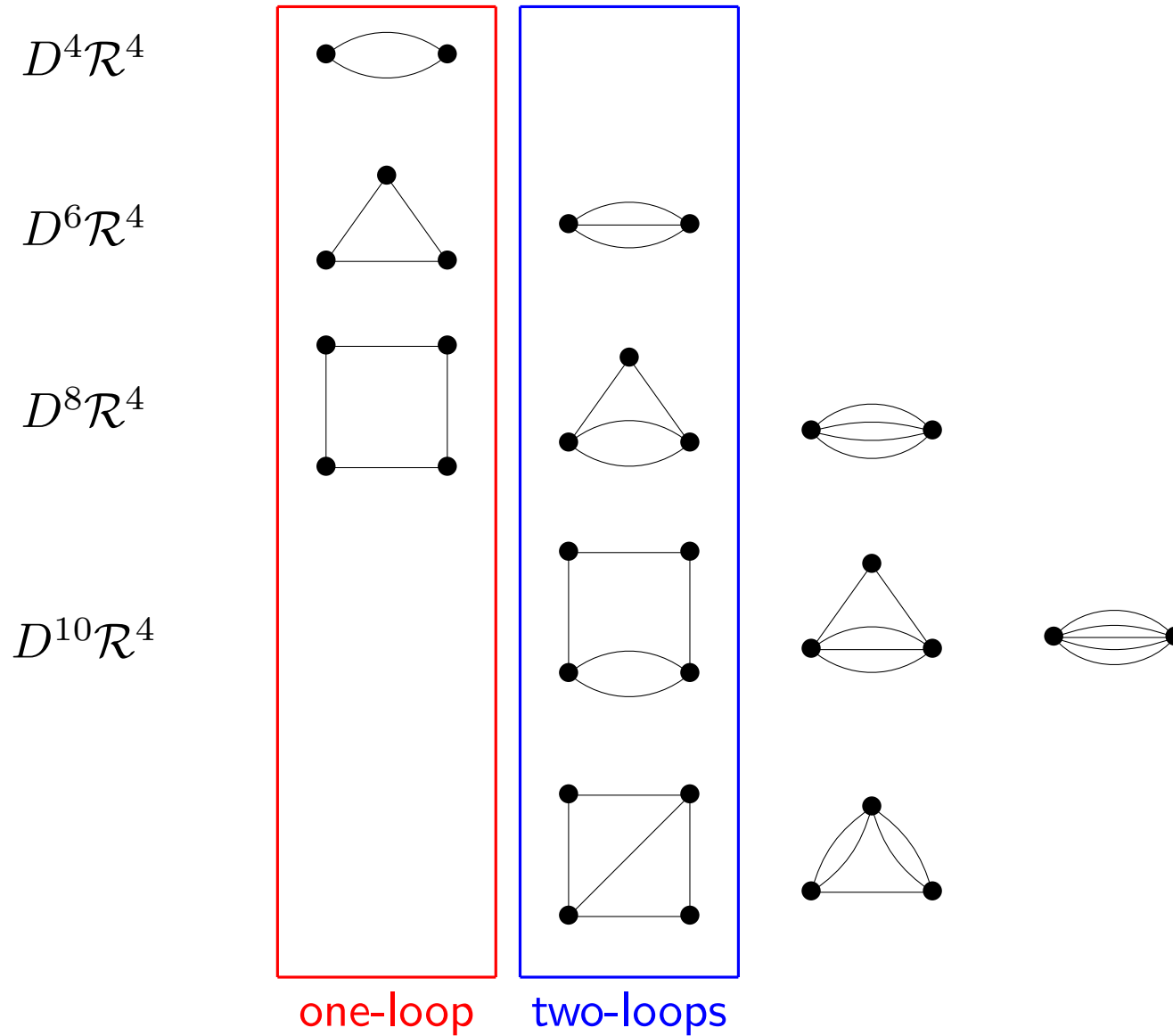
- Integrating $\mathcal{B}^{(1)}$ over τ produces branch cuts in s_{ij} starting at 0
 - extract non-analytic part in s_{ij} prior to extracting effective interactions

Modular graph functions

- Graph in the expansion of $D^{2w}\mathcal{R}^4 \implies$ Modular Function

 $D^4\mathcal{R}^4$

 $D^6\mathcal{R}^4$

 $D^8\mathcal{R}^4$

 $D^{10}\mathcal{R}^4$


Modular graph functions



One-loop : Eisenstein series

- One-loop worldsheet Feynman diagram with k bivalent vertices

$$\prod_{i=1}^k \int_{\Sigma} \frac{d^2 z_i}{\tau_2} G(z_i - z_{i+1} | \tau) = \sum_{\substack{p=m+\tau n \neq 0 \\ m, n \in \mathbb{Z}}} \frac{\tau_2^k}{\pi^k |p|^{2s}}$$

- Non-holomorphic Eisenstein series are defined by,

$$E_s(\tau) = \sum_{\substack{p=m+\tau n \neq 0 \\ m, n \in \mathbb{Z}}} \frac{\tau_2^s}{\pi^s |p|^{2s}}$$

- **Properties**

- absolutely convergent for $\text{Re}(s) > 1$; analytically continue to $s \in \mathbb{C}$
- reflection relation $\Gamma(s)E_s(\tau) = \Gamma(1-s)E_{1-s}(\tau)$
- modular invariant under $SL(2, \mathbb{Z})$, $E_s(\tau') = E_s(\tau)$ with $\tau' = \frac{a\tau+b}{c\tau+d}$
- Laplace-eigenvalue equation,

$$\left(\Delta - s(s-1) \right) E_s(\tau) = 0 \quad \Delta = 4\tau_2^2 \partial_{\tau} \partial_{\bar{\tau}}$$

Two-loops : modular graph functions

- Feynman diagrams evaluate to the modular functions

$$C_{a_1, a_2, a_3}(\tau) = \sum_{\substack{p_r = m_r + \tau n_r \neq 0 \\ m_r, n_r \in \mathbb{Z}, r=1,2,3}} \delta \left(\sum_{r=1}^3 p_r \right) \prod_{r=1}^3 \left(\frac{\tau_2}{\pi |p_r|^2} \right)^{a_r}$$

- contribute to $D^{2w} \mathcal{R}^4$ with the *weight* given by $w = a_1 + a_2 + a_3$
- satisfy (inhomogeneous) Laplace-eigenvalue equations

$$w = 3 \quad C_{1,1,1} = \text{Diagram} \quad (\Delta - 0)C_{1,1,1} = 6E_3$$

$$w = 4 \quad C_{2,1,1} = \text{Diagram} \quad (\Delta - 2)C_{2,1,1} = 9E_4 - E_2^2$$

$$w = 5 \quad C_{3,1,1} = \text{Diagram} \quad (\Delta - 6)C_{3,1,1} = 3C_{2,2,1} + 16E_5 - 4E_2E_3$$

$$w = 5 \quad C_{2,2,1} = \text{Diagram} \quad (\Delta - 0)C_{2,2,1} = 8E_5$$

- Note that eigenvalues are of the form $s(s - 1)$ for $s = 1, 2, 3$;

Structure Theorem

- $C_{a,b,c}(\tau)$ are linear combinations of modular functions $\mathfrak{C}_{w;s;p}(\tau)$ satisfying

$$(\Delta - s(s-1))\mathfrak{C}_{w;s;p} = \mathfrak{F}_{w;s;p}(E_{s'})$$

- $\mathfrak{C}_{w;s;p}$ and $\mathfrak{F}_{w;s;p}$ of weight $w = a + b + c$ (with $E_{s'}$ assigned weight s');
- \mathfrak{F} is a polynomial of total degree 2 in $E_{s'}$ with $2 \leq s' \leq w$;

$$s = w - 2m \qquad m = 1, \dots, \left\lfloor \frac{w-1}{2} \right\rfloor \qquad p = 0, \dots, \left\lfloor \frac{s-1}{3} \right\rfloor$$

- Examples at low weight

| | | |
|---------|------------------|--|
| $w = 3$ | $s = 1$ | $0^{(1)}$ |
| $w = 4$ | $s = 2$ | $2^{(1)}$ |
| $w = 5$ | $s = 1, 3$ | $0^{(1)} \oplus 6^{(1)}$ |
| $w = 6$ | $s = 2, 4$ | $2^{(1)} \oplus 12^{(2)}$ |
| $w = 7$ | $s = 1, 3, 5$ | $0^{(1)} \oplus 6^{(1)} \oplus 20^{(2)}$ |
| $w = 8$ | $s = 2, 4, 6$ | $2^{(1)} \oplus 12^{(2)} \oplus 30^{(2)}$ |
| $w = 9$ | $s = 1, 3, 5, 7$ | $0^{(1)} \oplus 6^{(1)} \oplus 20^{(2)} \oplus 42^{(3)}$ |

Constructive Proof

- A natural generating function is given by,

$$\mathcal{W}(t_1, t_2, t_2|\tau) = \sum_{a,b,c=1}^{\infty} t_1^{a-1} t_2^{b-1} t_3^{c-1} C_{a,b,c}(\tau)$$

Summing gives sunset diagram for three scalars with masses $-t_r$,

$$\mathcal{W}(t_1, t_2, t_2|\tau) = \sum_{\substack{p_r = m_r + \tau n_r \neq 0 \\ m_r, n_r \in \mathbb{Z}}} \delta\left(\sum_r p_r\right) \prod_{r=1}^3 \left(\frac{1}{\pi |p_r|^2 / \tau_2 - t_r}\right)$$

- Laplacian given by differential operator in the masses acting on \mathcal{W} ,

$$\Delta \mathcal{W} - \mathcal{L}^2 \mathcal{W} = \text{quadratic polynomial in } E_s$$

- Polynomials in t_r homogeneous of degree w on hyperbolic plane
- \mathcal{L}^2 is the quadratic Casimir of $SO(2, 1)$ acting on this hyperbolic plane
- $SO(2, 1)$ representation theory gives constructive proof of Structure Theorem.

Type IIB effective interactions at genus-two

- Recall Type II four-graviton amplitude at genus 2,

$$\mathcal{A}^{(2)}(\varepsilon_i, k_i) = \kappa \tilde{\kappa} \int_{\mathcal{M}_2} d\mu_2 \mathcal{B}^{(2)}(s_{ij}|\Omega)$$

$$\mathcal{B}^{(2)}(s_{ij}|\Omega) = \int_{\Sigma^4} \mathcal{Y} \wedge \bar{\mathcal{Y}} \exp \sum_{i < j} s_{ij} G(z_i, z_j)$$

- $\mathcal{Y} = (s - t)\Delta(z_1, z_3) \wedge \Delta(z_4, z_2) + 2$ permutations;
- $\Delta(z_i, z_j)$ is a holomorphic form independent of s, t, u .

- Contributions to local effective interactions,

- \mathcal{R}^4 : zero, since \mathcal{Y} vanishes for $s = t = u = 0$;
- $D^4\mathcal{R}^4$: non-zero, $\mathcal{B}^{(2)}$ constant on \mathcal{M}_2 ;
- $D^6\mathcal{R}^4$: non-zero, one power of G brought down in integral over Σ^4 ;

$$\mathcal{B}^{(2)}(s_{ij}|\Omega) = 32(s^2 + t^2 + u^2) + 192 stu \varphi(\Omega) + \mathcal{O}(s^4, \dots)$$

- $\varphi(\Omega)$ coincides with the Zhang -Kawazumi invariant [ED, Green 2013].

The Zhang-Kawazumi invariant for genus-two

- The ZK-invariant is given as follows

$$8\varphi(\Omega) = \sum_{I,J,K,L} \left(Y_{IJ}^{-1} Y_{KL}^{-1} - 2Y_{IL}^{-1} Y_{JK}^{-1} \right) \int_{\Sigma^2} G(x, y) \omega_I(x) \overline{\omega_J(x)} \omega_K(y) \overline{\omega_L(y)}$$

– equivalent to definition via Arakelov geometry [Zhang 2007, Kawazumi 2008]

- Coefficient of genus-two $D^6\mathcal{R}^4$ interaction involves $\int_{\mathcal{M}_2} d\mu_2 \varphi(\Omega)$

– Direct evaluation appeared completely out of reach ... until ...

The Zhang-Kawazumi invariant for genus-two

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- Coefficient of genus-two $D^6 \mathcal{R}^4$ interaction involves $\int_{\mathcal{M}_2} d\mu_2 \varphi(\Omega)$

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- Theorem [ED, Green, Pioline, R. Russo 2014]

$$(\Delta - 5)\varphi = -2\pi\delta_{SN}$$

- Δ is the Laplace-Beltrami operator on \mathcal{M}_2 with Siegel metric ds_2^2 ;
- δ_{SN} has support on separating node (into two genus-one surfaces)
- The integral over \mathcal{M}_2 reduces to an integral over $\partial\mathcal{M}_2$

$$\int_{\mathcal{M}_2} d\mu_2 \varphi = \frac{1}{5} \int_{\mathcal{M}_2} d\mu_2 (\Delta\varphi + 2\pi\delta_{SN}) = \frac{2\pi^3}{45}$$

Supersymmetry and S-duality

- Laplace-eigenvalue eq from space-time supersymmetry [Green, Sethi, 1998]
 - Eisenstein series = unique modular solution with polynomial growth at cusp

- Predicts vanishing contributions for high enough loop order,

| | | | |
|--------------------|---------|------------|---|
| \mathcal{R}^4 | 1/2 BPS | $h \geq 2$ | $E_{\frac{3}{2}}$ |
| $D^4\mathcal{R}^4$ | 1/4 BPS | $h \geq 3$ | $E_{\frac{5}{2}}$ |
| $D^6\mathcal{R}^4$ | 1/8 BPS | $h \geq 4$ | $(\Delta - 12)\mathcal{E}_{D^6\mathcal{R}^4} = (E_{\frac{3}{2}})^2$ |

[Green, Gutperle, Vanhove 1997; Green, Vanhove 2005]

- Predicts relations between non-vanishing contributions (e.g. with tree-level),

| | | |
|--------------------|--------------|----------------------------------|
| \mathcal{R}^4 | $h = 1$ | [Green, Gutperle 1997] |
| $D^4\mathcal{R}^4$ | $h = 2$ | [ED, Gutperle, Phong 2005] |
| $D^6\mathcal{R}^4$ | $h = 2$ (ZK) | [ED, Green, Pioline, Russo 2014] |
| | $h = 3$ | [Gomez, Mafra 2013] |

Speculation

- Can ambi-twistor prescription be obtained from chiral amplitudes ?

1. Moduli space compactified by Deligne-Mumford divisors $\Delta = \overline{\mathfrak{M}}_h - \mathfrak{M}_h$
 Reducible to separating and non-separating divisors with normal crossings
 Non-separating divisors Δ_0 define cohomology class
 \implies Instead of integrating over \mathfrak{M}_h , integrate over Δ_0
2. Rescale loop momenta $p_I^\mu \text{Im}(\Omega)_{IJ} p_J^\mu \rightarrow \ell_I^\mu \ell_I^\mu$
3. Superconformal structure deformations $\delta_{ww} \rightarrow$ insert T_{ww} and $\delta_{w\theta} \rightarrow$ insert $S_{w\theta}$

$$\mathcal{F} = \exp \left\{ i\pi p \Omega p + 2\pi i p \sum_i k_i \int^{z_i} \omega + \sum_{i < j} (k_i k_j \ln E(i, j) - i \varepsilon_i k_j \partial_i \ln E(i, j) + \dots) \right\}$$

Variational equations $\delta_{ww} \mathcal{F} = \Pi(w)^2 \mathcal{F}$

$$\Pi^\mu(w) = \pi i p_I \omega_I(w) + \sum_i (k_i \partial_w \ln E(w, i) + \varepsilon_i \partial_w \partial_i \ln E(w, i) + \dots)$$

requirement that the dependence on moduli be “topological”