Recent Advances in Two-loop Superstrings

Eric D'Hoker

Institut des Hautes Etudes Scientifiques, 2014 May 6

Outline

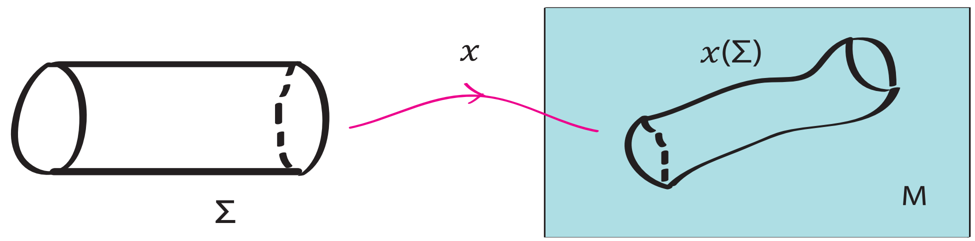
1. Overview of two-loop superstring methods, including global issues;

2. Applications to Vacuum Energy and Spontaneous Supersymmetry Breaking
   E. D'Hoker, D.H. Phong, arXiv:1307.1749,
   Two-Loop Vacuum Energy for Calabi-Yau orbifold models

3. Applications to Superstring Corrections to Type IIB Supergravity
   E. D'Hoker, M.B. Green, arXiv:1308.4597,
   Zhang-Kawazumi invariants and Superstring Amplitudes
   E. D'Hoker, M.B. Green, B. Pioline, R. Russo, arXiv:1405.6226,
   Matching the $D^6\mathcal{R}^4$ interaction at two-loops

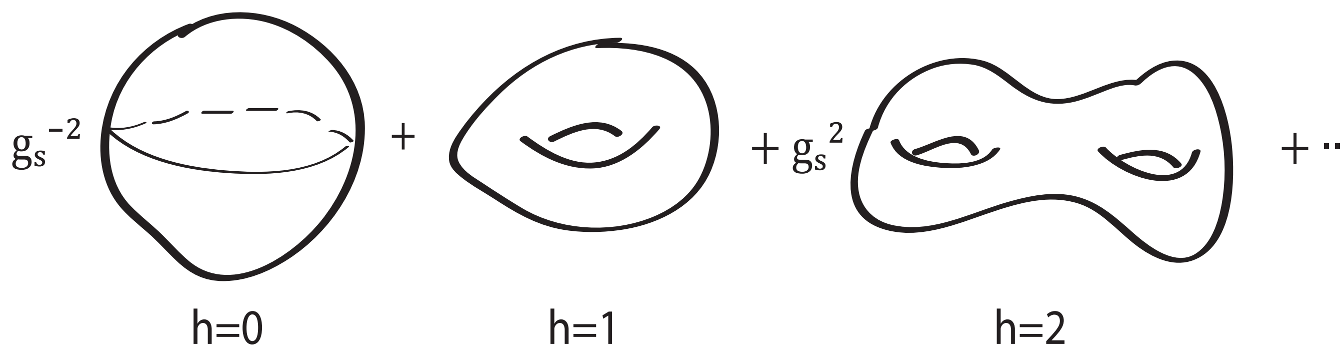
String Perturbation Theory

Quantum Strings: fluctuating surfaces in space-time M



Perturbative expansion of string amplitudes in powers of coupling constant g_s

$=$ sum over Riemann surfaces Σ of genus h



Bosonic string: • sum over maps $\{x\}$

• sum over conformal classes $[g]$ on Σ

$=$ integral over moduli space $\mathcal{M}_h$ of Riemann surfaces.

Superstrings

- Worldsheet = super Riemann surface

(x, ψ)	RNS-formulation ψ spinor on Σ
(g, χ)	superconformal geometry

- Worldsheet action invariant under local supersymmetry in addition to Diff(Σ)
  Absence of superconformal anomalies requires dim(M) =10

- Supermoduli Space $s\mathcal{M}_h$ = space of superconformal classes $[g, \chi]$,

$$\dim(s\mathcal{M}_h) = \begin{cases} (0|0) & h = 0 \\ (1|0)_{\rm even} \text{ or } (1|1)_{\rm odd} & h = 1 \\ (3h-3|2h-2) & h \geq 2 \end{cases}$$

- **Two-loops is lowest order at which odd moduli enter non-trivially.**

Independence of left and right chiralities

- Locally on Σ, worldsheet fields split into left & right chiralities

$$\partial_z \partial_{\bar z} x^\mu = 0 \quad \Longrightarrow \quad x^\mu = x_+^\mu(z) + x_-^\mu(\bar z)$$

$$\partial_z \psi_-^\mu = \partial_{\bar z} \psi_+^\mu = 0 \quad \Longrightarrow \quad \psi_+^\mu(z), \ \psi_-^\mu(\bar z)$$

Fundamental physical closed superstring theories

Type II ψ_+^μ and ψ_-^μ are independent (not complex conjugates)
with independent spin structure assignments
odd moduli for left and right are independent

Heterotic ψ_+^μ left chirality fermions with $\mu = 1, \cdots, 10$
ψ_-^A right chirality fermions with $A = 1, \cdots, 32$
odd moduli for left, but none for right chirality

Pairing prescription (Witten 2012)

- Separate moduli spaces for left and right chiralities
  - LEFT : $s\mathcal{M}_L$ of dim $(3h-3|2h-2)$ with local coordinates $(m_L, \bar{m}_L; \zeta_L)$
  - RIGHT: Type II string, $s\mathcal{M}_R$ of dim $(3h-3|2h-2)$, with $(m_R, \bar{m}_R; \zeta_R)$
    Heterotic string, $\mathcal{M}_R$ of dim $(3h-3|0)$, with $(m_R, \bar{m}_R)$
- Left and right odd moduli ζ_L, ζ_R are independent
- Even moduli must be related
  Heterotic string: integrate over a closed cycle $\Gamma \subset s\mathcal{M}_L \times \mathcal{M}_R$ such that
  - $\bar{m}_R = m_L +$ even nilpotent corrections dependent on ζ_L
  - certain conditions at the Deligne-Mumford compactification divisor
  - For $h \geq 5$ no natural projection $s\mathcal{M}_h \to \mathcal{M}_h$ exists (Donagi, Witten 2013)
  - but superspace Stokes's theorem guarantees independence of choice of Γ.

Superperiod matrix $\hat\Omega$ (ED & Phong 1988)

- For genus $h=2$ there is a natural projection $s\mathcal{M}_h \to \mathcal{M}_h$
  - provided by the super period matrix.

- Fix even spin structure δ, and canonical homology basis A_I, B_I for $H^1(\Sigma, \mathbb{Z})$
  - 1/2-forms $\hat\omega_I$ satisfying $\mathcal{D}_-\hat\omega_I = 0$ produce super period matrix $\hat\Omega$ (generalize holó 1-forms ω_I producing period matrix Ω_{IJ})

$$\oint_{A_I} \hat\omega_J = \delta_{IJ} \qquad\qquad \oint_{B_I} \hat\omega_J = \hat\Omega_{IJ}$$

  - Explicit formula in terms of (g, χ), and Szego kernel S_δ

$$\boxed{\hat\Omega_{IJ} = \Omega_{IJ} - \frac{i}{8\pi}\int\int \omega_I(z)\chi(z)S_\delta(z,w)\chi(w)\omega_J(w)}$$

  - $\hat\Omega_{IJ}$ is locally supersymmetric with $\hat\Omega_{IJ} = \hat\Omega_{JI}$ and $\mathrm{Im}\,\hat\Omega > 0$
  - Every $\hat\Omega$ corresponds to a Riemann surface, modulo $Sp(4,\mathbb{Z})$

$\Rightarrow$ Projection using $\hat\Omega$ is smooth and natural for genus 2.

The chiral measure in terms of ϑ-constants

Chiral measure on $s\mathcal{M}_2$ (with NS vertex operators) (ED & Phong 2001)

$$d\mu[\delta](\hat\Omega,\zeta) = \left(\mathcal{Z}[\delta](\hat\Omega) + \zeta^1\zeta^2 \frac{\Xi_6[\delta](\hat\Omega)\ \vartheta[\delta]^4(0,\hat\Omega)}{16\pi^6\ \Psi_{10}(\hat\Omega)} \right) d^2\zeta d^3\hat\Omega$$

- $\Psi_{10}(\hat\Omega)$ = Igusa's unique cusp modular form of weight 10
- $\mathcal{Z}[\delta]$ is known, but will not be given here.

The modular object $\Xi_6[\delta](\hat\Omega)$ may be defined, for genus 2 by

- Each even spin structure δ uniquely maps to a partition of the six odd spin structures ν_i. Let $\delta \equiv \nu_1+\nu_2+\nu_3 \equiv \nu_4+\nu_5+\nu_6$

$$\boxed{\Xi_6[\delta](\hat\Omega) = \sum_{1\leq i<j\leq 3} \langle \nu_i|\nu_j\rangle \prod_{k=4,5,6} \vartheta[\nu_i+\nu_j+\nu_k](0,\hat\Omega)^4}$$

- Symplectic pairing signature: $\langle \nu_i|\nu_j\rangle \equiv \exp 4\pi i(\nu_i'\nu_j'' - \nu_i''\nu_j') \in \{\pm 1\}$

Chiral Amplitudes

- **Chiral Amplitudes on $s\mathcal{M}_2$** (with NS vertex operators)
  - involve correlation functions which depend on $\hat{\Omega}$ and on ζ
  - Their effect multiplies the measure;

$$\mathcal{C}[\delta](\hat{\Omega},\zeta) = d\mu[\delta](\hat{\Omega},\zeta)\left(\mathcal{C}_0[\delta](\hat{\Omega}) + \zeta^1\zeta^2\mathcal{C}_2[\delta](\hat{\Omega})\right)$$

- **Projection to chiral amplitudes on $\mathcal{M}_2$**
  - by integrating over odd moduli ζ at fixed δ and fixed $\hat{\Omega}$

$$\mathcal{L}[\delta](\hat{\Omega}) = \int_\zeta \mathcal{C}[\delta](\hat{\Omega},\zeta) = \left(\mathcal{Z}[\delta]\mathcal{C}_2[\delta](\hat{\Omega}) + \frac{\Xi_6[\delta]\ \vartheta[\delta]^4}{16\pi^6\ \Psi_{10}}\mathcal{C}_0[\delta](\hat{\Omega})\right) d^3\hat{\Omega}$$

- **Gliozzi-Scherk-Olive projection (GSO)**
  - realized by summation over spin structures δ with constant phases;
  - separately in left and right chiral amplitudes for Type II and Heterotic;
  - phases determined uniquely from requirement of modular covariance.

Vacuum energy and susy breaking

Vacuum energy and susy breaking

- Vacuum energy observed in Universe is 10^{-120} smaller than QFT predicts.
- In supersymmetric theories, vacuum energy vanishes exactly
  (since fermion and boson contributions cancel one another)
- In Type II and Heterotic in flat $\mathbb{R}^{10}$
  - vanishing of vacuum energy conjectured for all h
  - well-known for $h = 1$ (Gliozzi-Scherk-Olive 1976)
  - proven for $h = 2$ using the chiral measure on $s\mathcal{M}_2$
    along with vanishing of amplitudes for ≤ 3 massless NS bosons.
    (ED & Phong 2005)

Vacuum energy and susy breaking (cont'd)

- Broken supersymmetry will lead to non-zero vacuum energy

- Supersymmetry spontaneously broken in perturbation theory
  - Superstring theory on Calabi-Yau preserves susy to tree-level
  - but one-loop corrections can break susy by Fayet-Iliopoulos mechanism if unbroken gauge group contains at least one $U(1)$ factor
    (Dine, Seiberg, Witten 1986; Dine, Ichinose, Seiberg 1987; Attick, Dixon, Sen 1987)

- Heterotic on 6-dim Calabi-Yau
  - holonomy $G \subset SU(3)$ embedded in gauge group to cancel anomalies
  - $E_8 \times E_8 \to E_6 \times E_8$ produces no $U(1)$
  - $Spin(32)/Z_2 \to U(1) \times SO(26)$ produces one $U(1)$

- Two-loop contributions to vacuum energy naturally decompose (Witten 2013)
  - interior of $s\mathcal{M}_2$ conjectured to vanish for both theories;
  - boundary of $s\mathcal{M}_2$, which vanish for $E_8 \times E_8$ but do not for $Spin(32)/Z_2$.
  - Leading order in α' using pure spinor formulation (Berkovits, Witten 2014)

$\mathbb{Z}_2 \times \mathbb{Z}_2$ Calabi-Yau orbifolds

- Prove conjecture for $\mathbb{Z}_2 \times \mathbb{Z}_2$ Calabi-Yau orbifolds of Heterotic strings.
  - using natural projection $s\mathcal{M}_2 \to \mathcal{M}_2$ provided by super period matrix

- $\mathbb{Z}_2 \times \mathbb{Z}_2$ Calabi-Yau orbifold of real dimension 6,

$$Y = (T_1 \times T_2 \times T_3)/G \qquad T_i = \mathbb{C}/(\mathbb{Z} \oplus t_i \mathbb{Z}), \qquad \mathrm{Im}(t_i) > 0$$

  - orbifold group $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \lambda_1, \lambda_2, \lambda_3 = \lambda_1 \lambda_2\}$ with $\lambda_1^2 = \lambda_2^2 = 1$

- Transformation laws of worldsheet fields x, ψ under $G \subset SU(3)$

$$\begin{aligned}
x \; &= (x^\mu, z^i, z^{\bar i}) & \lambda_i \, z^j &= (-)^{1-\delta_{ij}} z^j & \mu &= 0,1,2,3 \\
\psi_+ &= (\psi_+^\mu, \psi^i, \psi^{\bar i}) & \lambda_i \, \psi^j &= (-)^{1-\delta_{ij}} \psi^j & i, \bar i &= 1,2,3 \\
\psi_- &= (\psi_-^\alpha, \xi^i, \xi^{\bar i}) & \lambda_i \, \xi^j &= (-)^{1-\delta_{ij}} \xi^j & \alpha &= 1, \cdots, 26
\end{aligned}$$

  - while $x^\mu, \psi_+^\mu, \psi_-^\alpha$ are invariant.

Twisted fields

- Functional integral formulation of Quantum Mechanics prescribes
  - summation over all maps $\Sigma \to \mathbb{R}^4 \times Y$ with $Y = (T_1 \times T_2 \times T_3)/G$

- Fields on Σ obey identifications twisted by G,
  - On homologically trivial cycles, no twisting since G is Abelian.
  - On homologically non-trivial cycles, twists = half integer characteristics

$$(\varepsilon^i)'_I, (\varepsilon^i)''_I \in \left\{0, \tfrac{1}{2}\right\} \text{ for } I = 1, 2 \text{ and } i = 1, 2, 3.$$

Spinors ψ and ξ with spin structure $\delta = [\delta' \ \delta'']$ obey

$$\begin{aligned} \psi^i(w + A_I) &= (-)^{2(\varepsilon^i)'_I + 2\delta'_I} \, \psi^i(w) \\ \psi^i(w + B_I) &= (-)^{2(\varepsilon^i)''_I + 2\delta''_I} \, \psi^i(w) \end{aligned}$$

- twists must satisfy $\varepsilon^1 + \varepsilon^2 + \varepsilon^3 = 0$ so that $G \subset SU(3)$.

Summation over all Twisted Sectors

- Left chiral amplitude $\mathcal{L}[\vec{\varepsilon}, \delta](\hat{\Omega}, p_{\vec{\varepsilon}})$ now depends on
  - twist $\vec{\varepsilon} = (\varepsilon^1, \varepsilon^2, \varepsilon^3)$
  - left chirality spin structure δ
  - internal loop momenta $p_{\vec{\varepsilon}}$ (in the lattices $\Lambda_i + \Lambda_i^*$)

- Right chiral amplitude $\overline{\mathcal{R}[\vec{\varepsilon}, \delta_R](\hat{\Omega}, p)}$
  - twist $\vec{\varepsilon} = (\varepsilon^1, \varepsilon^2, \varepsilon^3)$
  - spin structure δ_R for $Spin(32)/Z_2$ and $\delta_R = (\delta_R^1, \delta_R^2)$ for $E_8 \times E_8$
  - internal loop momenta $p_{\vec{\varepsilon}}$ (in the lattices $\Lambda_i + \Lambda_i^*$)

- Full vacuum energy obtained by summing over all sectors,

$$\int_{\mathcal{M}_2} \sum_{\vec{\varepsilon}} \sum_{p_{\vec{\varepsilon}}} \left(\sum_{\delta} \mathcal{L}[\vec{\varepsilon}, \delta](\hat{\Omega}, p_{\vec{\varepsilon}}) \right) \left(\sum_{\delta_R} \overline{\mathcal{R}[\vec{\varepsilon}, \delta_R](\hat{\Omega}, p_{\vec{\varepsilon}})} \right)$$

- We prove that for fixed twist $\vec{\varepsilon}$ and fixed $\hat{\Omega}$ the left chirality sum vanishes,

$$\sum_{\delta} \mathcal{L}[\vec{\varepsilon}, \delta](\hat{\Omega}, p_{\vec{\varepsilon}}) = 0$$

Twist orbits under modular transformations

- Decompose summation over twists $\vec{\varepsilon} = (\varepsilon^1, \varepsilon^2, \varepsilon^3)$ into orbits under $Sp(4, \mathbb{Z})$
  - Triplets of twists $\vec{\varepsilon}$ with $\varepsilon^1 + \varepsilon^2 + \varepsilon^3 \equiv 0$ transform in 6 irreducible orbits,

$$\mathcal{O}_0 = \{(0,0,0)\}$$

$$\mathcal{O}_1 = \{(0,\varepsilon,\varepsilon)\}, \ \varepsilon \neq 0\} \qquad \mathcal{O}_2,\ \mathcal{O}_3 \ \text{with permuted entries}$$

$$\mathcal{O}_\pm = \{(\varepsilon, \eta, \varepsilon + \eta), \ \ \varepsilon, \eta \neq 0, \ \ \eta \neq \varepsilon, \ \ \langle \varepsilon | \eta \rangle = \pm 1\}$$

- $\mathcal{O}_0$ untwisted sector: vacuum energy cancels as in flat space-time;
- $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ effectively twisted by a single $\mathbb{Z}_2$;
  - vacuum energy was earlier shown to vanish (ED & Phong 2003)
- $\mathcal{O}_\pm$ genuinely twist by full $\mathbb{Z}_2 \times \mathbb{Z}_2$

Contributions from the orbits $\mathcal{O}_\pm$

- Concentrate on spin structure dependent contributions to left chiral amplitudes,
  - Each pair of Weyl fermions with spin structure δ and twist ε contributes a factor proportional to $\vartheta[\delta+\varepsilon](0,\Omega)$

- Contribution from twist $\vec{\varepsilon} = (\varepsilon^1, \varepsilon^2, \varepsilon^3)$ in orbits $\mathcal{O}_\pm$ is proportional to

$$\vartheta[\delta](0,\Omega) \prod_{i=1}^{3} \vartheta[\delta+\varepsilon^i](0,\Omega)$$

  - Vanishes unless δ as well as $\delta+\varepsilon^i$ are all even.
  - Define $\mathcal{D}[\vec{\varepsilon}] = \{\delta$ even, such that $\delta+\varepsilon^i$ is even for $i=1,2,3\}$

- For any $\vec{\varepsilon} \in \mathcal{O}_-$ we find $\#\mathcal{D}[\vec{\varepsilon}] = 0 \quad \Rightarrow \quad$ No contributions from orbit $\mathcal{O}_-$.

- For any $\vec{\varepsilon} \in \mathcal{O}_+$ we find $\#\mathcal{D}[\vec{\varepsilon}] = 4 \quad \Rightarrow \quad$ The only remaining contribution to left chiral amplitude $\mathcal{L}[\vec{\varepsilon},\delta](\Omega, p_{\vec{\varepsilon}})$ is from orbit $\mathcal{O}_+$.

A modular identity for $Sp(4,\mathbb{Z})/\mathbb{Z}_4$

- For fixed $\vec{\varepsilon} \in \mathcal{O}_+$ and fixed $\hat{\Omega}$ two terms contribute,

$$\sum_\delta \mathcal{L}[\vec{\varepsilon},\delta](\Omega,p_{\vec{\varepsilon}}) = \sum_\delta \left(\mathcal{Z}[\delta]\,\mathcal{C}_2[\vec{\varepsilon},\delta](\Omega,p_{\vec{\varepsilon}}) + \frac{\Xi_6[\delta]\ \vartheta[\delta]^4}{16\pi^6\ \Psi_{10}}\mathcal{C}_0[\vec{\varepsilon},\delta](\Omega,p_{\vec{\varepsilon}}) \right) d^3\Omega$$

  – $\mathcal{C}_0, \mathcal{C}_2$ calculated from orbifold construction

- Cancellation point-wise on $\mathcal{M}_2$ via the factorization identity

$$\boxed{\sum_{\delta\in\mathcal{D}[\vec{\varepsilon}]} \langle\delta_0|\delta\rangle\, \Xi_6[\delta](\Omega) = 6\Lambda[\vec{\varepsilon},\delta_0] \prod_{\delta\notin\mathcal{D}[\vec{\varepsilon}]} \vartheta[\delta](0,\Omega)^2}$$

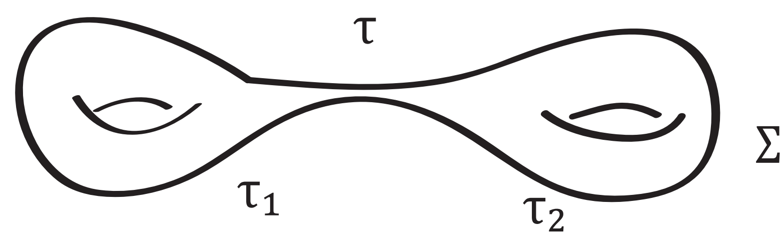
  for any $\delta_0 \in \mathcal{D}[\vec{\varepsilon}]$, and we have $\Lambda[\vec{\varepsilon},\delta_0]^2 = 1$.

- Proof includes Thomae map $\vartheta[\delta]^4$ to hyper-elliptic representation.

- Factorization identity is invariant under $Sp(4,\mathbb{Z})/\mathbb{Z}_4$
  – with $\mathbb{Z}_4 = \{I, J, -I, -J\}$ normal subgroup of $Sp(4,\mathbb{Z})$

Contributions from the boundary of $s\mathcal{M}_2$

• At separating degeneration node of $s\mathcal{M}_2$, integration is only conditionally convergent, due to right moving tachyon $\approx d\tilde{\tau}/\tilde{\tau}^2$
(Witten 2013)



• Regularization near separating node is required
  – consistent with physical factorization
  – produces a δ-function at separating node.

• To compute coefficient, decompose orbit $\mathcal{O}_+$ under modular subgroup
  – $Sp(2,\mathbb{Z}) \times Sp(2,\mathbb{Z}) \times \mathbb{Z}_2$ preserving separating degeneration
  – contributions only from $\vec{\varepsilon}$ such that $\mathcal{D}[\vec{\varepsilon}]$
  – contains one spin structure which decomposes to odd – odd

• Lengthy calculation shows
  – vanishing for $E_8 \times E_8$
  – non-vanishing for $Spin(32)/\mathbb{Z}_2$.

Superstring corrections to Type IIB supergravity

Superstring corrections to Type IIB supergravity

- String theory induces α' corrections to supergravity beyond $\mathcal{R}$
  - Local effective interactions from integrating out massive states
  - Non-analytic contributions from threshold effects

- Supersymmetry imposes strong constraints
  - supersymmetry e.g. prohibit $\mathcal{R}^2$, $\mathcal{R}^3$ corrections;
  - leading correction $\mathcal{R}^4$ subject to susy contraction of $R_{\mu\nu\rho\sigma}$

- S-duality requires axion/dilaton dependence through modular forms
  - S-duality in Type IIB on $\mathbb{R}^{10}$ is invariance under $SL(2,\mathbb{Z})$
  - axion-dilaton field $T \in \mathbb{C}$ with $T = \chi + i\, e^{-\phi}$ with $\mathrm{Im}(T) > 0$
  - $SL(2,\mathbb{Z})$ acts by $T \to (aT+b)/(cT+d)$
  - e.g. coefficient of $\mathcal{R}^4$ is a real Eisenstein series

$$\mathcal{E}_{(0,0)}(T) \sim \sum_{(m,n)\neq(0,0)} \frac{(\mathrm{Im}\, T)^{\frac{3}{2}}}{|m+nT|^3}$$

(Green, Gutperle 1997)

- Perturbative contributions only at tree-level and one-loop.

Superstring corrections of the form $D^{2p}\mathcal{R}^4$

- Accessible through 4-graviton amplitude

$$\mathcal{A}_4(\varepsilon_i, k_i; T) = \kappa^2 \, \mathcal{R}^4 \, \mathcal{I}_4(s,t,u;T)$$

  - ε_i, k_i are polarization tensor and momentum of gravitons;
  - $s = -\alpha' k_1 \cdot k_2/2$ etc are Lorentz invariants with $s+t+u=0$;
  - κ is 10-dimensional Newton constant.

- Expansions
  - Low energy for $|s|$, $|t|$, $|u| \ll 1$
    - ⋆ non-analytic part in s,t,u produced by massless states;
    - ⋆ analytic part in s,t,u producing local effective interactions.

$$\mathcal{I}_4(s,t,u;T)\Big|_{\rm analytic} = \sum_{m,n=0}^{\infty} \mathcal{E}_{(m,n)}(T) \left(s^2+t^2+u^2\right)^m \left(s^3+t^3+u^3\right)^n$$

    - ⋆ Coefficients $\mathcal{E}_{(m,n)}(T)$ are modular invariants in T.
  - Match with superstring perturbation theory for $g_s = ({\rm Im}\, T)^{-1} \to 0$

$$\mathcal{E}_{(m,n)}(T) = \sum_{h=0}^{\infty} g_s^{-2+2h} \, \mathcal{E}^{(h)}_{(m,n)} + \mathcal{O}(e^{-2\pi/g_s})$$

Predictions from Supersymmetry and S-duality

- Interplay of Type IIB and M-theory dualities from compactifications on $\mathbb{T}^d$

$\mathcal{R}^4$	$\mathcal{E}^{(0)}_{(0,0)} = 2\zeta(3)$	$\mathcal{E}^{(1)}_{(0,0)} = 4\zeta(2)$	$\mathcal{E}^{(h)}_{(0,0)} = 0, \quad h \geq 2$
$D^4\mathcal{R}^4$	$\mathcal{E}^{(0)}_{(1,0)} = \zeta(5)$	$\mathcal{E}^{(1)}_{(1,0)} = 0$	$\mathcal{E}^{(h)}_{(1,0)} = 0, \quad h \geq 3$
$D^6\mathcal{R}^4$	$\mathcal{E}^{(0)}_{(0,1)} = \frac{2}{3}\zeta(3)^2$	$\mathcal{E}^{(1)}_{(0,1)} = \frac{4}{3}\zeta(2)\zeta(3)$	$\mathcal{E}^{(h)}_{(0,1)} = 0, \quad h \geq 4$

- Non-vanishing coefficients at two and three loops

$$\mathcal{E}^{(2)}_{(1,0)} = \frac{4}{3}\zeta(4) \qquad \boxed{\mathcal{E}^{(2)}_{(0,1)} = \frac{8}{5}\zeta(2)^2} \qquad \mathcal{E}^{(3)}_{(0,1)} = \frac{4}{27}\zeta(6)$$

- Little is known beyond, for $D^8\mathcal{R}^4$, $D^{10}\mathcal{R}^4$ etc.
- basic references : (Green, Gutperle 1997)
  (Pioline; Green, Sethi 1998)
  (Green, Kwon, Vanhove; Green, Vanhove 1999)
  (Obers, Pioline 2000)
  (Green, Russo, Vanhove 2010) $\cdots$

Two-loop Type IIB 4-graviton amplitude

- Integral representation (ED & Phong 2001-2005)

$$\mathcal{I}_4^{(2)}(s,t,u;T) \sim g_s^2 \int_{\mathcal{M}_2} d\mu_2 \int_{\Sigma^4} \frac{|\mathcal{Y}|^2}{(\det Y)^2} \exp\Big(-\sum_{i<j} \alpha' k_i \cdot k_j \, G(z_i,z_j) \Big)$$

$$\mathcal{Y} = (k_1 - k_2)\cdot(k_3 - k_4)\,\omega_{[1}(z_1)\omega_{2]}(z_2)\,\omega_{[1}(z_3)\omega_{2]}(z_4) + 2 \text{ perm's}$$

  - $\omega_I(z)$ are the holomorphic Abelian differentials on Σ
  - $G(z,w)$ is a scalar Green function on Σ
  - $\Omega = X + iY$ with X, Y real matrices;
  - $d\mu_2$ canonical volume form on $\mathcal{M}_2$;

- $\mathcal{I}_4^{(2)}$ is defined by analytic continuation in s,t,u.

- Expansion for small s,t,u (using $\mathcal{Y}$ linear in s,t,u)
  - ⋆ As a result $\mathcal{R}^4$ coefficient $\mathcal{E}^{(2)}_{(0,0)} = 0$ (ED & Phong 2005)
  - ⋆ Confirm $D^4\mathcal{R}^4$ coefficient $\mathcal{E}^{(2)}_{(1,0)} = 4\zeta(4)/3$ (ED, Gutperle & Phong 2005)
  - ⋆ Calculating $D^6\mathcal{R}^4$ coefficient $\mathcal{E}^{(2)}_{(0,1)}$ requires integral with one power of G.

The Zhang-Kawazumi Invariant

- Integration over Σ^2 gives, (ED & Green 2013)

$$\mathcal{E}^{(2)}_{(0,1)} = \pi \int_{\mathcal{M}_2} d\mu_2 \, \varphi \qquad\qquad \varphi(\Sigma) \equiv -\frac{1}{8} \int_{\Sigma^2} P(x,y) \, G(x,y)$$

  – where P is a symmetric bi-form on Σ^2, defined by

$$P(x,y) = \sum_{I,J,K,L} \left(2Y^{-1}_{IL} Y^{-1}_{JK} - Y^{-1}_{IJ} Y^{-1}_{KL} \right) \omega_I(x) \overline{\omega_J(x)} \omega_K(y) \overline{\omega_L(y)}$$

  – φ conformal invariant, and modular invariant under $Sp(4,\mathbb{Z})$

- φ coincides with the invariant introduced by Zhang and Kawazumi (2008)

$$\varphi(\Sigma) = \sum_\ell \sum_{I,J} \frac{2}{\lambda_\ell} \left| \int_\Sigma \phi_\ell \, \omega'_I \wedge \overline{\omega'_J} \right|^2$$

  – ω'_I are holomorphic 1-forms normalized $\int_\Sigma \omega'_I \overline{\omega'_J} = -2i\delta_{IJ}$
  – ϕ_ℓ eigenfunction of the Arakelov Laplacian with eigenvalue λ_ℓ.
  – related to the Faltings δ-invariant (De Jong 2010)

Diff eqs from S-duality and Supersymmetry

- Direct integration of $\int_{\mathcal{M}_2} d\mu_2 \, \varphi$ appears out of reach.

- S-duality and supersymmetry lead to diff eqs in T (Pioline; Green, Sethi 1998)

$$(\Delta_T - 3/4) \, \mathcal{E}_{(0,0)}(T) = 0$$

  - satisfied by D-instanton sum in Type IIB (Green, Gutperle 1997)
  - Difficult to obtain diff eqs for higher coefficients

- Two-loop 11-d sugra on $\mathbb{T}^{d+1}$ for various d (Green, Kwon, Vanhove 2000)
  - conjecture diff eqs in perturbative and non-perturbative moduli m_d

$$\left(\Delta_{E_{d+1}} - \frac{3(d+1)(2-d)}{(8-d)} \right) \mathcal{E}_{(0,0)}(m_d) = 6\pi \, \delta_{d,2}$$

  - $\Delta_{E_{d+1}}$ Laplace operators on cosets $E_{d+1}(\mathbb{R})/K_{d+1}(\mathbb{R})$

$$E_1(\mathbb{R}) = SL(2,\mathbb{R})$$
$$E_2(\mathbb{R}) = SL(2,\mathbb{R}) \times \mathbb{R}^+ \qquad \cdots \qquad E_7(\mathbb{R}) = E_{7(7)}$$

  - $K_{d+1}(\mathbb{R})$ maximal compact subgroup of $E_{d+1}(\mathbb{R})$

Diff eqs from S-duality and Supersymmetry cont'd

- Expand differential equations for $\mathcal{E}_{(m,n)}(m_d)$ at weak string coupling
  - some moduli are not seen in perturbation theory (e.g. the axion)
  - moduli of torus $\mathbb{T}^d$ remain in perturbative limit: denote ρ_d

$$\mathcal{E}^{(2)}_{(0,1)}(\rho_d) = \pi \int_{\mathcal{M}_2} d\mu_2 \, \Gamma_{d,d,2}(\rho_d;\Omega) \, \varphi(\Omega)$$

  - where $\Gamma_{d,d,h}(\rho_d;\Omega)$ is the partition function on $\mathbb{T}^d$ for genus h
  - The perturbative part of $\mathcal{E}_{(0,1)}(m_d)$ satisfies,

$$\big(\Delta_{SO(d,d)} - (d+2)(5-d)\big) \, \mathcal{E}^{(2)}_{(0,1)}(\rho_d) = - \left(\mathcal{E}^{(1)}_{(0,0)}(\rho_d) \right)^2$$

  - For genus h and dimension d the torus partition function satisfies,

$$\left(\Delta_{SO(d,d)} - 2\Delta_\Omega + \frac{1}{2} dh(d-h-1) \right) \Gamma_{d,d,h}(\rho_d;\Omega) = 0$$

- Combining both implies the equation,

$$\int_{\mathcal{M}_2} d\mu_2 \, \varphi(\Omega) \, (\Delta_\Omega - 5) \, \Gamma_{d,d,2}(\rho_d,\Omega) = -\frac{\pi}{2} \left(\int_{\mathcal{M}_1} d\mu_1 \, \Gamma_{d,d,1}(\rho_d,\tau) \right)^2$$

  - This suggests $(\Delta_\Omega - 5)\varphi = 0$ in interior of $\mathcal{M}_2$.

Laplace eigenvalue equation for φ

- First prove the following Laplace eigenvalue equation,

$$(\Delta - 5)\varphi = -2\pi\, \delta^{(2)}_{SN}$$

  - where Δ is the Laplace-Beltrami operator on $\mathcal{M}_2$, represented as a fundamental domain for $Sp(4,\mathbb{Z})$ in Siegel upper half space.
  - and $\delta^{(2)}_{SN}$ is the volume form induced on the separating node of $\mathcal{M}_2$.

- Proven by methods of deformations of complex structures on Σ
  - derivatives with respect to Ω related to Beltrami differential μ

$$\delta_\mu \Omega_{IJ} = i \int_\Sigma \mu\, \omega_I \omega_J$$

  - Laplacian evaluated by computing $\delta_{\mu_1}\delta_{\bar\mu_2}\,\varphi$

Integrating φ over $\mathcal{M}_2$

- The integral $\int_{\mathcal{M}_2} d\mu_2 \, \varphi$ is absolutely convergent
  - to obtain a concrete relation, parametrize Ω by

$$\Omega = \begin{pmatrix} \tau_1 & \tau \\ \tau & \tau_2 \end{pmatrix} \qquad\qquad d\mu_2 = \frac{d^2\tau \, d^2\tau_1 \, d^2\tau_2}{(\det Y)^3}$$

  - asymptotics of φ near separating node where $\tau \to 0$

$$\varphi(\Omega) = -\ln \left| 2\pi\tau\eta(\tau_1)^2\eta(\tau_2)^2 \right| + \mathcal{O}(\tau^2)$$

  - near non-separating node where $\tau_2 \to i\infty$ using (Fay, Wentworth)

$$\varphi(\Omega) = \frac{\pi}{6}\mathrm{Im}\tau_2 + \frac{5\pi(\mathrm{Im}\tau)^2}{6\mathrm{Im}\tau_1} - \ln \left| \frac{\vartheta_1(\tau,\tau_1)}{\vartheta_1(0,\tau_1)} \right| + \mathcal{O}(1/\tau_2)$$

  - maximal non-separating ("supergravity" or "tropical") limit $\ell_i \to \infty$

$$\Omega = i \begin{pmatrix} \ell_1 + \ell_3 & \ell_3 \\ \ell_3 & \ell_2 + \ell_3 \end{pmatrix} \qquad\qquad \varphi(\Omega) = \frac{\pi}{6}\left(\ell_1 + \ell_2 + \ell_3 - \frac{5\, \ell_1\ell_2\ell_3}{\ell_1\ell_2 + \ell_2\ell_3 + \ell_3\ell_1} \right)$$

(Green, Russo, Vanhove 2008), (Tourkine 2013)

Integrating φ over $\mathcal{M}_2$ (cont'd)

- Integral on cut-off moduli space $\mathcal{M}_2^\varepsilon = \mathcal{M}_2 \cap \{|\tau| > \varepsilon\}$
  - using convergence of integral, and $(\Delta - 5)\varphi = -2\pi\, \delta_{SN}^{(2)}$

$$\int_{\mathcal{M}_2} d\mu_2\, \varphi = \lim_{\varepsilon \to 0} \int_{\mathcal{M}_2^\varepsilon} d\mu_2\, \varphi = \frac{1}{5} \lim_{\varepsilon \to 0} \int_{\mathcal{M}_2^\varepsilon} d\mu_2\, \Delta\varphi$$

  - reduces to integral over boundary

$$\partial \mathcal{M}_2^\varepsilon = \{|\tau| = \varepsilon\} \times \left(\mathcal{M}_1^{(1)} \times \mathcal{M}_1^{(2)} \right) / (\mathbb{Z}_2 \times \mathbb{Z}_2)$$

  - contribution from non-separating node vanishes
  - contribution from separating node governed by limit of,

$$d\mu_2\, \Delta\varphi = d \left\{ \frac{i}{2} \left(\frac{d\bar\tau}{\bar\tau} - \frac{d\tau}{\tau} \right) \wedge d\mu_1^{(1)} \wedge d\mu_1^{(2)} \right\}$$

  - using $\int_{\mathcal{M}_1} d\mu_1 = 2\pi/3$, and 4π from τ-integral, and $1/4$ from $\mathbb{Z}_2 \times \mathbb{Z}_2$

$$\int_{\mathcal{M}_2} d\mu_2\, \varphi = \frac{1}{5} \times \frac{1}{2} \times 4\pi \times \left(\frac{2\pi}{3} \right)^2 \times \frac{1}{4} = \frac{2\pi^3}{45}$$

  - Exact agreement with predictions from S-duality and supersymmetry

Outlook

$\surd$ Interplay between superstring perturbation theory, S-duality, supersymmetry

$\surd$ Integrated over $\mathcal{M}_2$ a non-trivial modular invariant φ

- For higher genus, $h \geq 3$, the ZK invariant exists,
  - but does not satisfy $(\Delta - \lambda)\varphi = 0$
  - string theory significance ?
  - Pure spinor calculation for $\mathcal{E}^{(3)}_{(0,1)}$ (Gomez, Mafra 2014)
- For $D^8\mathcal{R}^4$, $D^{10}\mathcal{R}^4, \cdots$ two-loop superstring perturbation theory
  - suggests new invariants (ED, Green 2013)
  - significance in theory of modular invariants – number theory ?
  - can one match with S-duality and supersymmetry in string theory ?