Recent Advances in Two-loop Superstrings

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Outline

1. Overview of two-loop superstring methods, including global issues;

2. Applications to Vacuum Energy and Spontaneous Supersymmetry Breaking
   *Two-Loop Vacuum Energy for Calabi-Yau orbifold models*

3. Applications to Superstring Corrections to Type IIB Supergravity
   E. D’Hoker, M.B. Green, arXiv:1308.4597,
   *Zhang-Kawazumi invariants and Superstring Amplitudes*
   E. D’Hoker, M.B. Green, B. Pioline, R. Russo, arXiv:1405.6226,
   *Matching the $D^6 \mathcal{R}^4$ interaction at two-loops*
**String Perturbation Theory**

**Quantum Strings**: fluctuating surfaces in space-time $M$

![Diagram showing fluctuating surfaces in space-time](image)

**Perturbative expansion** of string amplitudes in powers of coupling constant $g_s$

$= \sum$ over Riemann surfaces $\Sigma$ of genus $h$

$g_s^{-2}$ for $h=0$

$+ g_s^2$ for $h=2$

Bosonic string: 
- sum over maps $\{x\}$
- sum over conformal classes $[g]$ on $\Sigma$

$= \text{integral over moduli space } M_h \text{ of Riemann surfaces.}$
Superstrings

- **Worldsheet** = super Riemann surface

\[(x, \psi)\]  RNS-formulation \(\psi\) spinor on \(\Sigma\)

\[(g, \chi)\]  superconformal geometry

- **Worldsheet action** invariant under local supersymmetry in addition to \(\text{Diff}(\Sigma)\)
  
  Absence of superconformal anomalies requires \(\dim(M) = 10\)

- **Supermoduli Space** \(sM_h\) = space of superconformal classes \([g, \chi]\),

\[
\dim(sM_h) = \begin{cases} 
(0|0) & h = 0 \\
(1|0)_{\text{even}} \text{ or } (1|1)_{\text{odd}} & h = 1 \\
(3h - 3|2h - 2) & h \geq 2 
\end{cases}
\]

- **Two-loops** is lowest order at which odd moduli enter non-trivially.
Independence of left and right chiralities

- Locally on $\Sigma$, worldsheet fields split into left & right chiralities

$$\partial_{\bar{z}} \partial_z x^{\mu} = 0 \implies x^{\mu} = x^{\mu}_+(z) + x^{\mu}_-(\bar{z})$$

$$\partial_{\bar{z}} \psi^\mu_- = \partial_z \psi^\mu_+ = 0 \implies \psi^\mu_+(z), \psi^\mu_-(\bar{z})$$

Fundamental physical closed superstring theories

**Type II**

$\psi^\mu_+$ and $\psi^\mu_-$ are independent (not complex conjugates)

with independent spin structure assignments

odd moduli for left and right are independent

**Heterotic**

$\psi^{\mu}_+$ left chirality fermions with $\mu = 1, \cdots, 10$

$\psi^{A}_-$ right chirality fermions with $A = 1, \cdots, 32$

odd moduli for left, but none for right chirality
Pairing prescription  (Witten 2012)

• Separate moduli spaces for left and right chiralities

  – LEFT : $s\mathcal{M}_L$ of dim $(3h - 3|2h - 2)$ with local coordinates $(m_L, \bar{m}_L; \zeta_L)$

  – RIGHT: Type II string, $s\mathcal{M}_R$ of dim $(3h - 3|2h - 2)$, with $(m_R, \bar{m}_R; \zeta_R)$
    Heterotic string, $\mathcal{M}_R$ of dim $(3h - 3|0)$, with $(m_R, \bar{m}_R)$

• Left and right odd moduli $\zeta_L, \zeta_R$ are independent

• Even moduli must be related
  Heterotic string: integrate over a closed cycle $\Gamma \subset s\mathcal{M}_L \times \mathcal{M}_R$ such that
  – $\bar{m}_R = m_L + \text{even nilpotent corrections dependent on } \zeta_L$
  – certain conditions at the Deligne-Mumford compactification divisor

  – For $h \geq 5$ no natural projection $s\mathcal{M}_h \rightarrow \mathcal{M}_h$ exists (Donagi, Witten 2013)
  – but superspace Stokes’s theorem guarantees independence of choice of $\Gamma$. 
Superperiod matrix $\hat{\Omega}$ (ED & Phong 1988)

- For genus $h = 2$ there is a natural projection $s\mathcal{M}_h \to \mathcal{M}_h$
  - provided by the super period matrix.

- Fix even spin structure $\delta$, and canonical homology basis $A_I, B_I$ for $H^1(\Sigma, \mathbb{Z})$
  - $1/2$-forms $\hat{\omega}_I$ satisfying $\mathcal{D} \hat{\omega}_I = 0$ produce super period matrix $\hat{\Omega}$
    (generalize holó 1-forms $\omega_I$ producing period matrix $\Omega_{IJ}$)
    \[ \int_{A_I} \hat{\omega}_J = \delta_{IJ}, \quad \int_{B_I} \hat{\omega}_J = \hat{\Omega}_{IJ} \]
  - Explicit formula in terms of $(g, \chi)$, and Szego kernel $S_\delta$

\[
\hat{\Omega}_{IJ} = \Omega_{IJ} - \frac{i}{8\pi} \int \int \omega_I(z)\chi(z)S_\delta(z, w)\chi(w)\omega_J(w)
\]

- $\hat{\Omega}_{IJ}$ is locally supersymmetric with $\hat{\Omega}_{IJ} = \hat{\Omega}_{JI}$ and $\text{Im} \hat{\Omega} > 0$
- Every $\hat{\Omega}$ corresponds to a Riemann surface, modulo $Sp(4, \mathbb{Z})$

$\Rightarrow$ Projection using $\hat{\Omega}$ is smooth and natural for genus 2.
The chiral measure in terms of \( \vartheta \)-constants

**Chiral measure on** \( sM_2 \) (with NS vertex operators) (ED & Phong 2001)

\[
d\mu[\delta](\hat{\Omega}, \zeta) = \left( Z[\delta](\hat{\Omega}) + \zeta \zeta' \frac{\Xi_6[\delta](\hat{\Omega}) \vartheta[\delta]^4(0, \hat{\Omega})}{16\pi^6 \Psi_{10}(\hat{\Omega})} \right) d^2\zeta d^3\hat{\Omega}
\]

- \( \Psi_{10}(\hat{\Omega}) = \) Igusa’s unique cusp modular form of weight 10
- \( Z[\delta] \) is known, but will not be given here.

**The modular object** \( \Xi_6[\delta](\hat{\Omega}) \) may be defined, for genus 2 by

- Each even spin structure \( \delta \) uniquely maps to a partition of the six odd spin structures \( \nu_i \). Let \( \delta \equiv \nu_1 + \nu_2 + \nu_3 \equiv \nu_4 + \nu_5 + \nu_6 \)

\[
\Xi_6[\delta](\hat{\Omega}) = \sum_{1 \leq i < j \leq 3} \langle \nu_i | \nu_j \rangle \prod_{k=4,5,6} \vartheta[\nu_i + \nu_j + \nu_k](0, \hat{\Omega})^4
\]

- Symplectic pairing signature: \( \langle \nu_i | \nu_j \rangle \equiv \exp 4\pi i (\nu_i' \nu_j'' - \nu_i'' \nu_j') \in \{ \pm 1 \} \)
Chiral Amplitudes

• **Chiral Amplitudes on** $sM_2$ (with NS vertex operators)
  – involve correlation functions which depend on $\hat{\Omega}$ and on $\zeta$
  – Their effect multiplies the measure:

  $$C[\delta](\hat{\Omega}, \zeta) = d\mu[\delta](\hat{\Omega}, \zeta) \left( C_0[\delta](\hat{\Omega}) + \zeta^1 \zeta^2 C_2[\delta](\hat{\Omega}) \right)$$

• **Projection to chiral amplitudes on** $M_2$
  – by integrating over odd moduli $\zeta$ at fixed $\delta$ and fixed $\hat{\Omega}$

  $$\mathcal{L}[\delta](\hat{\Omega}) = \int_\zeta C[\delta](\hat{\Omega}, \zeta) = \left( Z[\delta] C_2[\delta](\hat{\Omega}) + \frac{\Xi_6[\delta] \vartheta[\delta]^4}{16\pi^6 \Psi_{10}} C_0[\delta](\hat{\Omega}) \right) d^3\hat{\Omega}$$

• **Gliozzi-Scherk-Olive projection (GSO)**
  – realized by summation over spin structures $\delta$ with constant phases;
  – separately in left and right chiral amplitudes for Type II and Heterotic;
  – phases determined uniquely from requirement of modular covariance.
Vacuum energy and susy breaking
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• Vacuum energy observed in Universe is $10^{-120}$ smaller than QFT predicts.

• In supersymmetric theories, vacuum energy vanishes exactly
  (since fermion and boson contributions cancel one another)

• In Type II and Heterotic in flat $\mathbb{R}^{10}$
  – vanishing of vacuum energy conjectured for all $h$
  – well-known for $h = 1$ (Gliozzi-Scherk-Olive 1976)
  – proven for $h = 2$ using the chiral measure on $sM_2$
    along with vanishing of amplitudes for $\leq 3$ massless NS bosons.
    (ED & Phong 2005)
Vacuum energy and susy breaking (cont’d)

• Broken supersymmetry will lead to non-zero vacuum energy

• Supersymmetry spontaneously broken in perturbation theory
  – Superstring theory on Calabi-Yau preserves susy to tree-level
  – but one-loop corrections can break susy by Fayet-Iliopoulos mechanism
    if unbroken gauge group contains at least one $U(1)$ factor
    (Dine, Seiberg, Witten 1986; Dine, Ichinose, Seiberg 1987; Attick, Dixon, Sen 1987)

• Heterotic on 6-dim Calabi-Yau
  – holonomy $G \subset SU(3)$ embedded in gauge group to cancel anomalies
  – $E_8 \times E_8 \rightarrow E_6 \times E_8 \quad \text{produces no } U(1)$
  – $Spin(32)/Z_2 \rightarrow U(1) \times SO(26) \quad \text{produces one } U(1)$

• Two-loop contributions to vacuum energy naturally decompose (Witten 2013)
  – interior of $sM_2$ conjectured to vanish for both theories;
  – boundary of $sM_2$, which vanish for $E_8 \times E_8$ but do not for $Spin(32)/Z_2$.
  – Leading order in $\alpha'$ using pure spinor formulation (Berkovits, Witten 2014)
\( \mathbb{Z}_2 \times \mathbb{Z}_2 \) Calabi-Yau orbifolds

- Prove conjecture for \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) Calabi-Yau orbifolds of Heterotic strings.
  - using natural projection \( sM_2 \rightarrow M_2 \) provided by super period matrix

- \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) Calabi-Yau orbifold of real dimension 6,
  \[
  Y = (T_1 \times T_2 \times T_3)/G \\
  T_i = \mathbb{C}/(\mathbb{Z} \oplus t_i \mathbb{Z}), \quad \text{Im}(t_i) > 0
  \]
  - orbifold group \( G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \lambda_1, \lambda_2, \lambda_3 = \lambda_1 \lambda_2\} \) with \( \lambda_1^2 = \lambda_2^2 = 1 \)

- Transformation laws of worldsheet fields \( x, \psi \) under \( G \subset SU(3) \)
  \[
  x = (x^\mu, z^i, \bar{z}^\bar{i}) \\
  \psi_+ = (\psi_+^\mu, \psi^i, \bar{\psi}^\bar{i}) \\
  \psi_- = (\psi_-^\alpha, \xi^i, \bar{\xi}^\bar{i})
  \]
  \[
  \lambda_i z^j = (-)^{1-\delta_{ij}} z^j, \quad \mu = 0, 1, 2, 3 \\
  \lambda_i \psi^j = (-)^{1-\delta_{ij}} \psi^j, \quad i, \bar{i} = 1, 2, 3 \\
  \lambda_i \xi^j = (-)^{1-\delta_{ij}} \xi^j, \quad \alpha = 1, \ldots, 26
  \]
  - while \( x^\mu, \psi_+^\mu, \psi_-^\alpha \) are invariant.
Twisted fields

- Functional integral formulation of Quantum Mechanics prescribes
  - summation over all maps $\Sigma \rightarrow \mathbb{R}^4 \times Y$ with $Y = (T_1 \times T_2 \times T_3)/G$

- Fields on $\Sigma$ obey identifications twisted by $G$,
  - On homologically trivial cycles, no twisting since $G$ is Abelian.
  - On homologically non-trivial cycles, twists = half integer characteristics

Spinors $\psi$ and $\xi$ with spin structure $\delta = [\delta', \delta'']$ obey

$$
\psi^i(w + A_I) = (-)^{2(\varepsilon^i)'_I + 2\delta'_I} \psi^i(w)
$$

$$
\psi^i(w + B_I) = (-)^{2(\varepsilon^i)''_I + 2\delta''_I} \psi^i(w)
$$

- twists must satisfy $\varepsilon^1 + \varepsilon^2 + \varepsilon^3 = 0$ so that $G \subset SU(3)$. 

Summation over all Twisted Sectors

- **Left chiral amplitude** $\mathcal{L}[\vec{\varepsilon}, \delta](\hat{\Omega}, p_{\vec{\varepsilon}})$ now depends on
  - twist $\vec{\varepsilon} = (\varepsilon^1, \varepsilon^2, \varepsilon^3)$
  - left chirality spin structure $\delta$
  - internal loop momenta $p_{\vec{\varepsilon}}$ (in the lattices $\Lambda_i + \Lambda_i^*$)

- **Right chiral amplitude** $\mathcal{R}[\vec{\varepsilon}, \delta_R](\hat{\Omega}, p)$
  - twist $\vec{\varepsilon} = (\varepsilon^1, \varepsilon^2, \varepsilon^3)$
  - spin structure $\delta_R$ for $Spin(32)/\mathbb{Z}_2$ and $\delta_R = (\delta^1_R, \delta^2_R)$ for $E_8 \times E_8$
  - internal loop momenta $p_{\vec{\varepsilon}}$ (in the lattices $\Lambda_i + \Lambda_i^*$)

- **Full vacuum energy obtained by summing over all sectors,**

  $$
  \int \mathcal{M}_2 \sum_{\vec{\varepsilon}} \sum_{p_{\vec{\varepsilon}}} \left( \sum_{\delta} \mathcal{L}[\vec{\varepsilon}, \delta](\hat{\Omega}, p_{\vec{\varepsilon}}) \right) \left( \sum_{\delta_R} \mathcal{R}[\vec{\varepsilon}, \delta_R](\hat{\Omega}, p_{\vec{\varepsilon}}) \right)
  $$

- We prove that for fixed twist $\vec{\varepsilon}$ and fixed $\hat{\Omega}$ the left chirality sum vanishes,

  $$
  \sum_{\delta} \mathcal{L}[\vec{\varepsilon}, \delta](\hat{\Omega}, p_{\vec{\varepsilon}}) = 0
  $$
Twist orbits under modular transformations

- Decompose summation over twists $\vec{\varepsilon} = (\varepsilon^1, \varepsilon^2, \varepsilon^3)$ into orbits under $Sp(4, \mathbb{Z})$
  - Triplets of twists $\vec{\varepsilon}$ with $\varepsilon^1 + \varepsilon^2 + \varepsilon^3 \equiv 0$ transform in 6 irreducible orbits,
    
    $O_0 = \{(0,0,0)\}$
    $O_1 = \{(0,\varepsilon,\varepsilon)\}, \varepsilon \neq 0$  
    $O_2, O_3$ with permuted entries
    $O_{\pm} = \{(\varepsilon,\eta,\varepsilon + \eta)\}, \varepsilon, \eta \neq 0, \eta \neq \varepsilon, \langle \varepsilon | \eta \rangle = \pm 1$

- $O_0$ untwisted sector: vacuum energy cancels as in flat space-time;

- $O_1, O_2, O_3$ effectively twisted by a single $\mathbb{Z}_2$;
  - vacuum energy was earlier shown to vanish (ED & Phong 2003)

- $O_{\pm}$ genuinely twist by full $\mathbb{Z}_2 \times \mathbb{Z}_2$
**Contributions from the orbits** $\mathcal{O}_\pm$

- Concentrate on spin structure dependent contributions to left chiral amplitudes,
  - Each pair of Weyl fermions with spin structure $\delta$ and twist $\varepsilon$ contributes a factor proportional to $\vartheta[\delta + \varepsilon](0, \Omega)$

- Contribution from twist $\vec{\varepsilon} = (\varepsilon^1, \varepsilon^2, \varepsilon^3)$ in orbits $\mathcal{O}_\pm$ is proportional to
  $$\vartheta[\delta](0, \Omega) \prod_{i=1}^{3} \vartheta[\delta + \varepsilon^i](0, \Omega)$$
  - Vanishes unless $\delta$ as well as $\delta + \varepsilon^i$ are all even.
  - Define $\mathcal{D}[\vec{\varepsilon}] = \{\delta \text{ even, such that } \delta + \varepsilon^i \text{ is even for } i = 1, 2, 3\}$

- For any $\vec{\varepsilon} \in \mathcal{O}_-$ we find $\#\mathcal{D}[\vec{\varepsilon}] = 0 \quad \Rightarrow \quad$ No contributions from orbit $\mathcal{O}_-$.

- For any $\vec{\varepsilon} \in \mathcal{O}_+$ we find $\#\mathcal{D}[\vec{\varepsilon}] = 4 \quad \Rightarrow \quad$ The only remaining contribution to left chiral amplitude $\mathcal{L}[\vec{\varepsilon}, \delta](\Omega, p_{\vec{\varepsilon}})$ is from orbit $\mathcal{O}_+$. 
A modular identity for $Sp(4, \mathbb{Z})/\mathbb{Z}_4$

- For fixed $\vec{\varepsilon} \in \mathcal{O}_+$ and fixed $\hat{\Omega}$ two terms contribute,

$$\sum_{\delta} \mathcal{L}[\vec{\varepsilon}, \delta](\Omega, p_{\vec{\varepsilon}}) = \sum_{\delta} \left( \Xi_6[\delta] C_2[\vec{\varepsilon}, \delta](\Omega, p_{\vec{\varepsilon}}) + \frac{\Xi_6[\delta] \vartheta[\delta]^4}{16\pi^6 \Psi_{10}} C_0[\vec{\varepsilon}, \delta](\Omega, p_{\vec{\varepsilon}}) \right) d^3\Omega$$

- $C_0, C_2$ calculated from orbifold construction

- Cancellation point-wise on $\mathcal{M}_2$ via the factorization identity

$$\sum_{\delta \in \mathcal{D}[\vec{\varepsilon}]} \langle \delta_0|\delta \rangle \Xi_6[\delta](\Omega) = 6\Lambda[\vec{\varepsilon}, \delta_0] \prod_{\delta \notin \mathcal{D}[\vec{\varepsilon}]} \vartheta[\delta](0, \Omega)^2$$

for any $\delta_0 \in \mathcal{D}[\vec{\varepsilon}]$, and we have $\Lambda[\vec{\varepsilon}, \delta_0]^2 = 1$.

- Proof includes Thomae map $\vartheta[\delta]^4$ to hyper-elliptic representation.

- Factorization identity is invariant under $Sp(4, \mathbb{Z})/\mathbb{Z}_4$
  - with $\mathbb{Z}_4 = \{I, J, -I, -J\}$ normal subgroup of $Sp(4, \mathbb{Z})$
Contributions from the boundary of $sM_2$

- At separating degeneration node of $sM_2$, integration is only conditionally convergent, due to right moving tachyon $\approx d\tilde{\tau}/\tilde{\tau}^2$ (Witten 2013)

- Regularization near separating node is required
  - consistent with physical factorization
  - produces a $\delta$-function at separating node.

- To compute coefficient, decompose orbit $O_+$ under modular subgroup
  - $Sp(2,\mathbb{Z}) \times Sp(2,\mathbb{Z}) \times \mathbb{Z}_2$ preserving separating degeneration
  - contributions only from $\bar{\varepsilon}$ such that $D[\bar{\varepsilon}]$
  - contains one spin structure which decomposes to odd – odd

- Lengthy calculation shows
  - vanishing for $E_8 \times E_8$
  - non-vanishing for $Spin(32)/\mathbb{Z}_2$. 
Superstring corrections to Type IIB supergravity
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- String theory induces $\alpha'$ corrections to supergravity beyond $\mathcal{R}$
  - Local effective interactions from integrating out massive states
  - Non-analytic contributions from threshold effects

- Supersymmetry imposes strong constraints
  - supersymmetry e.g. prohibit $\mathcal{R}^2, \mathcal{R}^3$ corrections;
  - leading correction $\mathcal{R}^4$ subject to susy contraction of $R_{\mu\nu\rho\sigma}$

- S-duality requires axion/dilaton dependence through modular forms
  - S-duality in Type IIB on $\mathbb{R}^{10}$ is invariance under $SL(2, \mathbb{Z})$
  - axion-dilaton field $T \in \mathbb{C}$ with $T = \chi + ie^{-\phi}$ with $\text{Im}(T) > 0$
  - $SL(2, \mathbb{Z})$ acts by $T \rightarrow (aT + b)/(cT + d)$
  - e.g. coefficient of $\mathcal{R}^4$ is a real Eisenstein series

\[ \mathcal{E}_{(0,0)}(T) \sim \sum_{(m,n)\neq(0,0)} \frac{(\text{Im } T)^{3/2}}{|m + nT|^3} \quad \text{(Green, Gutperle 1997)} \]

- Perturbative contributions only at tree-level and one-loop.
Superstring corrections of the form $D^2pR^4$

- Accessible through 4-graviton amplitude

$$A_4(\varepsilon_i, k_i; T) = \kappa^2 R^4 I_4(s, t, u; T)$$

- $\varepsilon_i, k_i$ are polarization tensor and momentum of gravitons;
- $s = -\alpha' k_1 \cdot k_2 / 2$ etc are Lorentz invariants with $s + t + u = 0$;
- $\kappa$ is 10-dimensional Newton constant.

- Expansions
  - Low energy for $|s|, |t|, |u| \ll 1$
    - $\star$ non-analytic part in $s, t, u$ produced by massless states;
    - $\star$ analytic part in $s, t, u$ producing local effective interactions.

$$I_4(s, t, u; T) \bigg|_{\text{analytic}} = \sum_{m,n=0}^{\infty} \mathcal{E}_{(m,n)}(T) \left( s^2 + t^2 + u^2 \right)^m \left( s^3 + t^3 + u^3 \right)^n$$

- $\star$ Coefficients $\mathcal{E}_{(m,n)}(T)$ are modular invariants in $T$.
- Match with superstring perturbation theory for $g_s = (\text{Im } T)^{-1} \to 0$

$$\mathcal{E}_{(m,n)}(T) = \sum_{h=0}^{\infty} g_s^{-2+2h} \mathcal{E}_{(m,n)}^{(h)} + \mathcal{O}(e^{-2\pi/g_s})$$
Predictions from Supersymmetry and S-duality

- Interplay of Type IIB and M-theory dualities from compactifications on $\mathbb{T}^d$

<table>
<thead>
<tr>
<th>$\mathcal{R}^4$</th>
<th>$\mathcal{E}^{(0)}_{(0,0)} = 2\zeta(3)$</th>
<th>$\mathcal{E}^{(1)}_{(0,0)} = 4\zeta(2)$</th>
<th>$\mathcal{E}^{(h)}_{(0,0)} = 0, \ h \geq 2$</th>
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<tbody>
<tr>
<td>$D^4\mathcal{R}^4$</td>
<td>$\mathcal{E}^{(0)}_{(1,0)} = \zeta(5)$</td>
<td>$\mathcal{E}^{(1)}_{(1,0)} = 0$</td>
<td>$\mathcal{E}^{(h)}_{(1,0)} = 0, \ h \geq 3$</td>
</tr>
<tr>
<td>$D^6\mathcal{R}^4$</td>
<td>$\mathcal{E}^{(0)}_{(0,1)} = \frac{2}{3}\zeta(3)^2$</td>
<td>$\mathcal{E}^{(1)}_{(0,1)} = \frac{4}{3}\zeta(2)\zeta(3)$</td>
<td>$\mathcal{E}^{(h)}_{(0,1)} = 0, \ h \geq 4$</td>
</tr>
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</table>

- Non-vanishing coefficients at two and three loops

$$\mathcal{E}^{(2)}_{(1,0)} = \frac{4}{3}\zeta(4) \quad \mathcal{E}^{(2)}_{(0,1)} = \frac{8}{5}\zeta(2)^2 \quad \mathcal{E}^{(3)}_{(0,1)} = \frac{4}{27}\zeta(6)$$

- Little is known beyond, for $D^8\mathcal{R}^4$, $D^{10}\mathcal{R}^4$ etc.
- basic references: (Green, Gutperle 1997)
  (Pioline; Green, Sethi 1998)
  (Green, Kwon, Vanhove; Green, Vanhove 1999)
  (Obers, Pioline 2000)
  (Green, Russo, Vanhove 2010) · · ·
Two-loop Type IIB 4-graviton amplitude

- Integral representation (ED & Phong 2001-2005)

\[
\mathcal{I}_4^{(2)}(s, t, u; T) \sim g_s^2 \int_{\mathcal{M}_2} d\mu_2 \int_{\Sigma^4} \frac{|\mathcal{Y}|^2}{(\det Y)^2} \exp\left(-\sum_{i<j} \alpha' k_i \cdot k_j G(z_i, z_j)\right)
\]

\[
\mathcal{Y} = (k_1 - k_2) \cdot (k_3 - k_4) \omega_{[1}(z_1)\omega_{2]}(z_2) \omega_{[1}(z_3)\omega_{2]}(z_4) + 2 \text{ perm's}
\]

- \(\omega_i(z)\) are the holomorphic Abelian differentials on \(\Sigma\)
- \(G(z, w)\) is a scalar Green function on \(\Sigma\)
- \(\Omega = X + iY\) with \(X, Y\) real matrices;
- \(d\mu_2\) canonical volume form on \(\mathcal{M}_2\);

- \(\mathcal{I}_4^{(2)}\) is defined by analytic continuation in \(s, t, u\).

- Expansion for small \(s, t, u\) (using \(\mathcal{Y}\) linear in \(s, t, u\))
  - As a result \(R^4\) coefficient \(\mathcal{E}^{(2)}_{(0,0)} = 0\) (ED & Phong 2005)
  - Confirm \(D^4R^4\) coefficient \(\mathcal{E}^{(2)}_{(1,0)} = 4\zeta(4)/3\) (ED, Gutperle & Phong 2005)
  - Calculating \(D^6R^4\) coefficient \(\mathcal{E}^{(2)}_{(0,1)}\) requires integral with one power of \(G\).
The Zhang-Kawazumi Invariant

• Integration over $\Sigma^2$ gives, (ED & Green 2013)

$$\mathcal{E}^{(2)}_{(0,1)} = \pi \int_{\mathcal{M}_2} d\mu_2 \varphi \quad \varphi(\Sigma) \equiv -\frac{1}{8} \int_{\Sigma^2} P(x, y) G(x, y)$$

– where $P$ is a symmetric bi-form on $\Sigma^2$, defined by

$$P(x, y) = \sum_{I,J,K,L} \left( 2Y_{IL}^{-1}Y_{JK}^{-1} - Y_{IJ}^{-1}Y_{KL}^{-1} \right) \omega_I(x)\overline{\omega_J(x)}\omega_K(y)\overline{\omega_L(y)}$$

– $\varphi$ conformal invariant, and modular invariant under $Sp(4, \mathbb{Z})$

• $\varphi$ coincides with the invariant introduced by Zhang and Kawazumi (2008)

$$\varphi(\Sigma) = \sum_{\ell} \sum_{I,J} \frac{2}{\lambda_{\ell}} \left| \int_{\Sigma} \phi_\ell \omega'_I \wedge \overline{\omega'_J} \right|^2$$

– $\omega'_I$ are holomorphic 1-forms normalized $\int_{\Sigma} \omega'_I \overline{\omega'_J} = -2i\delta_{IJ}$
– $\phi_\ell$ eigenfunction of the Arakelov Laplacian with eigenvalue $\lambda_{\ell}$.
– related to the Faltings $\delta$-invariant (De Jong 2010)
Diff eqs from S-duality and Supersymmetry

- Direct integration of $\int_{M_2} d\mu_2 \varphi$ appears out of reach.

- S-duality and supersymmetry lead to diff eqs in $T$ (Pioline; Green, Sethi 1998)

  $$(\Delta_T - 3/4) \mathcal{E}_{(0,0)}(T) = 0$$

  - satisfied by D-instanton sum in Type IIB (Green, Gutperle 1997)
  - Difficult to obtain diff eqs for higher coefficients

- Two-loop 11-d sugra on $T^{d+1}$ for various $d$ (Green, Kwon, Vanhove 2000)

  - conjecture diff eqs in perturbative and non-perturbative moduli $m_d$

    $$(\Delta_{E_{d+1}} - \frac{3(d + 1)(2 - d)}{(8 - d)}) \mathcal{E}_{(0,0)}(m_d) = 6\pi \delta_{d,2}$$

  - $\Delta_{E_{d+1}}$ Laplace operators on cosets $E_{d+1}(\mathbb{R})/K_{d+1}(\mathbb{R})$

    - $E_1(\mathbb{R}) = SL(2, \mathbb{R})$
    - $E_2(\mathbb{R}) = SL(2, \mathbb{R}) \times \mathbb{R}^+$
    - $\cdots$
    - $E_7(\mathbb{R}) = E_7(7)$

  - $K_{d+1}(\mathbb{R})$ maximal compact subgroup of $E_{d+1}(\mathbb{R})$
**Diff eqs from S-duality and Supersymmetry** cont’d

- Expand differential equations for $E_{(m,n)}(m_d)$ at weak string coupling
  - some moduli are not seen in perturbation theory (e.g. the axion)
  - moduli of torus $T^d$ remain in perturbative limit: denote $\rho_d$

\[
E_{(0,1)}^{(2)}(\rho_d) = \pi \int_{M_2} d\mu_2 \Gamma_{d,d,2}(\rho_d; \Omega) \varphi(\Omega)
\]

- where $\Gamma_{d,d,h}(\rho_d; \Omega)$ is the partition function on $T^d$ for genus $h$
- The perturbative part of $E_{(0,1)}(m_d)$ satisfies,

\[
\left( \Delta_{SO(d,d)} - (d + 2)(5 - d) \right) E_{(0,1)}^{(2)}(\rho_d) = - \left( E_{(0,0)}^{(1)}(\rho_d) \right)^2
\]

- For genus $h$ and dimension $d$ the torus partition function satisfies,

\[
\left( \Delta_{SO(d,d)} - 2\Delta_\Omega + \frac{1}{2}dh(d-h-1) \right) \Gamma_{d,d,h}(\rho_d; \Omega) = 0
\]

- Combining both implies the equation,

\[
\int_{M_2} d\mu_2 \varphi(\Omega) \left( \Delta_\Omega - 5 \right) \Gamma_{d,d,2}(\rho_d, \Omega) = -\frac{\pi}{2} \left( \int_{M_1} d\mu_1 \Gamma_{d,d,1}(\rho_d, \tau) \right)^2
\]

- This suggests $(\Delta_\Omega - 5) \varphi = 0$ in interior of $M_2$. 

Laplace eigenvalue equation for $\varphi$

- First prove the following Laplace eigenvalue equation,

$$(\Delta - 5)\varphi = -2\pi \delta_{SN}^{(2)}$$

- where $\Delta$ is the Laplace-Beltrami operator on $M_2$, represented as a fundamental domain for $Sp(4, \mathbb{Z})$ in Siegel upper half space.
- and $\delta_{SN}^{(2)}$ is the volume form induced on the separating node of $M_2$.

- Proven by methods of deformations of complex structures on $\Sigma$
  - derivatives with respect to $\Omega$ related to Beltrami differential $\mu$

$$\delta_{\mu} \Omega_{IJ} = i \int_{\Sigma} \mu \omega_I \omega_J$$

- Laplacian evaluated by computing $\delta_{\mu_1} \delta_{\bar{\mu}_2} \varphi$
Integrating $\varphi$ over $\mathcal{M}_2$

- The integral $\int_{\mathcal{M}_2} d\mu_2 \varphi$ is absolutely convergent
  
  - to obtain a concrete relation, parametrize $\Omega$ by
    $$\Omega = \begin{pmatrix} \tau_1 & \tau \\ \tau & \tau_2 \end{pmatrix}, \quad d\mu_2 = \frac{d^2\tau \, d^2\tau_1 \, d^2\tau_2}{(\det Y)^3}$$
  
  - asymptotics of $\varphi$ near separating node where $\tau \to 0$
    $$\varphi(\Omega) = -\ln \left| 2\pi \tau \eta(\tau_1)^2 \eta(\tau_2)^2 \right| + \mathcal{O}(\tau^2)$$
  
  - near non-separating node where $\tau_2 \to i\infty$ using (Fay, Wentworth)
    $$\varphi(\Omega) = \frac{\pi}{6} \text{Im} \tau_2 + \frac{5\pi (\text{Im} \tau)^2}{6\text{Im} \tau_1} - \ln \left| \frac{\vartheta_1(\tau, \tau_1)}{\vartheta_1(0, \tau_1)} \right| + \mathcal{O}(1/\tau_2)$$

- maximal non-separating (“supergravity” or “tropical”) limit $\ell_i \to \infty$
  
  $$\Omega = i \left( \begin{array}{ccc} \ell_1 + \ell_3 & \ell_3 \\ \ell_3 & \ell_2 + \ell_3 \end{array} \right), \quad \varphi(\Omega) = \frac{\pi}{6} \left( \ell_1 + \ell_2 + \ell_3 - \frac{5 \ell_1 \ell_2 \ell_3}{\ell_1 \ell_2 + \ell_2 \ell_3 + \ell_3 \ell_1} \right)$$

(Green, Russo, Vanhove 2008), (Tourkine 2013)
Integrating $\varphi$ over $\mathcal{M}_2$ (cont’d)

- Integral on cut-off moduli space $\mathcal{M}_2^\varepsilon = \mathcal{M}_2 \cap \{|\tau| > \varepsilon\}$

  - using convergence of integral, and $(\Delta - 5)\varphi = -2\pi \delta_{SN}^{(2)}$
  
  $$\int_{\mathcal{M}_2} d\mu_2 \varphi = \lim_{\varepsilon \to 0} \int_{\mathcal{M}_2^\varepsilon} d\mu_2 \varphi = \frac{1}{5} \lim_{\varepsilon \to 0} \int_{\mathcal{M}_2^\varepsilon} d\mu_2 \Delta \varphi$$

  - reduces to integral over boundary

  $$\partial \mathcal{M}_2^\varepsilon = \{|\tau| = \varepsilon\} \times (\mathcal{M}_1^{(1)} \times \mathcal{M}_1^{(2)}) / (\mathbb{Z}_2 \times \mathbb{Z}_2)$$

  - contribution from non-separating node vanishes

  - contribution from separating node governed by limit of,

  $$d\mu_2 \Delta \varphi = d \left\{ \frac{i}{2} \left( \frac{d\bar{\tau}}{\bar{\tau}} - \frac{d\tau}{\tau} \right) \wedge d\mu_1^{(1)} \wedge d\mu_1^{(2)} \right\}$$

  - using $\int_{\mathcal{M}_1} d\mu_1 = 2\pi/3$, and $4\pi$ from $\tau$-integral, and $1/4$ from $\mathbb{Z}_2 \times \mathbb{Z}_2$

  $$\int_{\mathcal{M}_2} d\mu_2 \varphi = \frac{1}{5} \times \frac{1}{2} \times 4\pi \times \left(\frac{2\pi}{3}\right)^2 \times \frac{1}{4} = \frac{2\pi^3}{45}$$

  - Exact agreement with predictions from S-duality and supersymmetry
Outlook

√ Interplay between superstring perturbation theory, S-duality, supersymmetry

√ Integrated over $\mathcal{M}_2$ a non-trivial modular invariant $\varphi$

• For higher genus, $h \geq 3$, the ZK invariant exists,
  – but does not satisfy $(\Delta - \lambda)\varphi = 0$
  – string theory significance ?
  – Pure spinor calculation for $E^{(3)}_{(0,1)}$ (Gomez, Mafra 2014)

• For $D^8\mathcal{R}^4$, $D^{10}\mathcal{R}^4$, · · · two-loop superstring perturbation theory
  – suggests new invariants (ED, Green 2013)
  – significance in theory of modular invariants – number theory ?
  – can one match with S-duality and supersymmetry in string theory ?