Some Geometrical Problems in AdS/CFT

Eric D'Hoker

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Outline

I. What is the AdS/CFT correspondence ?
   – $\mathcal{N} = 4$ Super Yang-Mills theory;
   – Type IIB String Theory and Super Gravity;

II. Generalization (1) Geometries with supersymmetry
   – Sasaki-Einstein manifolds

III. Generalization (2) Geometries with a planar interface
   – exact solutions without/with supersymmetry
   – relation with integrable system

IV. Some open problems

I. The AdS/CFT Correspondence

The AdS/CFT correspondence is a conjectured equivalence between,

$\mathcal{N} = 4$ Super Yang-Mills $\Leftrightarrow$ Type IIB string theory
on flat Minkowski $\mathbf{R}^4$ on $AdS_5 \times S^5$

Yang-Mills Theory $\Leftrightarrow$ A Theory of gravity

Maldacena (1997);
Gubser, Klebanov, Polyakov (1998);
Witten (1998)

$\mathcal{N}=4$ Super Yang-Mills

- Yang-Mills theory with gauge group $SU(N)$ on flat Minkowski $\mathbf{R}^4$;

A		$SU(N)$ connection	$\mathbf{1}$ of $SU(4)_R$
ψ^a	$a=1,\cdots,4$	4 Weyl gauginos	$\mathbf{4}$ of $SU(4)_R$
ϕ^i	$i=1,\cdots,6$	6 real scalars	$\mathbf{6}$ of $SU(4)_R$

- All fields transform in the adjoint representation of $SU(N)$;
  - field strength $F=dA+A\wedge A$
  - covariant derivative $D\phi=d\phi+[A,\phi]$
- The Lagrangian is given by

$$\mathcal{L}=-\frac{1}{2g^2}\mathrm{tr}\Big(|F|^2+|D\phi^i|^2+i\bar{\psi}\gamma\cdot D\psi-\frac{1}{2}[\phi^i,\phi^j]^2-\psi^t\mathcal{C}\rho^i[\phi^i,\psi]\Big)$$

- The theory is invariant under extended Poincaré supersymmetry,

$$\{Q^a_\alpha, \bar Q^b_\beta\} = \delta^{ab}\gamma^\mu_{\alpha\beta}P_\mu \qquad\qquad a,b = 1,\cdots,\mathcal{N}=4$$

which maps the fields as follows, $A \xrightarrow{Q} \psi \xrightarrow{Q} \phi \xrightarrow{Q} \psi \xrightarrow{Q} A$

- $\mathcal{N}=4$ is the maximal Poincaré supersymmetry in YM theory.
  - $SU(4)_R$ is the automorphism group of the superalgebra.

- $\mathcal{L}$ invariant under conformal transformations $SO(2,4) \sim SU(2,2)$,
  - classically and quantum mechanically
  - adding conformal supersymmetries S^a_α and $\bar S^a_\alpha$
  - Full invariance group is $SU(2,2|4) \supset SO(2,4) \times SU(4)_R$

- Adding to $\mathcal{L}$ the second Chern class $\theta\, \mathrm{tr} F \wedge F$,
  - the complex coupling $\tau = \theta/2\pi + 4\pi i/g^2$ has
  - $SL(2,Z)$ acting by $\tau \to (a\tau+b)/(c\tau+d)$ (Mononen-Olive)

Type IIB String Theory

- Type IIA and Type IIB have maximal $\mathcal{N} = 2$ supersymmetry;
  - The IIA and IIB massive spectra coincide.
  - IIA has massless forms of rank 1, 3, ($D0$, $D2$, $D4$, $D6$ branes)
  - IIB has massless forms of rank 0,2,4, ($D1$, $D3$, $D5$, $D7$ branes)
- The massive string states are heavy $(\alpha')^{-\frac{1}{2}} \sim 10^{19} \times$ proton mass.
  - At energy scales $\ll (\alpha')^{-\frac{1}{2}}$, only the massless states matter;
- At low energy, Type IIB string theory reduces to Type IIB supergravity.
  - Advantage : supergravity may be described by local fields.

Type IIB Supergravity

- The fields of Type IIB supergravity on M_{10} are

$g_{\mu\nu}$	metric	$\mu, \nu = 0, 1, \cdots, 9$
τ	complex axion/dilaton	$T = d\tau / \mathrm{Im}\tau$
$B_{(2)}$	complex anti − symm	field strength $G_{(3)}$
$C_{(4)}$	real anti − symm	field strength $F_{(5)}$
ψ_μ	gravitino	Weyl spinor
λ	dilatino	Weyl spinor

- The anti-symmetric tensor fields are generalized gauge fields, eg

$$G_{(3)} = (dB_{(2)} - \tau d\bar{B}_{(2)})/\mathrm{Im}\tau \qquad\qquad \delta B_{(2)} = db_{(1)}$$

- Supersymmetry tranf. for the spinors are (vanishing fermions)

$$\delta\psi_\mu = \nabla_\mu \varepsilon + \frac{i}{4}(F_{(5)} \cdot \Gamma)\,\Gamma_\mu \varepsilon - \frac{1}{16}(\Gamma_\mu(G_{(3)} \cdot \Gamma) + 2(G_{(3)} \cdot \Gamma)\Gamma_\mu)\varepsilon^*$$

- Here, Γ^μ form a representation of Cliff(1,9), and

$$H_{(n)} \cdot \Gamma = \frac{1}{n!} H_{\mu_1 \cdots \mu_n} \Gamma^{[\mu_1} \Gamma^{\mu_2} \ldots \Gamma^{\mu_n]}$$

- The field eqs consist of the self-duality relation $*F_{(5)} = F_{(5)}$ and, eg

$$R_{\mu\nu} = 2T_{\{\mu}\bar{T}_{\nu\}} + \frac{1}{6}(F^2_{(5)})_{\mu\nu} + \frac{1}{24}(G_{(3)}\bar{G}_{(3)})_{\mu\nu}$$

Anti-de Sitter space AdS_5

- AdS_{n+1} is the Minkowski signature hyperbolic upper half space.
  - A global parametrization is by coordinates $(x_{-1}, x_0, \vec{x}) \in \mathbf{R}^{n+2}$,

$$
\begin{aligned}
-L^2 &= -(x_{-1})^2 - (x_0)^2 + (\vec{x})^2 \\
ds^2_{AdS_{n+1}} &= -(dx_{-1})^2 - (dx_0)^2 + (d\vec{x})^2
\end{aligned}
$$

  - The isometry is $SO(2, n)$, specifically $SO(2, 4)$ for AdS_5.

AdS/CFT – mapping parameters

SYM on $\mathbf{R}^4$	$\Leftrightarrow$	Type IIB strings	on $AdS_5 \times S^5$
gauge group $SU(N)$	$\Leftrightarrow$	$\int_{S^5} F_{(5)} = N$	
gauge coupling g^2	$=$	$g_s = 4\pi g^2$	string coupling
't Hooft coupling g^2N	$\Leftrightarrow$	radius $L^4 = g_s N(\alpha')^2$	of AdS_5 and S^5
large N (fixed g^2N)	$\Leftrightarrow$	classical strings	(still unsolved)
also large g^2N	$\Leftrightarrow$	Type IIB sugra	on $AdS_5 \times S^5$ (this is tractable)

AdS/CFT – mapping symmetries

SYM in d=4	$\Leftrightarrow$	Type IIB strings	on $AdS_5 \times S^5$
$SO(2,4)$ conformal	$=$	$SO(2,4)$	isometry of AdS_5
$SU(4)_R$ automorph.	$=$	$SU(4) \sim SO(6)$	isometry of S^5
$SU(2,2\|4)$ superconf.	$=$	$SU(2,2\|4)$	"isometry"
$SL(2,Z)$ duality	$=$	S-duality strings	$SL(2,R)$ (in supergravity)

Holography

- The $AdS_5 \times S^5$ metric is $ds^2 = g^{AdS}_{\mu\nu} dz^\mu dz^\nu + g^S_{ab} dy^a dy^b$

- Fields ϕ on $AdS_5 \times S^5$ are decomposed in spherical harm. Y_Δ on S^5,

$$\phi(z,y) = \sum_{\Delta=0}^{\infty} \phi_\Delta(z) Y_\Delta(y)$$

- The fields ϕ_Δ couple via the boundary of AdS_5

$$\varphi_\Delta(\vec{z}) = \lim_{z_0 \to 0} z_0^{\Delta-4} \phi_\Delta(z_0, \vec{z})$$

II. Generalizing the AdS/CFT correspondence

- Only asymptotically conformal, asymptotically $AdS_5 \times M_5$
  - Even with supersymmetry, this is the hardest case
  - Physically the most interesting
  - (Klebanov Witten; Polchinski Strassler; Klebanov Strassler)
- Still conformal $SO(2,4)$ invariant, hence $AdS_5 \times M_5$,
  $+$ Less supersymmetry (e.g. $\mathcal{N} = 1$)
- Conformal subgroup $SO(2,3)$ invariant, hence $AdS_4 \times M_6$,
  $+$ Yang-Mills theories with a planar INTERFACE

AdS/CFT with conformal supersymmetry

- Seek general solution with
  - $SO(2,4)$ conformal invariance
  - at least $\mathcal{N}=1$ supersymmetry, $SU(2,2|1)$
  - constant dilaton τ, $B_{(2)}=0$
- Of the form $AdS_5 \times M_5$.
  - Field equations require M_5 to be Einstein.
  - Supersymmetry requires M_5 to be Sasaki.
- The dual Yang-Mills theory has $\mathcal{N}=1$ supersymmetry;

  $$\{Q_\alpha, \bar{Q}_\beta\} = \gamma^\mu_{\alpha\beta} P_\mu$$

  - always has a $U(1)_R$ symmetry
    $=$ the automorphism group of supersymmetry algebra

M_5 is Einstein

- Express the metric in terms of an orthonormal frame e^a
  - $ds^2_{AdS_5} = \eta_{ab}\, e^a \otimes e^b$, with $a, b = 0, 1, 2, 3, 4$
  - $ds^2_{M_5} = \delta_{ab}\, e^a \otimes e^b$, with $a, b = 5, 6, 7, 8, 9$
- Self-duality $*F_{(5)} = F_{(5)}$ requires

$$F_{(5)} = (-e^{01234} + e^{56789}), \quad \text{using notation } e^{a_1 \cdots a_n} = e^{a_1} \wedge \cdots \wedge e^{a_n}$$

- The Type IIB supergravity field equations then reduce to

$$R_{ab} = \frac{1}{6}(F^2_{(5)})_{ab} \quad = \quad \begin{cases} -4\eta_{ab} & a, b = 0, 1, 2, 3, 4 \\ +4\delta_{ab} & a, b = 5, 6, 7, 8, 9 \end{cases}$$

M_5 is Sasaki

- Supersymmetry of a field configuration requires
  - vanishing gravitino variation $\delta\psi_\mu = 0$,

$$\nabla_\mu \varepsilon + \frac{i}{4}(F_{(5)} \cdot \Gamma)\,\Gamma_\mu \varepsilon = 0, \qquad\qquad F_{(5)} \cdot \Gamma = -\Gamma^{01234} + \Gamma^{56789}$$

- On M_5, reduces to $\nabla_a \varepsilon \pm \frac{i}{2}\Gamma_a \varepsilon = 0$ having non-zero solutions ε;
- This may be used as a definition of M_5 being Sasaki. Alternatively,
  - the cone $dr^2 + r^2 ds^2_{M_5}$ over M_5 is Kähler.
  - holonomy group is reduced to a subgroup of $SU(3)$.
- Sasaki-Einstein manifolds have at least one Killing vector
  - which generates the $U(1)_R$ of the YM theory

New Sasaki-Einstein manifolds

- There are infinite series of $\mathcal{N}=1$ superconformal Yang-Mills theories.
  - now having product gauge groups $\prod SU(N_i)$,
  - with various assignments of $U(1)_R$ charges

    e.g. Benvenuti, Franco, Hanany, Martelli, Wecht
- Thus, one expects discrete infinite series of Sasaki-Einstein spaces M_5;
- Foliation of M_5 by $U(1)_R$: leafs $\hat{M}_4$ are locally Kähler-Einstein;
  - REGULAR: free action by a circle; all $\hat{M}_4$ known (Tian, Yau 1987)

    $\hat{M}_4 = CP^2$, then $M_5 = S^5$ original $AdS_5 \times S^5$

    $\hat{M}_4 = CP^1 \times CP^1$, then $M_5 = T^{1,1} = (SU(2) \times SU(2))/U(1)$
  - QUASI-REGULAR: orbifold of circle action; rational $U(1)_R$ charges
  - IRREGULAR: $U(1)_R$ orbits do not close, irrational $U(1)_R$ charges
- Inhomogeneous quasi-regular Sasaki-Einstein metrics exist on $S^2 \times S^3$

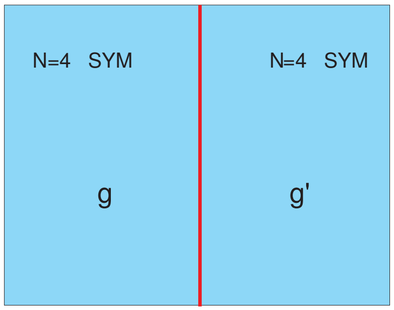
  (Boyer, Galicki, Nakamaye 2001)

- Explicit construction, including for the metric, exists for an infinite series,
- Theorem (Gauntlett, Martelli, Sparks, Waldram 2004)
  - For every positive curvature $2n$-dimensional Kähler-Einstein B_{2n}, There exits a countable infinite class of compact Sasaki-Einstein manifolds X_{2n+3} of dimension $2n+3$;
  - The holonomy group of X_{2n+3} is a subgroup of $SU(n+2)$;
  - The isometry group $\supset G \times T^2$ as subgroup, ($G =$ isometry of B_{2n}).
- Locally, the metrics are given explicitly,

$$ds^2 = \rho^2 ds^2_{B_{2n}} + u(\rho)^{-1} d\rho^2 + \rho^2 u(\rho)(d\tau - A)^2 + (d\psi - \sigma)^2$$

  - dA is the Kähler form of B_{2n},
  - $u(\rho) = a + b\rho^2 + c\rho^{-2-2n}$, with a, b, c discrete
  - $d\sigma = \rho^2 dA + \rho(d\tau - A) \wedge d\rho$
  - some of these are of irregular type

III. Interface AdS/CFT



- YM theory has $SO(2,3)$ conformal symmetry of planar interface;
- Type IIB solution given by AdS_4-slicing, which is asymptotically AdS_5;
- Varying YM coupling, hence varying dilaton;
- Closely related to boundary CFT (Cardy; Sen)
- Developed in Bak, Gutperle, Hirano (2003);
  D'Hoker, Estes, Gutperle (2006).

Classification of supersymmetries on YM side

- Generically, interface breaks supersymmetry;
- Restored by adding interface operators, modifying transformation law;

0 supercharges $SO(2,3) \times SU(4)$, (non-supersymmetric Janus);

2 supercharges $OSp(4|1) \times SU(3)$; (Freedman, Karch, Schnabl)

4 supercharges $OSp(4|2) \times SU(2)$; (D'Hoker, Estes, Gutperle)

8 supercharges $OSp(4|4)$; (")

The Janus solution with no supersymmetry

- The solution is a 1-parameter deformation of $AdS_5 \times S^5$;

$$ds^2 = f(\mu)^2 (d\mu^2 + ds^2_{AdS_4}) + ds^2_{S^5} \qquad \phi = \phi(\mu)$$

$AdS_5 \times S^5$ has $f(\mu) = (\cos\mu)^{-1}$, with $\mu \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and ϕ constant.

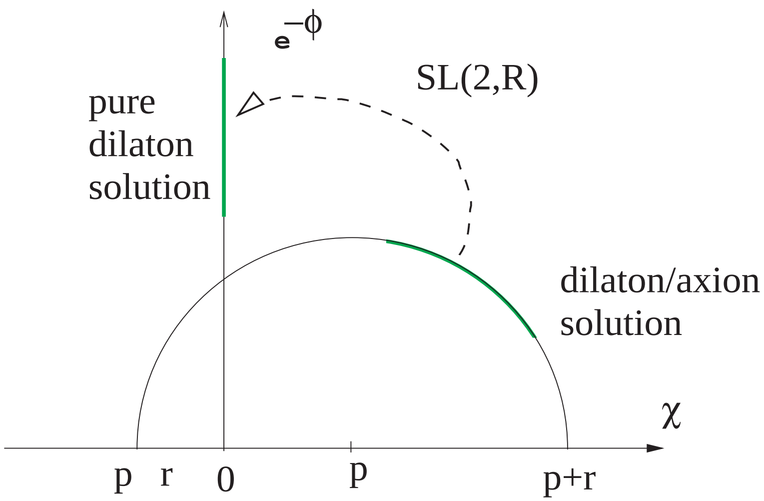
- Janus is an exact solution, given by an elliptic integral,

$$4f(\mu)d\mu = \frac{f^2 df^2}{\sqrt{f^8 - f^6 + c^2/24}} \qquad \phi'(\mu) = cf(\mu)^{-3}$$

where $\mu \in [-\mu_c, \mu_c]$, with $\mu_c > \pi/2$; non-singular for $0 < c^2 < 81/32$.

Janus with varying axion

- Type IIB supergravity has $SL(2,\mathbf{R})$ symmetry, mixing dilaton - axion.
- All solutions with axion are $SL(2,\mathbf{R})$ images of solutions without axion.



The Janus solution for the $SU(3)$ theory

- $SO(2,3)$ requires AdS_4-slicing of AdS_5;
- $SU(3)$ singles out S^1 fibration over CP_2;
- $SL(2,\mathbf{R})$ symmetry of Type IIB realized on family of Ansätze

$$
\begin{aligned}
ds^2 &= f_4^2(d\mu^2 + ds^2_{AdS_4}) + f_1^2(d\beta + A_1)^2 + f_2^2 ds^2_{CP_2} \\
B_{(2)} &= f_3 A_2 + g_3 \bar{A}_2 + f_6 K \\
F_{(5)} &= f_4^{-5} g_5' \, (-e^{01234} \; + \; e^{56789})
\end{aligned}
$$

- $K = dA_1$ is the Kähler form on CP_2,
- A_2 is the unique $SU(3)$-invariant form on S^5,
- The dilaton ϕ, the coefficients $f_{1,2,3,4,6}$ and $g_{3,5}$ are functions of μ only.

Conditions for supersymmetry

- Recall the vanishing of the susy variation (or BPS) equations,

$$
\begin{aligned}
0 &= iT \cdot \Gamma \, \mathcal{B}^{-1} \varepsilon^* - \frac{i}{4} (G_{(3)} \cdot \Gamma) \varepsilon \\
0 &= D_\mu \varepsilon + \frac{i}{4} (F_{(5)} \cdot \Gamma) \, \Gamma_\mu \varepsilon - \frac{1}{16} (\Gamma_\mu (G_{(3)} \cdot \Gamma) + 2 (G_{(3)} \cdot \Gamma) \Gamma_\mu) \mathcal{B}^{-1} \varepsilon^*
\end{aligned}
$$

- The antisymmetric 3-form is given by complex functions a, b, c, d,

$$
\begin{aligned}
G_{(3)} &= (a e^5 - i b e^4) \wedge A_2 + (c e^5 - i d e^4) \wedge \bar{A}_2 \\
a &= -3 f_1^{-1} f_2^{-2} f (f_3 - B g_3) \qquad c = a(f_3 \leftrightarrow \bar{g}_3) \\
b &= -f_4^{-1} f_2^{-2} f (f_3' - B g_3') \qquad d = b(f_3 \leftrightarrow \bar{g}_3)
\end{aligned}
$$

- The remaining condition from the dilatino equation is

$$(c+d)(c-d)^* = f^4 |B'|^2 (f_4)^{-2} \qquad\qquad f \equiv (1-|B|^2)^{-1/2}$$

- The gravitino equation along the directions $4,5,6,7,8,9$ require

$$\frac{f_1'}{f_1} = \frac{f_4^2}{2f^2 B'}(3ac - bd) \qquad\qquad \frac{f_1 f_4}{f_2^2} - f_4 f_5 = \frac{f_4^2}{2f^2 B'}(ad+bc)$$

$$\frac{f_2'}{f_2} = \frac{f_4^2}{2f^2 B'}(ac + bd) \qquad\qquad \frac{f_4}{f_1} - \frac{f_1 f_4}{f_2^2} = \frac{f_4^2}{3f^2 B'}(ad-bc)$$

- The gravitino equation along the directions of AdS_4 require

$$\frac{f_1^2 f_4^2}{f_2^4} - \left(\frac{f_2'}{f_2} + \frac{f_4'}{f_4}\right)^2 = 1$$

- This is closely related to a Toda integrable system.

Exact regular supersymmetric Janus solutions

- Numerical analysis of the BPS eqs indicates singularities unless $a = 0$.
- For $a = 0$, the solution is regular, and asymptotically AdS_5;
- BPS eqs decouple, may be integrated in terms of a single function ψ,

$$(f_4)^4 = \rho^2(\psi^{-4} + \sigma^2\psi^2) \qquad f_1 = \psi/f_4 \qquad c = 9\sigma^2\psi^4(f_4)^6$$

$$f_2 = \rho/(f_4\psi) \qquad \text{etc}$$

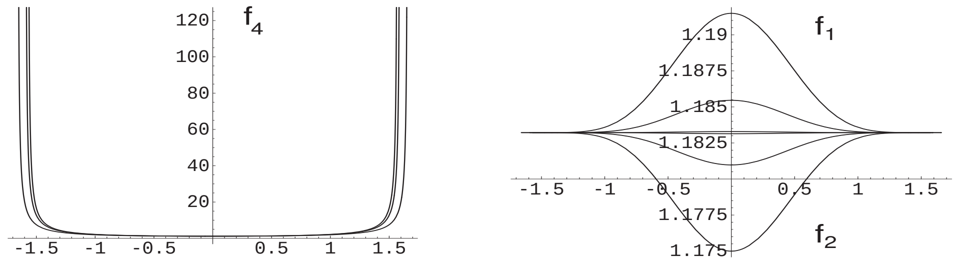
- The function ψ is given by an Abelian integral of genus 3.

$$(\psi')^2 = (1 + \sigma^2\psi^6)^2 - \psi^2$$

- The family of solutions has 2 continuous parameters ρ, σ.

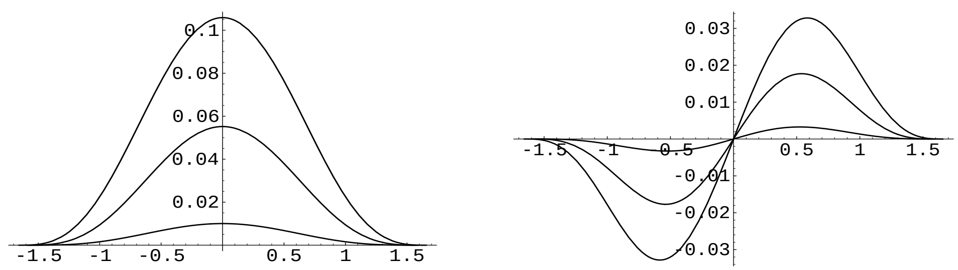
Numerical Results

- For 3 sets of parameters ρ, σ, plots of f_4 (left) and f_1, f_2 (right),



(a) (b)

- For 3 sets of parameters ρ, σ, plots of c (left) and d (right),



(a) (b)

Summary and Open problems

- The space-time part of a Type IIB solution restricts the internal space
  - $\mathbf{R}^4 \times K_6$ requires K_6 Calabi-Yau;
  - $AdS_5 \times M_5$ requires M_5 Sasaki-Einstein;
  - $AdS_4 \times \mathbf{R} \times N_5$ interface requires N_5 as deformation of S^5;
- General case of interest includes supersymmetric solutions of the form
  - $\Sigma_5 \times \Lambda_5$ and Σ_5 has Poincaré $ISO(1,3)$ isometry;
- Especially interesting is Σ_5 interpolating between
  - two asymptotic AdS_5 with different radii;
  - which represents two end points of renormalization group flow.
- What is the general relation between Σ_5 and Λ_5 ?