# New genus-two modular invariants & string theory

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Automorphic forms, mock modular forms and string theory

# Bibliography

## Based on


- ED, Michael Green, Boris Pioline, in preparation (2017),
  *Asymptotic properties of genus-two modular graph functions, I & II*


## and earlier work


- ED, Michael Green, arXiv:1308.4597, Journal of Number Theory, Vol 144 (2014) 111-150,
  *Zhang-Kawazumi invariants and Superstring Amplitudes*
- ED, Michael Green, Boris Pioline, Rudolfo Russo, arXiv:1405.6226, JHEP 1501 (2015) 031,
  *Matching the $D^6\mathcal{R}^4$ interaction at two-loops*
- Boris Pioline, arXiv:1504.04182, Journal of Number Theor. **163**, 520 (2016),
  *A Theta lift representation for the Kawazumi-Zhang and Faltings invariants of genus-two Riemann surfaces*

# Introduction

- **String theory naturally generalizes non-holomorphic Eisenstein series**
  - modular graph functions for genus one surfaces,
    - ⋆ multiple Kronecker-Eisenstein sums,
    - ⋆ multiple integrations of Green function on the torus.

- **String theory includes contributions from surfaces of all genera**
  $\Longrightarrow$ **expect modular graph functions for higher genus surfaces.**

- **Focus of this talk is on genus two**
  - simplest is Kawazumi-Zhang invariant (to be explained below)
    (ED, Green, Pioline, Russo 2014)
  - string theory predicts an infinite number of higher string invariants
    (ED, Green 2013)
  - I will give an account of what we know and do not know to date.

# Genus-one string amplitude

- **Effective $\mathcal{R}^4$-type interactions in Type II** (see Michael Green's talk)
  - Generated by a multiple integral over a torus $\Sigma_1 = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$, of modulus $\tau \in \mathbb{H}$, namely $\tau = \tau_1 + i\tau_2$ with $\tau_1, \tau_2 \in \mathbb{R}$ and $\tau_2 > 0$,

$$\mathcal{B}^{(1)}(s_{ij}|\tau) = \prod_{i=1}^{N} \int_{\Sigma_1} \frac{d^2 z_i}{\tau_2} \, \exp\left\{ \sum_{1 \leq i < j \leq N} s_{ij} \, g(z_i - z_j|\tau) \right\}$$

- **Mathematically, one may consider this integral for arbitrary $N$**
  - $g(z|\tau)$ is the translation-invariant Green function on $\Sigma_1$,

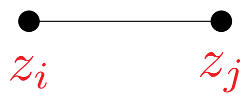
$$\tau_2 \partial_{\bar z} \partial_z g(z|\tau) = -\pi \delta^{(2)}(z) + \pi \qquad\qquad \int_{\Sigma_1} \frac{d^2 z}{\tau_2} g(z|\tau) = 0$$

  - Complex parameters $s_{ij} = s_{ji}$ satisfy $\sum_{j=1}^N s_{ij} = 0$ for all $i$;
  - Integrals absolutely convergent for $|s_{ij}| < 1$; analytic near $s_{ij} = 0$;
  - $\mathcal{B}^{(1)}(s_{ij}|\tau)$ is invariant under the modular group $SL(2,\mathbb{Z})$.

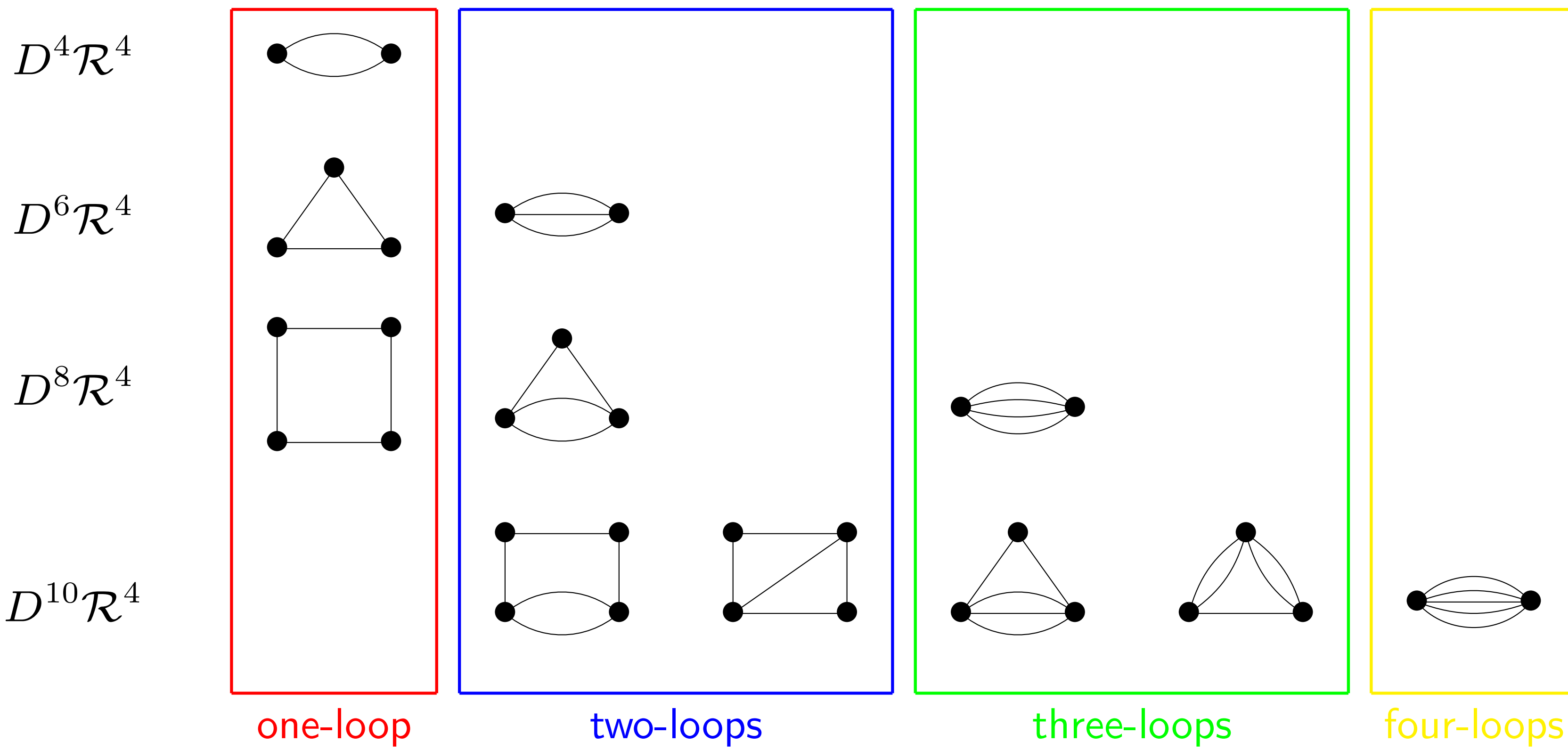
- **String amplitude obtained by further integral over $\tau \in \mathbb{H}/SL(2,\mathbb{Z})$.**

# Genus-one modular graph functions

- **Taylor series expansion of $\mathcal{B}^{(1)}(s_{ij}|\tau)$ for fixed $\tau$ in powers of $s_{ij}$**
  - An integration point $z_i$ is represented by a vertex •
  - A Green function is represented by an edge

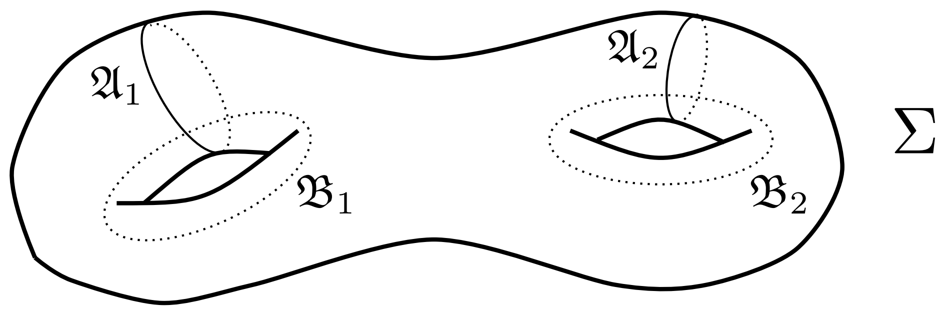



$= \quad g(z_i - z_j|\tau)$



# Genus-two surfaces

- $\Sigma$ **is a compact Riemann surface of genus two**
  - Key difference with genus-one: no translation symmetry




- **Homology and cohomology**
  - One-cycles $H_1(\Sigma, \mathbb{Z}) \approx \mathbb{Z}^4$ with intersection pairing $\mathfrak{J}(\cdot,\cdot) \to \mathbb{Z}$
  - Canonical basis $\mathfrak{J}(\mathfrak{A}_I, \mathfrak{A}_J) = 0, \ \mathfrak{J}(\mathfrak{B}_I, \mathfrak{B}_J) = 0$ with $I, J = 1, 2$

$$\mathfrak{J}(\mathfrak{A}_I, \mathfrak{B}_J) = \delta_{IJ}, \ \mathfrak{J}(\mathfrak{B}_I, \mathfrak{A}_J) = -\delta_{IJ}$$

  - Canonical dual basis of holomorphic one-forms $\omega_I$ in $H^{(1,0)}(\Sigma)$

$$\oint_{\mathfrak{A}_I} \omega_J = \delta_{IJ} \qquad\qquad \oint_{\mathfrak{B}_I} \omega_J = \Omega_{IJ}$$

  - Period matrix $\Omega$ obeys Riemann relations $\Omega^t = \Omega, \ \mathrm{Im}(\Omega) > 0$

# Modular transformations and geometry

- **Transformation** $Sp(4,\mathbb{Z}) : H_1(\Sigma,\mathbb{Z}) \to H_1(\Sigma,\mathbb{Z})$ **leaves** $\mathfrak{J}(\cdot,\cdot)$ **invariant**
  - action on basis cycles given by

$$\begin{pmatrix} \mathfrak{B}_I \\ \mathfrak{A}_I \end{pmatrix} \to \sum_J M_{IJ} \begin{pmatrix} \mathfrak{B}_J \\ \mathfrak{A}_J \end{pmatrix} \qquad\qquad M^t \mathfrak{J} M = \mathfrak{J}$$

  - action on 1-forms $\omega_I$ and periods $\Omega_{IJ}$ given by

$$\begin{aligned} \omega &\to \omega\,(C\Omega + D)^{-1} \\ \Omega &\to (A\Omega + B)\,(C\Omega + D)^{-1} \end{aligned} \qquad\qquad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

- **Siegel upper half space** $\mathcal{S}_2$

$$\mathcal{S}_2 = \Big\{ \Omega_{IJ} \in \mathbb{C} \text{ with } \Omega^t = \Omega \text{ and } Y = \mathrm{Im}(\Omega) > 0 \Big\}$$

  - $\mathcal{S}_2 = \frac{Sp(4,\mathbb{R})}{SU(2)\times U(1)} = \frac{SO(3,2)}{SO(3)\times SO(2)}$ is Kähler with invariant metric

$$ds_2^2 = \sum_{I,J,K,L=1,2} Y^{-1}_{IJ}\, d\bar\Omega_{JK}\, Y^{-1}_{KL}\, d\Omega_{LI}$$

  - Moduli space of genus-two surfaces is $\mathcal{S}_2/Sp(4,\mathbb{Z})$ (minus diagonal $\Omega$)

# Green function and volume form

- **How to generalize the genus-one formula to a genus-two formula ?**
  - recall the genus-one formula

$$\mathcal{B}^{(1)}(s_{ij}|\tau) = \prod_{i=1}^{N} \int_{\Sigma_1} \frac{d^2 z_i}{\tau_2} \exp\Big\{ \sum_{1 \leq i < j \leq N} s_{ij}\, g(z_i - z_j|\tau) \Big\}$$

- **Natural "Arakelov metric" and volume form for genus-two** $\Sigma$
  - modular invariant and smooth

$$\kappa_A = \frac{i}{4} \sum_{I,J} Y^{-1}_{IJ} \omega_I \wedge \overline{\omega_J} \qquad \qquad \int_\Sigma \kappa_A = 1$$

- **Natural "Arakelov Green function"** $G_A(w,z|\Omega) = G_A(z,w|\Omega)$
  - Inverse of scalar Laplace operator for Arakelov metric

$$\begin{aligned} \partial_{\bar w} \partial_w G_A(w,z|\Omega) &= -\pi \delta(w,z) + \pi \kappa_A(w) \\ \int_\Sigma \kappa_A(w) G_A(w,z|\Omega) &= 0 \end{aligned}$$

# A natural genus-two candidate

- **A natural candidate formula for a string amplitude would be**

$$\mathcal{C}^{(2)}(s_{ij}|\Omega) = \prod_{i=1}^{N} \int_\Sigma \kappa_A(z_i) \, \exp\Big\{ \sum_{1\leq i<j\leq N} s_{ij} \, G_A(z_i,z_j|\Omega) \Big\}$$

  - Integrals absolutely convergent for $|s_{ij}|<1$; analytic near $s_{ij}=0$,
  - Expanding in powers of $s_{ij}$ gives *genus-two modular graph functions*.

- **But ... Genus-two string amplitudes are NOT given by $\mathcal{C}^{(2)}(s_{ij}|\Omega)$**

- **For integration over a single copy of $\Sigma$**
  - $\kappa_A$ is the only natural modular invariant volume form.

- **For integration over multiple copies of $\Sigma$**
  - $Sp(4,\mathbb{Z})$ modular invariants other than $\prod_i \kappa_A(z_i)$ allowed.
  - For example, when $N=2$ we can have $\kappa_A(z_1)\kappa_A(z_2)$ as well as

$$\sum_{I,J,K,L} Y^{-1}_{IL} \, Y^{-1}_{JK} \, \omega_I(z_1) \, \overline{\omega_J(z_1)} \, \omega_K(z_2) \, \overline{\omega_L(z_2)}$$

# Genus-two string amplitude

- **Instead the $N = 4$ graviton amplitude was calculated** (ED, Phong 2005)

$$\mathcal{B}^{(2)}(s_{ij}|\Omega) = \int_{\Sigma^4} \frac{\mathcal{Y} \wedge \bar{\mathcal{Y}}}{(\det Y)^2} \exp\left\{ \sum_{1\leq i<j\leq 4} s_{ij}\, G_A(z_i, z_j) \right\}$$

- **The key difference with the candidate $\mathcal{C}^{(2)}$ is the structure of $\mathcal{Y}$**

$$\begin{aligned} 3\mathcal{Y} \quad = \quad & (t-u)\Delta(z_1,z_2) \wedge \Delta(z_3,z_4) & \qquad s = s_{12} = s_{34} \\ & +(s-t)\Delta(z_1,z_3) \wedge \Delta(z_4,z_2) & \qquad t = s_{13} = s_{24} \\ & +(u-s)\Delta(z_1,z_4) \wedge \Delta(z_2,z_3) & \qquad u = s_{14} = s_{23} \end{aligned}$$

  – where $\Delta$ is a holomorphic $(1,0)_i \times (1,0)_j$ form on $\Sigma \times \Sigma$

$$\Delta(z_i, z_j) = \varepsilon^{IJ} \omega_I(z_i) \wedge \omega_J(z_j)$$

  – The combination $\mathcal{Y} \wedge \bar{\mathcal{Y}}/(\det Y)^2$ is $Sp(4,\mathbb{Z})$-invariant,
  – and produces a modular invariant $\mathcal{B}^{(2)}(s_{ij}|\Omega)$.

# Low energy expansion

- **Contributions to local effective interactions**
  - Expand $\mathcal{B}^{(2)}(s_{ij}|\Omega)$ in powers of $s_{ij}$ and integrate over $\mathcal{M}_2 = \mathcal{S}_2/Sp(4,\mathbb{Z})$

| | |
|---|---|
| $\mathcal{R}^4, D^2\mathcal{R}^4$ | zero, since $\mathcal{Y}$ vanishes for $s=t=u=0$ |
| $D^4\mathcal{R}^4$ | Siegel volume form on $\mathcal{M}_2$ |
| $D^6\mathcal{R}^4$ | one factor of $G_A$ in expansion in powers of $s_{ij}$ |

$$\mathcal{B}^{(2)}(s_{ij}|\Omega) = 32(s^2+t^2+u^2) + 192\, stu\, \varphi(\Omega) + \mathcal{O}(s_{ij}^4)$$

$$\varphi(\Omega) = -\frac{1}{4}\sum_{I,J,K,L} Y^{-1}_{IL} Y^{-1}_{JK} \int_{\Sigma^2} G_A(x,y)\omega_I(x)\overline{\omega_J(x)}\omega_K(y)\overline{\omega_L(y)}$$

- $\varphi(\Omega)$ **coincides with the Kawazumi-Zhang invariant** (ED, Green 2013)
  - introduced as a spectral invariant (Kawazumi 0801.4218 and Zhang 0812.0371)
  - related to the genus-two Faltings invariant (De Jong 2010)
  - formulated in terms of modular tensors (Kawazumi OIST lecture notes 2016)

$$\mathcal{A}_{IJ;KL} = \int_{\Sigma^2} G_A(x,y)\omega_I(x)\overline{\omega_J(x)}\omega_K(y)\overline{\omega_L(y)}$$

# Higher string-invariants

- **The KZ-invariant exists for all genera $\geq 2$** (Kawazumi 2008; Zhang 2008)
  - but is probably not the correct object for string theory at genus $\geq 3$.
- **But the Taylor expansion coefficients of $\mathcal{B}^{(2)}(s_{ij}|\Omega)$**
  - are *modular graph functions at genus-two*;
  - do naturally emerge from string theory at genus-two;
  - provide a string-motivated generalization of KZ-invariants at genus-two.
- **Higher string-invariants** (contribute to $D^8\mathcal{R}^4$ and $D^{10}\mathcal{R}^4$) (ED, Green 2013)

$$
\begin{aligned}
\mathcal{B}^{(2)}_{(2,0)} &= \int_{\Sigma^4} \frac{|\Delta(1,2)\Delta(3,4)|^2}{(\det Y)^2} \Big( G_A(1,4) + G_A(2,3) - G_A(1,3) - G_A(2,4) \Big)^2 \\
\mathcal{B}^{(2)}_{(1,1)} &= \int_{\Sigma^4} \frac{|\Delta(1,2)\Delta(3,4) - \Delta(1,4)\Delta(2,3)|^2}{(\det Y)^2} \Big( G_A(1,2) + G_A(3,4) \\
&\qquad + G_A(1,4) + G_A(2,3) - 2G_A(1,3) - 2G_A(2,4) \Big)^3 \\
\cdots &= \cdots
\end{aligned}
$$

# Differential equations and Asymptotics

- **Genus-one modular graph functions satisfy** (see Michael's talk)
  - System of inhomogeneous Laplace-eigenvalue equations
  - Laurent polynomial behavior near the cusp of degree bounded by weight

- **Genus-two KZ-invariant $\varphi$ satisfies**
  - Characteristic class relations and role in Johnson homomorphism
    (Kawazumi 2008; Zhang 2008; DeJong 2010)
  - Eigenvalue equations for $Sp(4,\mathbb{R})$-invariant differential operators
    (ED, Green, Pioline, Russo 2014; Pioline 2015)
  - Theta-lift analogous to Borcherds for Igusa cusp form (Pioline 2015)
  - Laurent polynomial of degree $(1,1)$ near non-separating divisor
    (De Jong 2010; Pioline 2015; ED, Green, Pioline 2017)

- **Genus-two higher string-invariants satisfy**
  - Laurent polynomial of bounded degree near non-separating degeneration
    (in progress ED, Green, Pioline 2017)
  - System of differential equations for invariant differential operators
    (in progress ED, Green, Pioline 201?)

# Differential equations

- **Theorem 1** *Laplace eigenvalue equation* (ED, Green, Pioline, Russo 2014)

$$(\Delta - 5)\varphi = -2\pi\delta_{SN}$$

  – $\delta_{SN}$ has support on separating node (into two genus-one surfaces)
  – $\Delta$ is the Laplace-Beltrami operator on $\mathcal{S}_2$ with Siegel metric

$$\Delta = 4 \sum_{I,J,K,L} Y_{IK} Y_{JL} \bar\partial^{IJ} \partial^{KL} \qquad \partial^{IJ} = \frac{1}{2}(1+\delta^{IJ}) \frac{\partial}{\partial \Omega_{IJ}}$$

  – proven by theory and methods of deformations of complex structures

- **Theorem 2** *Quartic differential operator eigenvalue equation* (Pioline 2015)

$$(32\, \Box_2^* \Box_0 - 15)\varphi = 0$$

  – $\Box_0, \Box_2$ are Maass-Siegel operators

$$\Box_0 = \varepsilon^{IJ} \varepsilon^{KL} \nabla_{IK} \nabla_{JL} \qquad \nabla_{IJ} = Y_{IK} Y_{JL} \partial^{KL}$$

- **System of differential equations satisfied by higher string invariants ?**

  (in progress ED, Green, Pioline 201?)

# Asymptotics

**For genus one, there is only one type of degeneration, as modulus $\tau \to i\infty$**

- *Behavior of holomorphic Eisenstein series*

$$\mathbb{G}_{2k}(\tau) = -\frac{B_{2k}}{2k} + \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n \qquad q = e^{2\pi i \tau}$$

- *Behavior of non-holomorphic Eisenstein series,* $y = \pi \mathrm{Im}(\tau)$

$$E_k(\tau) = \frac{2\zeta(2k)}{\pi^{2k}} y^k + 4 \begin{pmatrix} 2k-3 \\ k-1 \end{pmatrix} \frac{\zeta(2k-1)}{(4y)^{k-1}} + \mathcal{O}(|q|)$$

- *Behavior of general genus-one modular graph functions.*
  - Finite degree Laurent polynomial in $y$ plus exponentials,
  - eg two-loop modular graph functions (ED, Bill Duke 2017)

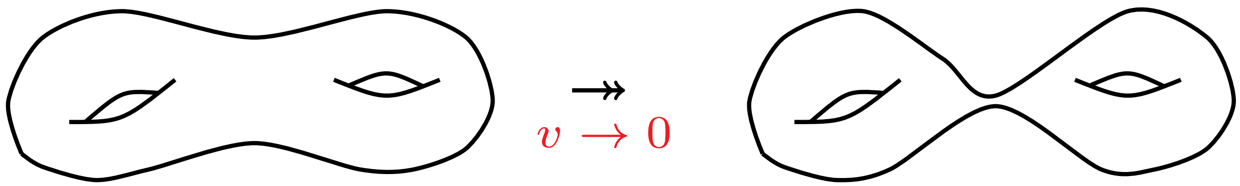
$$C_{a,b,c}(\tau) = c_w y^w + \frac{c_{2-w}}{y^{w-2}} + \sum_{k=1}^{w-1} \frac{c_{w-2k-1} \zeta(2k+1)}{y^{2k+1-w}} + \mathcal{O}(|q|)$$

$$c_{2-w} = \sum_{m=1}^{w-2} \gamma_m \, \zeta(2m+1) \, \zeta(2w-2m-3)$$

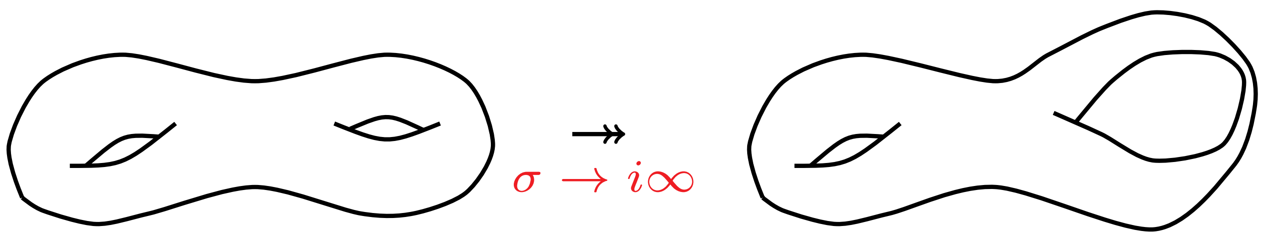
  - with $w = a+b+c$ and $c_w, \gamma_m, c_{w-2k-1} \in \mathbb{Q}$.

# Degenerations of genus-two Riemann surfaces $\Omega = \begin{pmatrix} \tau & v \\ v & \sigma \end{pmatrix}$

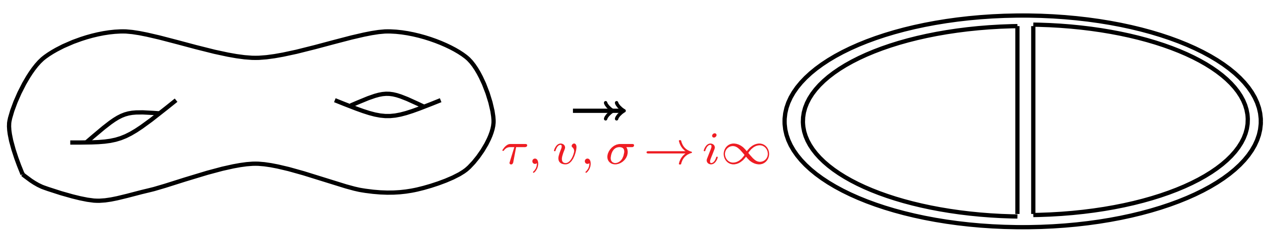
– *Separating degeneration*




– *Non-separating degeneration*




– *Maximal degeneration (or "tropical limit")*



# Non-separating degeneration

- **A genus-two surface $\Sigma$ degenerates to a torus $\Sigma_1$ with two punctures $p_a, p_b$**
  - keep the cycles $\mathfrak{A}_1, \mathfrak{B}_1, \mathfrak{A}_2$ fixed, and let $\mathfrak{B}_2 \to \infty$

$$\Omega = \begin{pmatrix} \tau & v \\ v & \sigma \end{pmatrix} \qquad\qquad \mathrm{Im}(\sigma) \to \infty$$

  - $\tau$ is the modulus of $\Sigma_1$ and $v = \int_{p_a}^{p_b} \omega_1 = p_b - p_a$

- **The genus-two modular group $Sp(4,\mathbb{Z})$ restricts to $SL(2,\mathbb{Z})$**
  - subgroup of $M \in Sp(4,\mathbb{Z})$ such that $M\mathfrak{B}_2 = \mathfrak{B}_2$ is $SL(2,\mathbb{Z})$

$$M = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \left\{ \begin{array}{lll} \tau' & = & (a\tau + b)/(c\tau + d) \\ v' & = & v/(c\tau + d) \\ \sigma' & = & \sigma - cv^2/(c\tau + d) \end{array} \right.$$

  - The degeneration parameter $\sigma$ is not invariant under $SL(2,\mathbb{Z})$

# Degeneration of Siegel modular forms

- **A genus-two** (**"rank two"**) **Siegel modular form** $S$ **of modular weight** $k$
  - is holomorphic and transforms under $Sp(4,\mathbb{Z})$ by,

$$S(\Omega') = \det(C\Omega + D)^k S(\Omega) \qquad\qquad \Omega' = (A\Omega + B)(C\Omega + D)^{-1}$$

  - Fourier expansion in powers of $e^{2\pi i\sigma}$ near non-separating node

$$S(\Omega) = \sum_{m=0}^{\infty} e^{2\pi i m\sigma} \phi_m(v|\tau)$$

  - $\phi_m(v|\tau)$ is a *Jacobi form* of modular weight $k$ and index $m$
  - transforms under the residual modular group $SL(2,\mathbb{Z})$ by

$$\phi_m\left(\frac{v}{c\tau + d}\middle|\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k e^{2\pi i m c v^2/(c\tau+d)} \phi_m(v|\tau)$$

(Eichler and Zagier, "The theory of Jacobi forms", 1985)

# Degeneration of the KZ-invariant

- **Non-separating degeneration of non-holomorphic modular functions**
  - is governed by a *real* $SL(2,\mathbb{Z})$-invariant parameter $t > 0$

$$t \equiv \frac{\det(\mathrm{Im}\,\Omega)}{\mathrm{Im}\,\tau} \qquad\qquad \Omega = \begin{pmatrix} \tau & v \\ v & \sigma \end{pmatrix}$$

  - with the non-separating node being characterized by $t \to \infty$

- **Theorem 3** Non-separating degeneration of KZ-invariant (Pioline 2015)

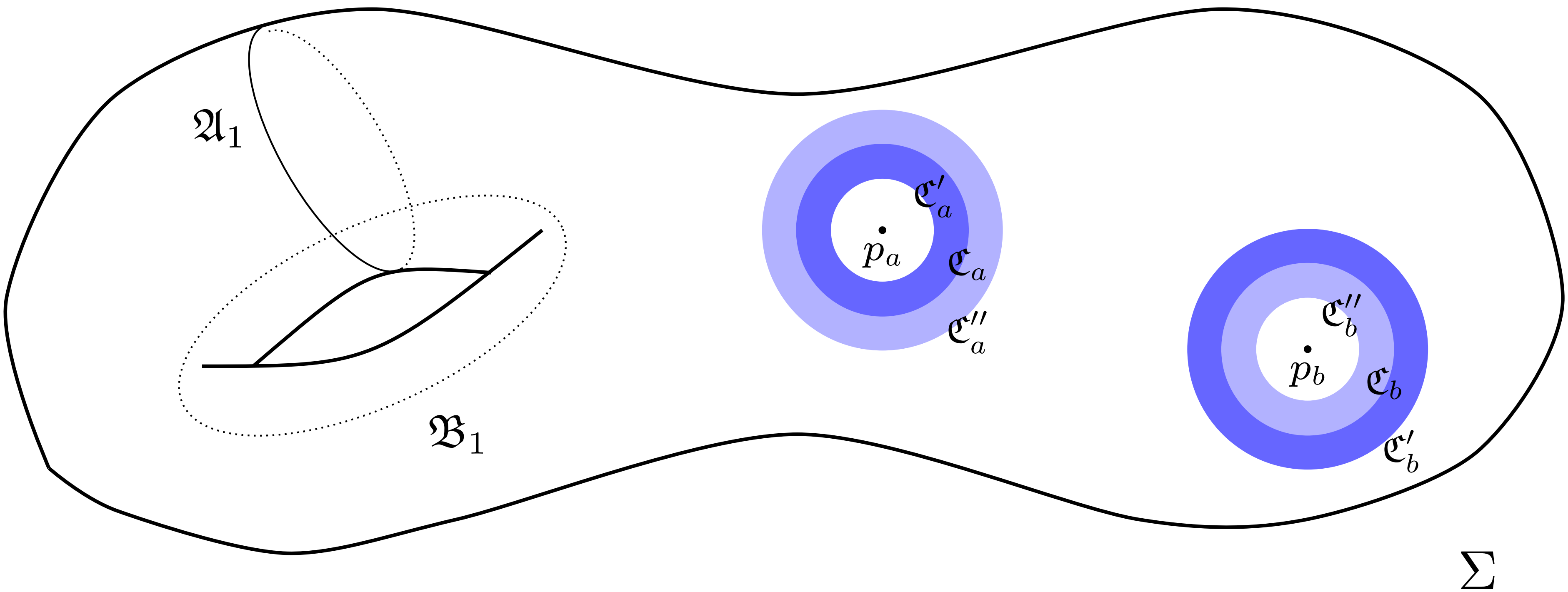
$$\varphi(\Omega) = \frac{\pi t}{6} + \frac{1}{2} g(v|\tau) + \frac{1}{4\pi t}\Big(E_2(\tau) - g_2(v|\tau)\Big) + \mathcal{O}(e^{-2\pi t})$$

  - $g(v|\tau)$ is the torus Green function
  - $E_2$ is the non-holo Eisenstein series
  - $g_2(v|\tau) = \int_{\Sigma_1} d^2z/\tau_2\, g(v-z|\tau) g(z|\tau)$
  - Derived using the Laplace-eigenvalue equation for $\varphi$.

- **The Laurent polynomial is of finite degree in the variable $t$**
  - but it is not of finite degree in, say, $t+1$

# Non-separating node from a punctured surface

- **Standard construction of a surface near a non-separating node** (Fay 1973)
  - start from a genus-one surface with two punctures $p_a, p_b$
  - local coordinates $z_a, z_b$ which vanish respectively at $p_a, p_b$
  - identify points $z_a z_b = \mathfrak{t}$ between two annuli $\mathfrak{C}'_a = \mathfrak{C}'_b$ and $\mathfrak{C}''_a = \mathfrak{C}''_b$




  - in practice not easy to implement, unless one can define cycles $\mathfrak{C}$ naturally

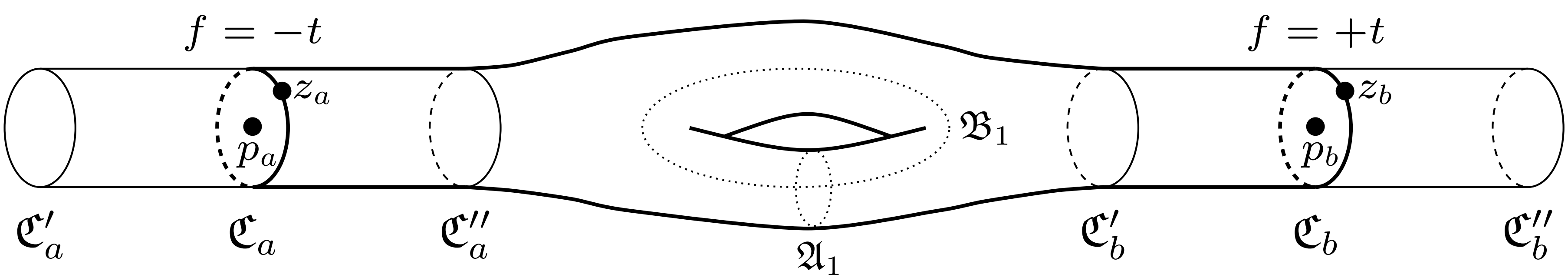
# Non-separating node from a punctured surface (cont'd)

- **Key is the existence of a real single-valued harmonic function $f(z)$ on $\Sigma$**
  - such that in the degeneration limit

    $f(z) \to -\infty$ as $z \to p_a$ and $f(z) \to +\infty$ as $z \to p_b$
  - for large $t$ cycles prescribed by $f(\mathfrak{C}_a) \to -2\pi t$ and $f(\mathfrak{C}_b) \to +2\pi t$

$$f(z) = -2\pi t + 4\pi \, \mathrm{Im} \int_{z_a}^{z} \Big( \omega_2 - \mathrm{Im}(v)\, \omega_1 / \mathrm{Im}(\tau) \Big)$$

- **Funnel construction of the non-separating degeneration**




  - The cycle $\mathfrak{A}_2$ is homologous to the cycles $\mathfrak{C}_a, \mathfrak{C}_a', \mathfrak{C}_a''$ and $\mathfrak{C}_b, \mathfrak{C}_b', \mathfrak{C}_b''$;
  - The cycles are pairwise identified by $\mathfrak{C}_a \approx \mathfrak{C}_b$, $\mathfrak{C}_a' \approx \mathfrak{C}_b'$ and $\mathfrak{C}_a'' \approx \mathfrak{C}_b''$;
  - The points are pairwise identified by $z_a \approx z_b$;
  - The cycle $\mathfrak{B}_2$ may be chosen to be a simple curve connecting $z_a$ to $z_b$.

# Degeneration of higher string invariants

- **Consider the full genus-two string amplitude**

$$\mathcal{B}(s_{ij}|\Omega) = \int_{\Sigma^4} \frac{\mathcal{Y} \wedge \overline{\mathcal{Y}}}{(\det Y)^2} \exp\left\{ \sum_{i<j} s_{ij} G_A(z_i, z_j|\Omega) \right\}$$

  – Expanding in powers of $s_{ij}$ and collecting all terms homogeneous of degree $w$
  – gives modular graph functions of weight $w$ for genus-two surfaces $\Sigma$

$$\mathcal{B}_w(s_{ij}|\Omega) = \int_{\Sigma^4} \frac{\mathcal{Y} \wedge \overline{\mathcal{Y}}}{(\det Y)^2} \left( \sum_{i<j} s_{ij} G_A(z_i, z_j|\Omega) \right)^w$$

- **Theorem 4** *Under non-separating degeneration of* $\Sigma$ (ED, Green, Pioline 2017)
  – $\mathcal{B}_w(s_{ij}|\Omega)$ *has a Laurent polynomial of degree* $(w,w)$ *in* $t$

$$\mathcal{B}_w(s_{ij}|\Omega) = \sum_{k=-w}^{w} \mathcal{B}_w^{(k)}(s_{ij}|v,\tau)\, t^k + \mathcal{O}(e^{-2\pi t})$$

  – $\mathcal{B}_w^{(k)}(s_{ij}|v,\tau)$ *are invariant under* $SL(2,\mathbb{Z}) \subset Sp(4,\mathbb{Z})$

$$\mathcal{B}_w^{(k)}\left( s_{ij} \middle| \frac{v}{c\tau+d}, \frac{a\tau+b}{c\tau+d} \right) = \mathcal{B}_w^{(k)}(s_{ij}|v,\rho)$$

- Note – *similarity with the Laurent polynomial expansion for genus one;*
  – $\mathcal{B}_w^{(k)}(s_{ij}|v,\tau)$ generalize Jacobi forms to a non-holomorphic setting;
  – combine genus-one modular graph functions and *elliptic polylogarithms*.

# Ingredients in proof of Theorem 4

- **Uniform asymptotics of the Green function**

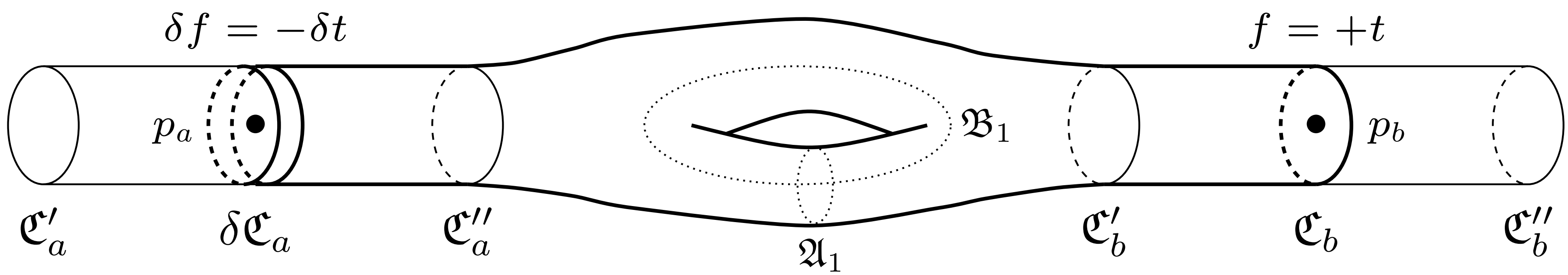
$$G_A(x,y|\Omega) = g(x-y|\tau) + \frac{1}{8\pi t}(f(x)-f(y))^2 + \mathcal{O}(e^{-2\pi t})$$

  – up to immaterial addition of $h(x)+h(y)$ (improving on Wentworth 1991)

- **Measure**

$$4\pi^2 \mathcal{Y} = dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4 \sum_{i<j} s_{ij} \partial_{z_i} f(z_i) \partial_{z_j} f(z_j)$$

- **The cycles $\mathfrak{C}_a, \mathfrak{C}_b$ have exponentially vanishing coordinate radius**
  – $z \in \mathfrak{C}_a$ satisfies $|z-p_a|^2 \approx e^{-2\pi t}$
  – Power dependence in $t$ arise from poles at $p_a, p_b$ in the integrand
  – Extract variation in $t$-dependence (cfr RG with cut-off $t$)



# Summary and outlook

- **Low energy expansion of string theory has revealed a rich structure of**
  - Modular graph functions for genus-one Riemann surfaces;
  - Kawazumi-Zhang and higher string invariants for genus-two surfaces.
- **Asymptotics for higher string invariants**
  - Remarkable amount of structure in Laurent polynomial
  - Being tested versus maximal degeneration limit (ED, Green, Pioline 2017)
  - Crucial for the discovery of relations between modular graph functions
- **Differential equations for higher string invariants**
  - In progress (ED, Green, Pioline 201?)
  - Asymptotics is expected to be a key guide
- **Theta-lift of higher string invariants**
  - In progress (ED, Green, Pioline 20??)